Exercise 24.2

1. Solution:

Let $\vec{o}$, $\vec{a}$ and $\vec{b}$ be the position vector of the O, A and B.

P and Q are points of trisection of AB.

Position vector of point P = $\frac{2\vec{a} + \vec{b}}{3}$

Position vector of point Q = $\frac{\vec{a} + 2\vec{b}}{3}$

Now,

$$OP = \frac{2\vec{a} + \vec{b}}{3} - \vec{o} = \frac{2\vec{a} + \vec{b} - 3\vec{o}}{3} = \frac{2OA + OB}{3}$$

$$OQ = \frac{\vec{a} + 2\vec{b}}{3} - \vec{o} = \frac{\vec{a} + 2\vec{b} - 3\vec{o}}{3} = \frac{OA + 2OB}{3}$$

$$OP^2 + OQ^2 = \left(\frac{2OA + OB}{3}\right)^2 + \left(\frac{OA + 2OB}{3}\right)^2$$

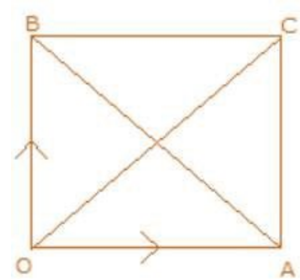
$$= \frac{5(OA^2 + OB^2) + 8(OA)(OB)\cos 90^\circ}{9}$$

$$= \frac{5}{9}AB^2 \ldots\ldots\ldots\left[\because OA^2 + OB^2 = AB^2 \text{ and } \cos 90^\circ = 0\right]$$

$\therefore OP^2 + OQ^2 = 5/9\ AB^2$

2. Solution:

Let OACB be a quadrilateral such that its diagonal bisect each other at right angles.
We know that, if the diagonals of a quadrilateral bisect each other then it's a parallelogram.
Thus, OACB is a parallelogram
So,
OA = BC and OB = AC
Now,
Taking O as the origin. Let $\vec{a}$ and $\vec{b}$ be the position vector of A and B
AB and OC be the diagonals of quadrilateral which bisect each other at right angles.



$$\Rightarrow (\overrightarrow{OC}).(\overrightarrow{AB}) = 0$$

$$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$$

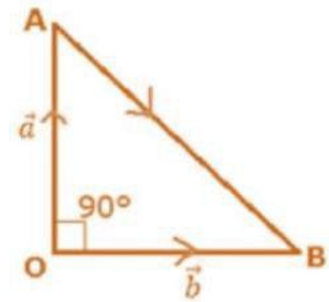
$$|a|^2 + \vec{a}.\vec{b} - \vec{a}.\vec{b} - |b|^2 = 0$$

$$|a|^2 = |b|^2$$

$$OA = OB$$

Similarly,
OA = OB = BC = CA
Therefore, OACB is a rhombus.

3. Solution:



Let ΔAOB be a right-angle triangle with right angle at O.
Required to prove: $AB^2 = OA^2 + OB^2$

Taking O as the origin, we have

$\vec{a}$ and $\vec{b}$ to be the position vector of A and B respectively.
Now, as OB is perpendicular to OA their dot product equals to zero
So, we have

$$(\overrightarrow{OA}).(\overrightarrow{OB}) = 0$$

$$\vec{a}.\vec{b} = 0 \text{ ......(i)}$$

And,

$$(\overrightarrow{AB})^2 = (\vec{b} - \vec{a})^2$$

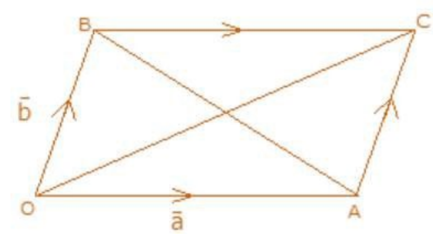
$$\Rightarrow (\overrightarrow{AB})^2 = a^2 + b^2 - 2\vec{a}.\vec{b}$$

From equation (i), we have

$$\Rightarrow (\overrightarrow{AB})^2 = a^2 + b^2 - 0$$

Therefore,
$AB^2 = OA^2 + OB^2$
- Hence proved

4. Solution:



Let OAC be a right triangle, right angled at O.
Now, taking O as the origin

Let $\vec{a}$ and $\vec{b}$ be the position vector of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$

$\therefore \overrightarrow{OA} \bullet \overrightarrow{OB} = 0$

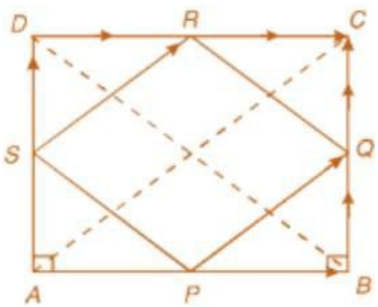
$\vec{a} \bullet \vec{b} = 0$

Now,

$$\overrightarrow{AB}^2 = (\vec{b} - \vec{a})^2 = (\vec{a})^2 + (\vec{b})^2 - 2\vec{a} \bullet \vec{b} = (\vec{a})^2 + (\vec{b})^2 - 0 = (\overrightarrow{OA})^2 + (\overrightarrow{OB})^2$$

- Hence proved

5. Solution:



Given, ABCD is a rectangle
Let P, Q, R and S be the mid points of the sides AB, BC, CD and DA respectively.
Now,

$$\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \frac{1}{2}\overrightarrow{AC}\ldots\ldots\ldots(i)$$

$$\overrightarrow{SR} = \overrightarrow{SD} + \overrightarrow{DR} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{DC}\right) = \frac{1}{2}\overrightarrow{AC}\ldots\ldots\ldots(ii)$$

From (i) and (ii), we have

$\overrightarrow{PQ} = \overrightarrow{SR}$ i.e. sides PQ and SR are equal and parallel.

∴ PQRS is a parallelogram.

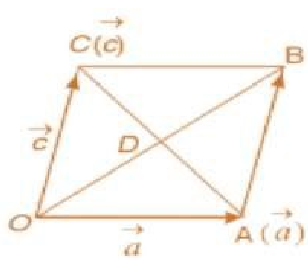
Now,

$$(PQ)^2 = \overrightarrow{PQ}\bullet\overrightarrow{PQ} = \left(\overrightarrow{PB} + \overrightarrow{BQ}\right)\bullet\left(\overrightarrow{PB} + \overrightarrow{BQ}\right) = |PB|^2 + |BQ|^2\ldots\ldots\ldots(iii)$$

$$(PS)^2 = \overrightarrow{PS}\bullet\overrightarrow{PS} = \left(\overrightarrow{PA} + \overrightarrow{PS}\right)\bullet\left(\overrightarrow{PA} + \overrightarrow{PS}\right) = |PA|^2 + |AS|^2 = |PB|^2 + |BQ|^2\ldots\ldots\ldots(iv)$$

From (iii) and (iv), we get
$(PQ)^2 = (PS)^2$
⇒ PQ = PS
So, the adjacent sides of PQRS are equal
Hence, PQRS is rhombus.

6. Solution:



Let OABC be a rhombus, whose diagonals OB and AC intersect at point D
And, let O be the origin
Let the position vector of A and C be $\vec{a}$ and $\vec{c}$ respectively then,

$\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OC} = \vec{c}$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OC} = \vec{a} + \vec{c}$............$\left[\because \overrightarrow{AB} = \overrightarrow{OC}\right]$

Position vector of mid-point of $\overrightarrow{OB} = \frac{1}{2}(\vec{a} + \vec{c})$

Position vector of mid-point of $\overrightarrow{AC} = \frac{1}{2}(\vec{a} + \vec{c})$

$\therefore$ Midpoints of OB and AC coincide.

$\therefore$ Diagonal OB and AC bisect each other.

$\overrightarrow{OB} \bullet \overrightarrow{AC} = (\vec{a} + \vec{c}) \bullet (\vec{c} - \vec{a}) = (\vec{c} + \vec{a}) \bullet (\vec{c} - \vec{a}) = |\vec{c}|^2 - |\vec{a}|^2 = \overrightarrow{OC} - \overrightarrow{OA} = 0$

[$\because$ OC and OA are sides of the rhombus]

$\Rightarrow \overrightarrow{OB} \perp \overrightarrow{AC}$

7. Solution:

Let ABCD be a rectangle

Taking A as the origin, we have position vectors of point B and D to be $\vec{a}$ and $\vec{b}$ respectively

By parallelogram law,

$\overrightarrow{AC} = \vec{a} + \vec{b}$ and $\overrightarrow{BD} = \vec{a} - \vec{b}$

As ABCD is a rectangle, AB $\perp$ AD

$\Rightarrow \vec{a} \bullet \vec{b} = 0$..........(i)

Now, diagonals AC and BD are perpendicular iff $\overrightarrow{AC} \bullet \overrightarrow{BD} = 0$

$(\vec{a} + \vec{b})(\vec{a} - \vec{b}) = 0$

$(\vec{a})^2 - (\vec{b})^2 = 0$

$|\overrightarrow{AB}|^2 = |\overrightarrow{AD}|^2$

$\Rightarrow |AB| = |AD|$

Hence, ABCD is a square.