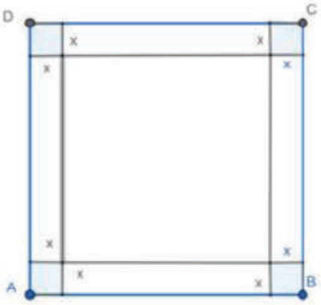


Let us consider,

- ‘x’ be the length and breadth of the piece cut from each vertex of the piece.
- Side of the box now will be (12-2x)
- The height of the new formed box will also be ‘x’.

Let the volume of the newly formed box is :

$V = (12\text{-}2x)^2 \times (x)$

$V = (144 + 4x^2 - 48x)\ x$

$V = 4x^3 - 48x^2 + 144x$ ------ (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dV}{dx} = \frac{d}{dx}[4x^3 - 48x^2 + 144x]$$

$$\frac{dV}{dx} = 12x^2 - 96x + 144 \text{ -------- } (2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dV}{dx} = 12x^2 - 96x + 144 = 0$$

$x^2 - 8x + 12 = 0$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(12)}}{2(1)} = \frac{8 \pm \sqrt{64 - 48}}{2} = \frac{8 \pm \sqrt{16}}{2}$$

$$x = = \frac{8 \pm 4}{2}$$

x = 6 or x =2

x= 2

[as x = 6 is not a possibility, because 12-2x = 12-12= 0]

Now to check if this critical point will determine the maximum area of the box, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{dx^2} = \frac{d}{dx}[12x^2 - 96x + 144]$$

$$\frac{d^2V}{dx^2} = 24x - 96 \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dx^2}\right)_{x=2} = 24\,(2) - 96 = 48 - 96 = -48$$

As $\left(\frac{d^2V}{dx^2}\right)_{x=2} = -48 < 0$, so the function A is maximum at x = 2

Now substituting x = 2 in 12 – 2x, the side of the considered box:

Side = 12-2x = 12 - 2(2) = 12-4= 8cms

Therefore side of wanted box is 8cms and height of the box is 2cms.

Now, the volume of the box is

$V = (8)^2 \times 2 = 64 \times 2 = 128cm^3$

Hence maximum volume of the box formed by cutting the given 12cms sheet is $128cm^3$ with 8cms side and 2cms height.

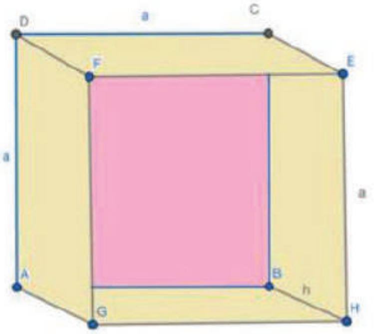
17. Question

An open box with a square base is to be made out of a given cardboard of area c^2 (square) units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ (cubic) units.

Answer

Given,

- The open box has a square base
- The area of the box is c^2 square units.
- The volume of the box is maximum.



Let us consider,

- The side of the square base of the box be ‘a’ units. (pink coloured in the figure)
- The breadth of the 4 sides of the box will also be ‘a’units (skin coloured part).
- The depth of the box or the length of the sides be ‘h’ units (skin coloured part).

Now, the area of the box =

(area of the base) + 4 (area of each side of the box)

So as area of the box is given c^2,

$c^2 = a^2 + 4ah$

$h = \frac{c^2 - a^2}{4a}$ ---- (1)

Let the volume of the newly formed box is :

$V = (a)^2 \times (h)$

[substituting (1) in the volume formula]

$$V = a^2 \times \left(\frac{c^2 - a^2}{4a}\right)$$

$V = \left(\frac{ac^2 - a^3}{4}\right)$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with a and then equating it to zero. This is because if the function f(a) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to a:

$$\frac{dV}{da} = \frac{d}{da}\left[\left(\frac{ac^2 - a^3}{4}\right)\right]$$

$\frac{dV}{da} = \frac{c^2}{4} - \frac{3a^2}{4}$ ------- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{da} = \frac{c^2}{4} - \frac{3a^2}{4} = 0$$

$c^2 - 3a^2 = 0$

$$a^2 = \frac{c^2}{3}$$

$$a = \pm\sqrt{\frac{c^2}{3}}$$

$$a = \frac{c}{\sqrt{3}}$$

[as 'a' cannot be negative]

Now to check if this critical point will determine the maximum Volume of the box, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{da^2} = \frac{d}{dx}\left[\frac{c^2}{4} - \frac{3a^2}{4}\right]$$

$\frac{d^2V}{da^2} = 0 - \frac{3\times2\times a}{4} = -\frac{3a}{2}$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{da^2}\right)_{a=\frac{c}{\sqrt{3}}} = -\frac{3\left(\frac{c}{\sqrt{3}}\right)}{2} = -\frac{c\sqrt{3}}{2}$$

As $\left(\frac{d^2V}{da^2}\right)_{a=\frac{c}{\sqrt{3}}} = -48 - \frac{c\sqrt{3}}{2} < 0$, so the function V is maximum at $a = \frac{c}{\sqrt{3}}$

Now substituting a in equation (1)

$$h = \frac{c^2 - \left(\frac{c}{\sqrt{3}}\right)^2}{4\left(\frac{c}{\sqrt{3}}\right)} = \frac{\frac{2c^2}{3}}{\frac{4c}{\sqrt{3}}} = \frac{c\sqrt{3}}{6} = \frac{c}{2\sqrt{3}}$$

$$\therefore h = \frac{c}{2\sqrt{3}}$$

Therefore side of wanted box has a base side, $a = \frac{c}{\sqrt{3}}$ is and height of the box, $h = \frac{c}{2\sqrt{3}}$.

Now, the volume of the box is

$$V = \left(\frac{c}{\sqrt{3}}\right)^2 \times \left(\frac{c}{2\sqrt{3}}\right)$$

$$V = \frac{c^2}{3} \times \left(\frac{c}{2\sqrt{3}}\right) = \frac{c^3}{6\sqrt{3}}$$

$$\therefore V = \frac{c^3}{6\sqrt{3}}$$

18. Question

A cylindrical can is to be made to hold 1 litre of oil. Find the dimensions which will minimize the cost of the metal to make the can.

Answer

Given,

- The can is cylindrical with a circular base
- The volume of the cylinder is 1 litre = 1000 cm^2.
- The surface area of the box is minimum as we need to find the minimum dimensions.

Let us consider,

- The radius base and top of the cylinder be ‘r’ units. (skin coloured in the figure)
- The height of the cylinder be ‘h’units.
- As the Volume of cylinder is given, V = $1000cm^3$

The Volume of the cylinder= πr^2h

$1000 = \pi r^2h$

$h = \frac{1000}{\pi r^2}$ ---- (1)

The Surface area cylinder is = area of the circular base + area of the circular top + area of the cylinder

$S = \pi r^2 + \pi r^2 + 2\pi rh$

$S = 2\pi r^2 + 2\pi rh$

[substituting (1) in the volume formula]

$$S = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$S = 2\left[\pi r^2 + \left(\frac{1000}{r}\right)\right] \text{------ (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function f(r) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\left[\pi r^2 + \left(\frac{1000}{r}\right)\right]\right]$$

$$\frac{dS}{dr} = 2(2\pi r) + \left(\frac{1000}{r^2}\right)(-1)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{dS}{dr} = 2(2\pi r) - 2\left(\frac{1000}{r^2}\right) \text{------- (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2(2\pi r) - 2\left(\frac{1000}{r^2}\right) = 0$$

$$2(2\pi r) - 2\left(\frac{1000}{r^2}\right) = 0$$

$$2\pi r = \frac{1000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

Now to check if this critical point will determine the minimum surface area of the box, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2(2\pi r) - 2\left(\frac{1000}{r^2}\right)\right]$$

$$\frac{d^2S}{dr^2} = 4\pi - \frac{2\times 1000\times(-2)}{r^3} = 4\pi + \frac{4000}{r^3} \text{----- (4)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{500}{\pi}}} = 4\pi + \frac{4000}{\left(\sqrt[3]{\frac{500}{\pi}}\right)^3} = 4\pi + \frac{4000\times\pi}{500} = 4\pi + 8\pi = 12\pi$$

As $\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{500}{\pi}}} = 12\pi > 0$, so the function S is minimum at $r = \sqrt[3]{\frac{500}{\pi}}$

Now substituting r in equation (1)

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi^{\frac{1}{3}}\,(500)^{\frac{2}{3}}}$$

$$\therefore\ h = \frac{1000}{\pi^{\frac{1}{3}}\,(500)^{\frac{2}{3}}}$$

Therefore the radius of base of the cylinder, $r = \sqrt[3]{\frac{500}{\pi}}$ and height of the cylinder, $h = \frac{1000}{\pi^{\frac{1}{3}}\,(500)^{\frac{2}{3}}}$ where the surface area of the cylinder is minimum.

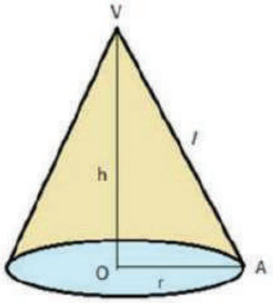
19. Question

Show that the right circular cone of the least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

Answer

Given,

- The volume of the cone.
- The cone is right circular cone.
- The cone has least curved surface.



Let us consider,

- The radius of the circular base be ‘r’ cms.
- The height of the cone be ‘h’ cms.
- The slope of the cone be ‘l’ cms.

Given the Volume of the cone = $\pi r^2 l$

$$V = \frac{\pi r^2 h}{3}$$

$$h = \frac{3v}{\pi r^2} \text{ ----- } (1)$$

The Surface area cylinder is = πrl

$S = \pi rl$

$$S = \pi r\left(\sqrt{h^2 + r^2}\right)$$

[substituting (1) in the Surface area formula]

$$S = \pi r\left[\sqrt{\left(\frac{3V}{\pi r^2}\right)^2 + r^2}\right]$$

[squaring on both sides]

$$Z = S^2 = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$$

$$Z = \pi^2\left(\frac{9V^2}{\pi^2 r^2} + r^4\right) \text{----- (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function Z has a maximum/minimum at a point c then $Z'(c) = 0$.

Differentiating the equation (2) with respect to r:

$$\frac{dZ}{dr} = \frac{d}{dr}\left[\pi^2\left(\frac{9V^2}{\pi^2 r^2} + r^4\right)\right]$$

$$\frac{dZ}{dr} = \pi^2\left(\frac{9V^2}{\pi^2}\right)\frac{d}{dr}\left(\frac{1}{r^2}\right) + \pi^2\frac{d}{dr}(r^4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{dZ}{dr} = \left(\frac{-18V^2}{r^3}\right) + \pi^2(4r^3) \text{------- (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dr} = \left(\frac{-18V^2}{r^3}\right) + \pi^2(4r^3) = 0$$

$$\pi^2(4r^3) = \frac{18V^2}{r^3}$$

$$2\pi^2 r^6 = 9V^2 \text{ ---- (4)}$$

Now to check if this critical point will determine the minimum surface area of the cone, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with r:

$$\frac{d^2Z}{dr^2} = \frac{d}{dr}\left[\left(\frac{-18V^2}{r^3}\right) + \pi^2(4r^3)\right]$$

$$\frac{d^2Z}{dr^2} = \frac{-18V^2(-3)}{r^4} + \pi^2(4\times 3r^2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{d^2Z}{dr^2} = \frac{54V^2}{r^4} + \pi^2(12r^2)$$

Now let us find the value of

$$\left(\frac{d^2Z}{dr^2}\right) = \frac{54V^2}{r^4} + \pi^2(12r^2) > 0$$

As $\left(\frac{d^2Z}{dr^2}\right) > 0$, so the function $Z = S^2$ is minimum

Now consider, the equation (4),

$$9V^2 = 2\pi^2 r^6$$

Now substitute the volume of the cone formula in the above equation.

$$9\left(\frac{\pi r^2 h}{3}\right)^2 = 2\pi^2 r^6$$

$\pi^2 r^4 h^2 = 2\ \pi^2 r^6$

$2r^2 = h^2$

$h = r\sqrt{2}$

Hence, the relation between h and r of the cone is proved when S is the minimum.

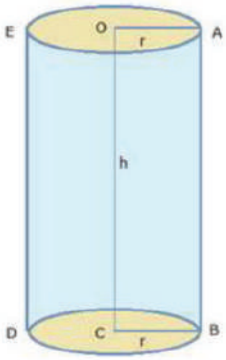
20. Question

Find the radius of a closed right circular cylinder of volume 100cm^3 which has the minimum total surface area.

Answer

Given,

- The closed is cylindrical can with a circular base and top.
- The volume of the cylinder is 1 litre = 100 cm^3.
- The surface area of the box is minimum.



Let us consider,

- The radius base and top of the cylinder be ‘r’ units. (skin coloured in the figure)
- The height of the cylinder be ‘h’units.
- As the Volume of cylinder is given, V = 100cm^3

The Volume of the cylinder= $\pi r^2 h$

$100 = \pi r^2 h$

$h = \frac{100}{\pi r^2}$ ---- (1)

The Surface area cylinder is = area of the circular base + area of the circular top + area of the cylinder

$S = \pi r^2 + \pi r^2 + 2\pi r h$

$S = 2\ \pi r^2 + 2\pi r h$

[substituting (1) in the volume formula]

$$S = 2\pi r^2 + 2\pi r\left(\frac{100}{\pi r^2}\right)$$

$S = 2\left[\pi r^2 + \left(\frac{100}{r}\right)\right]$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. <u>This is because if the function f(r) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr}=\frac{d}{dr}\left[2\left[\pi r^2+\left(\frac{100}{r}\right)\right]\right]$$

$$\frac{dS}{dr}=2\,(2\pi r)+\left(\frac{100}{r^2}\right)(-1)$$

[Since $\frac{d}{dx}(x^n)=nx^{n-1}$ and $\frac{d}{dx}(x^{-n})=-nx^{-n-1}$]

$$\frac{dS}{dr}=2\,(2\pi r)-2\left(\frac{100}{r^2}\right) \text{------- (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr}=2\,(2\pi r)-2\left(\frac{100}{r^2}\right)=0$$

$$2\,(2\pi r)-2\left(\frac{100}{r^2}\right)=0$$

$$2\pi r=\frac{100}{r^2} \text{---- (4)}$$

<u>Now to check if this critical point will determine the minimum surface area of the box, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2}=\frac{d}{dr}\left[2\,(2\pi r)-2\left(\frac{100}{r^2}\right)\right]$$

$$\frac{d^2S}{dr^2}=4\pi-\frac{2\times100\times(-2)}{r^3}=4\pi+\frac{400}{r^3} \text{------ (5)}$$

[Since $\frac{d}{dx}(x^n)=nx^{n-1}$ and $\frac{d}{dx}(x^{-n})=-nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{50}{\pi}}}=4\pi+\frac{400}{\left(\sqrt[3]{\frac{50}{\pi}}\right)^3}=4\pi+\frac{400\times\pi}{50}=4\pi+8\pi=12\pi$$

As $\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{50}{\pi}}}=12\pi>0$, so the function S is minimum at $r=\sqrt[3]{\frac{50}{\pi}}$

As S is minimum from equation (4)

$$2\pi r=\frac{100}{r^2}$$

$$2\pi r=\frac{V}{r^2}$$

$V = 2\pi r^3$

Now in equation (1) we have,

$$h=\frac{V}{\pi r^2}=\frac{2\pi r^3}{\pi r^2}$$

h = 2r = diameter

Therefore when the total surface area of a cone is minimum, then height of the cone is equal to twice the radius or equal to its diameter.

21. Question

Show that the height of a closed cylinder of given volume and the least surface area is equal to its diameter.

Answer

Let r be the radius of the base and h the height of a cylinder.

The surface area is given by,

$S = 2\pi r^2 + 2\pi rh$

$h = \frac{S-2\pi r^2}{2\pi r}$.........(1)

Let V be the volume of the cylinder.

Therefore, $V = \pi r^2 h$

$V = \pi r^2\left(\frac{S-2\pi r^3}{2\pi r}\right)$.......Using equation 1

$$V = \frac{Sr - 2\pi r^3}{2}$$

Differentiating both sides w.r.t r, we get,

$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$..........(2)

For maximum or minimum, we have,

$$\frac{dV}{dr} = 0$$

$\Rightarrow \frac{S}{2} - 3\pi r^2 = 0$

$\Rightarrow S = 6\pi r^2$

$2\pi r^2 + 2\pi rh = 6\pi r^2$

$h = 2r$

Differentiating equation 2, with respect to r to check for maxima and minima, we get,

$$\frac{d^2V}{dr^2} = -6\pi r < 0$$

Hence, V is maximum when h = 2r or h = diameter

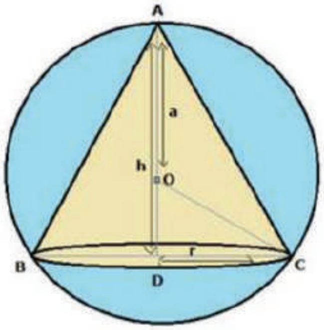
22. Question

Prove that the volume of the largest cone that can be inscribed in a sphere is $\frac{8}{27}$ of the volume of the sphere.

Answer

Given,

- Volume of the sphere.
- Volume of the cone.
- Cone is inscribed in the sphere.
- Volume of cone is maximum.



Let us consider,

- The radius of the sphere be 'a' units.
- Volume of the inscribed cone be 'V'.
- Height of the inscribed cone be 'h'.
- Radius of the base of the cone is 'r'.

Given volume of the inscribed cone is,

$$V = \frac{\pi r^2 h}{3}$$

Consider OD = (AD-OA) =(h-a)

Now let $OC^2 = OD^2 + DC^2$, here OC = a, OD = (h-a), DC = r,

So $a^2 = (h-a)^2 + r^2$

$r^2 = a^2 - (h^2 + a^2 - 2ah)$

$r^2 = h(2a - h)$ ----- (1)

Let us consider the volume of the cone:

$$V = \frac{1}{3}(\pi r^2 h)$$

Now substituting (1) in the volume formula,

$$V = \frac{1}{3}(\pi h(2a - h)h)$$

$$V = \frac{1}{3}(2\pi h^2 a - \pi h^3) \text{ ---- (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with h and then equating it to zero. This is because if the function V(r) has a maximum/minimum at a point c then V'(c) = 0.

Differentiating the equation (2) with respect to h:

$$\frac{dV}{dh} = \frac{d}{dh}\left[\frac{1}{3}(2\pi h^2 a - \pi h^3)\right]$$

$$\frac{dV}{dh} = \frac{1}{3}(2\pi a)(2h) - \frac{1}{3}(\pi)(3h^2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dV}{dh} = \frac{1}{3}[4\pi ah - 3\pi h^2] \text{ ------- (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{dh} = \frac{1}{3}[4\pi ah - 3\pi h^2] = 0$$

$4\pi ah - 3\pi h^2 = 0$

$h(4\pi a-3\pi h)= 0$

h = 0 (or) $h = \frac{4\pi a}{3\pi} = \frac{4a}{3}$

$h = \frac{4a}{3}$

[as h cannot be zero]

Now to check if this critical point will determine the maximum volume of the inscribed cone, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with h:

$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[\frac{1}{3}[4\pi ah - 3\pi h^2\right]$$

$\frac{d^2V}{dh^2} = \frac{1}{3}[4\pi a - (3\pi)(2h)] = \frac{\pi}{3}[4a - 6h]$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4a}{3}} = \frac{\pi}{3}\left[4a - 6\left(\frac{4a}{3}\right)\right] = \frac{4a\pi}{3}[1-2] = -\frac{4a\pi}{3}$$

As $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4a}{2}} = -\frac{4a\pi}{3} < 0$, so the function V is maximum at $h = \frac{4a}{3}$

Substituting h in equation (1)

$$r^2 = \left(\frac{4a}{3}\right)\left(2a - \frac{4a}{3}\right)$$

$$r^2 = \left(\frac{4a}{3}\right)\left(2a - \frac{4a}{3}\right)$$

$$r^2 = \frac{8a^2}{9}$$

As V is maximum, substituting h and r in the volume formula:

$$V = \frac{1}{3}\pi\left(\frac{8a^2}{9}\right)\left(\frac{4a}{3}\right)$$

$$V = \frac{8}{27}\left(\frac{4}{3}\pi a^3\right)$$

$$V = \frac{8}{27}(\text{volume of the sphere})$$

Therefore when the volume of a inscribed cone is maximum, then it is equal to $\frac{8}{27}$ times of the volume of the sphere in which it is inscribed.

23. Question

Which fraction exceeds its pth power by the greatest number possible?

Answer

Given,

The pth power of a number exceeds by a fraction to be the greatest.

Let us consider,

- 'x' be the required fraction.

- The greatest number will be $y = x - x^p$ ------ (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function y(x)has a maximum/minimum at a point c then y'(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}(x - x^p)$$

$$\frac{dy}{dx} = 1 - px^{p-1} \text{ ---- } (2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dy}{dx} = 1 - px^{p-1} = 0$$

$$1 = px^{p-1}$$

$$x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$$

Now to check if this critical point will determine the if the number is the greatest, we need to check with second differential which needs to be negative.

Consider differentiating the equation (2) with x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[1 - px^{p-1}]$$

$$\frac{d^2y}{dx^2} = -p(p-1)x^{p-2} \text{ ----- } (3)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2y}{dx^2}\right)_{x=\left(\frac{1}{p}\right)^{\frac{1}{p-1}}} = -p(p-1)\left(\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right)^{p-2}$$

As $\left(\frac{d^2y}{dx^2}\right)_{x=\left(\frac{1}{p}\right)^{\frac{1}{p-1}}} = -p(p-1)\left(\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right)^{p-2} < 0$, so the number y is greatest at $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

Hence, the y is the greatest number and exceeds by a fraction $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

24. Question

Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, −8).

Answer

Given,

- A point is present on a curve $y^2 = 4x$

- The point is near to the point (2,-8)

Let us consider,

- The co-ordinates of the point be P(x,y)

- As the point P is on the curve, then $y^2 = 4x$

$$x = \frac{y^2}{4}$$

- The distance between the points is given by,

$D^2 = (x-2)^2 + (y+8)^2$

$D^2 = x^2 - 4x + 4 + y^2 + 64 + 16y$

Substituting x in the distance equation

$$D^2 = \left(\frac{y^2}{4}\right)^2 - 4\left(\frac{y^2}{4}\right) + y^2 + 16y + 68$$

$$Z = D^2 = \frac{y^4}{16} + 16y + 68 \text{ ---- } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with y and then equating it to zero. <u>This is because if the function Z(x) has a maximum/minimum at a point c then Z'(c) = 0.</u>

Differentiating the equation (2) with respect to y:

$$\frac{dZ}{dy} = \frac{d}{dy}\left(\frac{y^4}{16} + 16y + 68\right)$$

$$\frac{dZ}{dy} = \frac{4y^3}{16} + 16 = \frac{y^3}{4} + 16 \text{ ---- } (2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dZ}{dy} = \frac{y^3}{4} + 16 = 0$$

$y^3 + 64 = 0$

$(y + 4)(y^2 - 4y + 16) = 0$

$(y+4) = 0$ (or) $y^2 - 4y + 16 = 0$

$y = -4$

(as the roots of the $y^2 - 4y + 16$ are imaginary)

<u>Now to check if this critical point will determine the distance is mimimum, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (2) with y:

$$\frac{d^2Z}{dy^2} = \frac{d}{dy}\left[\frac{y^3}{4} + 16\right]$$

$$\frac{d^2Z}{dy^2} = \frac{3y^2}{4} \text{ ----- } (3)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2Z}{dy^2}\right)_{y=-4} = \frac{3}{4}(-4)^2 = 12$$

As $\left(\frac{d^2Z}{dy^2}\right)_{y=-4} = 12 > 0$, so the Distance D^2 is minimum at $y = -4$

Now substituting y in x, we have

$$x = \frac{(-4)^2}{4} = 4$$

So, the point P on the curve $y^2 = 4x$ is (4,-4) which is at nearest from the (2,-8)

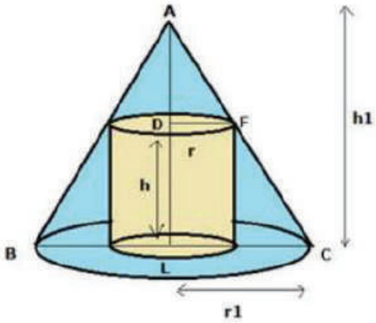
25. Question

A right circular cylinder is inscribed in a cone. Show that the curved surface area of the cylinder is maximum when the diameter of the cylinder is equal to the radius of the base of the cone.

Answer

Given,

- A right circular cylinder is inscribed inside a cone.
- The curved surface area is maximum.



Let us consider,

- 'r_1' be the radius of the cone.
- 'h_1' be the height of the cone.
- 'r' be the radius of the inscribed cylinder.
- 'h' be the height of the inscribed cylinder.

DF = r, and AD = AL - DL = $h_1 - h$

Now, here ΔADF and ΔALC are similar,

Then

$$\frac{AD}{AL} = \frac{DF}{LC} \Rightarrow \frac{h_1 - h}{h_1} = \frac{r}{r_1}$$

$$h_1 - h = \frac{rh_1}{r_1}$$

$$h = h_1 - \frac{rh_1}{r_1} = h_1\left(1 - \frac{r}{r_1}\right)$$

$$h = h_1\left(1 - \frac{r}{r_1}\right) \text{----- (1)}$$

Now let us consider the curved surface area of the cylinder,

S = 2πrh

Substituting h in the formula,

$$S = 2\pi r\left[h_1\left(1 - \frac{r}{r_1}\right)\right]$$

$S = 2\pi r h_1 - \frac{2\pi h_1 r^2}{r_1}$ ---- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function S(r) has a maximum/minimum at a point c then S’(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\pi r h_1 - \frac{2\pi h_1 r^2}{r_1}\right]$$

$$\frac{dS}{dr} = 2\pi h_1 - \frac{2\pi h_1 (2r)}{r_1}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dS}{dr} = 2\pi h_1 - \frac{4\pi h_1 r}{r_1}$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\pi h_1 - \frac{4\pi h_1 r}{r_1} = 0$$

$$\frac{4\pi h_1 r}{r_1} = 2\pi h_1$$

$$r = \frac{2\pi h_1 r_1}{4\pi h_1}$$

$$r = \frac{r_1}{2}$$

Now to check if this critical point will determine the maximum volume of the inscribed cylinder, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\pi h_1 - \frac{4\pi h_1 r}{r_1}\right]$$

$\frac{d^2S}{dr^2} = 0 - \frac{4\pi h_1}{r_1} = -\frac{4\pi h_1}{r_1}$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\frac{d^2S}{dr^2}_{r=\frac{r_1}{2}} = -\frac{4\pi h_1}{r_1}$$

As $\frac{d^2S}{dr^2}_{r=\frac{r_1}{2}} = -\frac{4\pi h_1}{r_1} < 0$, so the function S is maximum at $r = \frac{r_1}{2}$

Substituting r in equation (1)

$$h = h_1\left(1 - \frac{\frac{r_1}{2}}{r_1}\right)$$

$h = h_1\left(1 - \frac{1}{2}\right) = \frac{h_1}{2}$ --- (5)

As S is maximum, from (5) we can clearly say that $h_1 = 2h$ and

$r_1 = 2r$

this means the radius of the cone is twice the radius of the cylinder or equal to diameter of the cylinder.

26. Question

Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.

Answer

Given,

- Closed cuboid has square base.
- The volume of the cuboid is given.
- Surface area is minimum.

Let us consider,

- The side of the square base be ‘x’.
- The height of the cuboid be ‘h’.
- The given volume, $V = x^2h$

$h = \frac{V}{x^2}$ ----- (1)

Consider the surface area of the cuboid,

Surface Area =

2(Area of the square base) + 4(areas of the rectangular sides)

$S = 2x^2 + 4xh$

Now substitute (1) in the Surface Area formula

$S = 2x^2 + 4x\left(\frac{V}{x^2}\right)$

$S = 2x^2 + \left(\frac{4V}{x}\right)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function S(x) has a maximum/minimum at a point c then S’(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$\frac{dS}{dx} = \frac{d}{dx}\left[2x^2 + \left(\frac{4V}{x}\right)\right]$

$\frac{dS}{dx} = 2(2x) + 4V\left(\frac{-1}{x^2}\right)$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{dS}{dx} = 4x - \frac{4V}{x^2}$ ------- (3)

To find the critical point, we need to equate equation (3) to zero.

$\frac{dS}{dx} = 4x - \frac{4V}{x^2} = 0$

$4x = \frac{4V}{x^2}$

$x^3 = V$

<u>Now to check if this critical point will determine the minimum surface area, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[4x - \frac{4V}{x^2}\right]$$

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3} \text{----- (4)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\frac{d^2S}{dx^2}_{x=V^{\frac{1}{3}}} = 4 + \frac{8V}{V} = 12$$

As $\frac{d^2S}{dx^2}_{x=V^{\frac{1}{3}}} = 12 > 0$, so the function S is minimum at $x = \sqrt[3]{V}$

Substituting x in equation (1)

$$h = \frac{V}{x^2} = \frac{x^3}{x^2} = x$$

h = x

As S is minimum and h = x, this means that the cuboid is a cube.

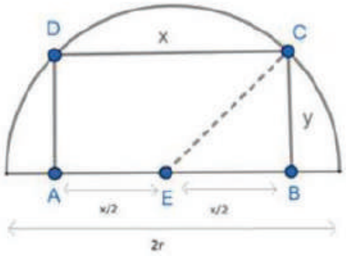
27. Question

A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle so that its area is maximum. Find also this area.

Answer

Given,

- Radius of the semicircle is ‘r’.
- Area of the rectangle is maximum.



Let us consider,

- The base of the rectangle be ‘x’ and the height be ‘y’.

Consider the ΔCEB,

$CE^2 = EB^2 + BC^2$

As CE = r, $EB = \frac{x}{2}$ and CB = y

$$r^2 = \left(\frac{x}{2}\right)^2 + y^2$$

$$y^2 = r^2 - \left(\frac{x}{2}\right)^2 \text{---- (1)}$$

Now the area of the rectangle is

A = x × y

Squaring on both sides

$A^2 = x^2\, y^2$

Substituting (1) in the above Area equation

$$A^2 = x^2 \left[r^2 - \left(\frac{x}{2}\right)^2\right]$$

$$Z = A^2 = x^2r^2 - x^2\frac{x^2}{4} = x^2r^2 - \frac{x^4}{4} \text{ ----- } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function Z(x) has a maximum/minimum at a point c then Z'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[x^2r^2 - \frac{x^4}{4}\right]$$

$$\frac{dZ}{dx} = r^2\,(2x) - \frac{4x^3}{4}$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$]

$$\frac{dZ}{dx} = 2xr^2 - x^3 \text{ ------- } (3)$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 2xr^2 - x^3 = 0$$

$x(2r^2 - x^2) = 0$

$x = 0$ (or) $x^2 = 2r^2$

$x = 0$ (or) $x = r\sqrt{2}$

$x = r\sqrt{2}$

[as x cannot be zero]

<u>Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[2xr^2 - x^3]$$

$$\frac{d^2Z}{dx^2} = 2r^2 - 3x^2 \text{ ----- } (4)$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$]

Now let us find the value of

$$\frac{d^2Z}{dx^2}_{x=r\sqrt{2}} = 2r^2 - 3(r\sqrt{2})^2 = 2r^2 - 6r^2 = -4r^2$$

As $\frac{d^2Z}{dx^2}_{x=r\sqrt{2}} = -4r^2 < 0$, so the function Z is maximum at $x = r\sqrt{2}$

Substituting x in equation (1)

$$y^2 = r^2 - \left(\frac{r\sqrt{2}}{2}\right)^2 = r^2 - \frac{r^2}{2} = \frac{r^2}{2}$$

$y = \sqrt{\frac{r^2}{2}} = \frac{r}{\sqrt{2}} = \frac{r\sqrt{2}}{2}$

As the area of the rectangle is maximum, and $x = r\sqrt{2}$ and $y = \frac{r\sqrt{2}}{2}$

So area of the rectangle is

$A = r\sqrt{2} \times \frac{r\sqrt{2}}{2}$

$A = r^2$

Hence the maximum area of the rectangle inscribed inside a semicircle is r^2 square units.

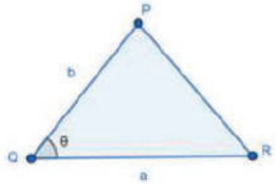
28. Question

Two sides of a triangle have lengths a and b and the angle between them is θ. What value of θ will maximize the area of the triangle?

Answer

Given,

- The length two sides of a triangle are 'a' and 'b'
- Angle between the sides 'a' and 'b' is θ.
- The area of the triangle is maximum.



Let us consider,

The area of the ΔPQR is given be

$A = \frac{1}{2}$ ab sinθ ---- (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with θ and then equating it to zero. This is because if the function A (θ) has a maximum/minimum at a point c then A'(c) = 0.

Differentiating the equation (1) with respect to θ:

$\frac{dA}{d\theta} = \frac{d}{d\theta}\left[\frac{1}{2} ab \sin\theta\right]$

$\frac{dA}{d\theta} = \frac{1}{2} ab\cos\theta$ ---- (2)

[Since $\frac{d}{dx}(\sin\theta) = \cos\theta$]

To find the critical point, we need to equate equation (2) to zero.

$\frac{dA}{d\theta} = \frac{1}{2} ab\cos\theta = 0$

Cos θ = 0

$\theta = \frac{\pi}{2}$

Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.

Consider differentiating the equation (2) with θ :

$$\frac{d^2A}{d\theta^2} = \frac{d}{d\theta}\left[\frac{1}{2}\,ab\cos\theta\right]$$

$\frac{d^2A}{d\theta^2} = -\frac{1}{2}\,ab\sin\theta$ ----- (2)

[Since $\frac{d}{dx}(\cos\theta) = -\sin\theta$]

Now let us find the value of

$$\frac{d^2A}{d\theta^2}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\,ab\sin\left(\frac{\pi}{2}\right) = -\frac{1}{2}\,ab$$

As $\frac{d^2A}{d\theta^2}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\,ab < 0$, so the function A is maximum at $\theta = \frac{\pi}{2}$

As the area of the triangle is maximum when $\theta = \frac{\pi}{2}$

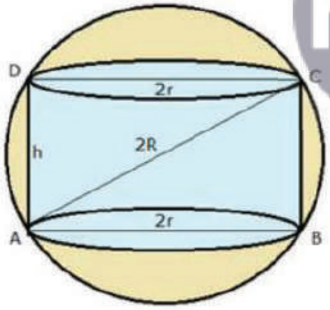
29. Question

Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is (500π)cm^3.

Answer

Given,

- Radius of the sphere is $5\sqrt{3}$.
- Volume of cylinder is maximum.



Let us consider,

- The radius of the sphere be ‘R’ units.
- Volume of the inscribed cylinder be ‘V’.
- Height of the inscribed cylinder be ‘h’.
- Radius of the cylinder is ‘r’.

Now let $AC^2 = AB^2 + BC^2$, here AC = 2R, AB =2r, BC = h,

So $4R^2 = 4r^2 + h^2$

$r^2 = \frac{1}{4}\left[4R^2 - h^2\right]$ ----- (1)

Let us consider, the volume of the cylinder:

$V = \pi r^2 h$

Now substituting (1) in the volume formula,

$$V = \pi h \left(\frac{1}{4}\left[4R^2 - h^2\right]\right)$$

$$V = \frac{\pi}{4}\left(4R^2 h - h^3\right) \text{ ---- } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with h and then equating it to zero. This is because if the function V(h) has a maximum/minimum at a point c then V'(c) = 0.

Differentiating the equation (2) with respect to h:

$$\frac{dV}{dh} = \frac{d}{dh}\left[\frac{\pi}{4}\left(4R^2 h - h^3\right)\right]$$

$$\frac{dV}{dh} = \frac{4R^2\pi}{4} - \frac{\pi}{4}\left(3h^2\right)$$

[Since $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$]

$$\frac{dV}{dh} = R^2\pi - \frac{3h^2\pi}{4} \text{ ------- } (3)$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{dh} = R^2\pi - \frac{3h^2\pi}{4} = 0$$

$$3h^2\pi = 4R^2\pi$$

$$h^2 = \frac{4}{3}R^2 = \frac{4}{3}(5\sqrt{3})^2 = \frac{4}{3}(25 \times 3) = 100$$

$$h = 10$$

[as h cannot be negative]

Now to check if this critical point will determine the maximum volume of the inscribed cone, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with h:

$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[R^2\pi - \frac{3h^2\pi}{4}\right]$$

$$\frac{d^2V}{dh^2} = 0 - \frac{3(2h)\pi}{3} = -2h\pi \text{ ----- } (4)$$

[Since $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dh^2}\right)_{h=10} = -2h\pi = -2(10)\pi = -20\pi$$

As $\left(\frac{d^2V}{dh^2}\right)_{h=10} = -20\pi < 0$, so the function V is maximum at h=10

Substituting h in equation (1)

$$r^2 = \frac{1}{4}\left[4(5\sqrt{3})^2 - (10)^2\right]$$

$$r^2 = \frac{1}{4}\left[4(25 \times 3) - 100\right]$$

$$r^2 = \frac{300 - 100}{4} = \frac{200}{4} = 50$$

As V is maximum, substituting h and r in the volume formula:

$V = \pi\ (50)\ (10)$

$V = 500\pi\ cm^3$

Therefore when the volume of a inscribed cylinder is maximum and is equal $500\pi\ cm^3$

30. Question

A square tank of capacity 250cubic meters has to be dug out. The cost of the land is Rs. 50 per square metre. The cost of digging increases with the depth and for the whole tank, it is Rs.$\left(400 \times h^2\right)$, where h metres is the depth of the tank. What should be the dimensions of the tank so that the cost is minimum?

Answer

Given,

- Capacity of the square tank is 250 cubic metres.
- Cost of the land per square meter Rs.50.
- Cost of digging the whole tank is Rs. $(400 \times h^2)$.
- Where h is the depth of the tank.

Let us consider,

- Side of the tank is x metres.
- Cost of the digging is; $C = 50x^2 + 400h^2$ ---- (1)
- Volume of the tank is; $V = x^2h$; $250 = x^2h$

$h = \frac{250}{x^2}$ ----- (2)

Substituting (2) in (1),

$$C = 50x^2 + 400\left(\frac{250}{x^2}\right)^2$$

$$C = 50x^2 + \frac{400 \times 62500}{x^4} \text{ ----- (3)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function C(x) has a maximum/minimum at a point c then C'(c) = 0.

Differentiating the equation (3) with respect to x:

$$\frac{dC}{dx} = \frac{d}{dx}\left[50x^2 + \frac{400\ \times 62500}{x^4}\right]$$

$$\frac{dC}{dx} = 50\ (2x) + \frac{25000000\ (-4)}{x^5}$$

[Since $\frac{d}{dx}\ (x^n) = nx^{n-1}$]

$$\frac{dC}{dx} = 100x - \frac{10^8}{x^5} \text{ ------- (4)}$$

To find the critical point, we need to equate equation (4) to zero.

$$\frac{dC}{dx} = 100x - \frac{10^8}{x^5} = 0$$

$x^6 = 10^6$

$x = 10$

Now to check if this critical point will determine the minimum volume of the tank, we need to check with second differential which needs to be positive.

Consider differentiating the equation (4) with x:

$$\frac{d^2C}{dx^2} = \frac{d}{dx}\left[100x - \frac{10^8}{x^5}\right]$$

$\frac{d^2C}{dx^2} = 100 - \frac{10^8(-5)}{x^6} = 100 + \frac{10^8(5)}{x^6}$ ------ (5)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2C}{dx^2}\right)_{x=10} = 100 + \frac{10^8(5)}{(10)^6} = 100 + 500 = 600$$

As $\left(\frac{d^2C}{dx^2}\right)_{x=10} = 600 > 0$, so the function C is minimum at x=10

Substituting x in equation (2)

$$h = \frac{250}{(10)^2} = \frac{250}{100} = \frac{5}{2}$$

h = 2.5 m

Therefore when the cost for the digging is minimum, when x = 10m and h = 2.5m

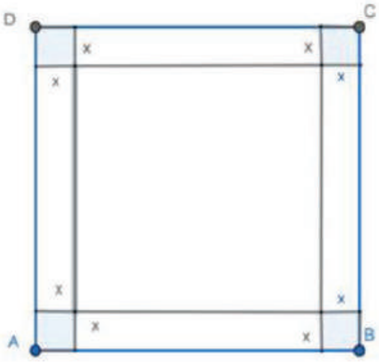
31. Question

A square piece of tin of side 18 cm is to be made into a box without the top, by cutting a square piece from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find the maximum volume of the box.

Answer

Given,

- Side of the square piece is 18 cms.
- the volume of the formed box is maximum.



Let us consider,

- ‘x’ be the length and breadth of the piece cut from each vertex of the piece.
- Side of the box now will be (18-2x)
- The height of the new formed box will also be ‘x’.

Let the volume of the newly formed box is :

$V = (18-2x)^2 \times (x)$

$V = (324 + 4x^2 - 72x)\, x$

$V = 4x^3 - 72x^2 + 324x$ ------ (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function V(x) has a maximum/minimum at a point c then V'(c) = 0.</u>

Differentiating the equation (1) with respect to x:

$$\frac{dV}{dx}=\frac{d}{dx}[4x^3-72x^2+324x]$$

$\frac{dV}{dx}=12x^2-144x+324$ -------- (2)

[Since $\frac{d}{dx}(x^n)=nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dV}{dx}=12x^2-144x+324=0$$

$x^2 - 12x + 27 = 0$

$$x=\frac{-(-12)\pm\sqrt{(-12)^2-4(1)(27)}}{2(1)}=\frac{12\pm\sqrt{144-108}}{2}=\frac{12\pm\sqrt{36}}{2}$$

$$x==\frac{12\pm6}{2}$$

x = 9 or x =3

x= 2

[as x = 9 is not a possibility, because 18-2x = 18-18= 0]

<u>Now to check if this critical point will determine the maximum area of the box, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{dx^2}=\frac{d}{dx}[12x^2-144x+324]$$

$\frac{d^2V}{dx^2}=24x-144$ ----- (4)

[Since $\frac{d}{dx}(x^n)=nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dx^2}\right)_{x=3}=24(3)-144=72-144=-72$$

As $\left(\frac{d^2V}{dx^2}\right)_{x=3}=-72<0$, so the function V is maximum at x = 3cm

Now substituting x = 3 in 18 – 2x, the side of the considered box:

Side = 18-2x = 18 - 2(3) = 18-6= 12cm

Therefore side of wanted box is 12cms and height of the box is 3cms.

Now, the volume of the box is

$V = (12)^2 \times 3 = 144 \times 3 = 432cm^3$

Hence maximum volume of the box formed by cutting the given 18cms sheet is $432cm^3$ with 12cms side and 3cms height.

32. Question

An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width.

Answer

Given,

- The tank is square base open tank.
- The cost of the construction to be least.

Let us consider,

- Side of the tank is x metres.
- Height of the tank be ‘h’ metres.
- Volume of the tank is; $V = x^2h$
- Surface Area of the tank is $S = x^2 + 4xh$
- Let Rs.P is the price per square.

Volume of the tank,

$h = \frac{V}{x^2}$ ---- (1)

Cost of the construction be:

$C = (x^2 + 4xh)P$ ---- (2)

Substituting (1) in (2),

$$C = \left[x^2 + 4x\frac{V}{x^2}\right]P$$

$C = [x^2 + \frac{4V}{x}]P$ ----- (3)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function C(x) has a maximum/minimum at a point c then C′(c) = 0.

Differentiating the equation (3) with respect to x:

$$\frac{dC}{dx} = \frac{d}{dx}\left[x^2 + \frac{4V}{x}\right]P$$

$$\frac{dC}{dx} = \left[(2x) + \frac{4V(-1)}{x^2}\right]P$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{dC}{dx} = \left[2x - \frac{4V}{x^2}\right]P$ ------- (4)

To find the critical point, we need to equate equation (4) to zero.

$$\frac{dC}{dx} = \left[2x - \frac{4V}{x^2}\right]P = 0$$

$x^3 = 2V$

Now to check if this critical point will determine the minimum volume of the tank, we need to check with second differential which needs to be positive.

Consider differentiating the equation (4) with x:

$$\frac{d^2C}{dx^2} = P\frac{d}{dx}\left[2x - \frac{4V}{x^2}\right]$$

$\frac{d^2C}{dx^2} = \left[2 - \frac{4V(-2)}{x^3}\right]P = \left[2 + \frac{8V}{x^3}\right]P$ ----- (5)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2C}{dx^2}\right)_{x=(2V)^{\frac{1}{3}}} = \left[2 + \frac{8V}{2V}\right]P = [2+4]P = 6P$$

As $\left(\frac{d^2C}{dx^2}\right)_{x=(2V)^{\frac{1}{3}}} = 6P > 0$, so the function C is minimum at $x = \sqrt[3]{2V}$

Substituting x in equation (2)

$$h = \frac{V}{(2V)^{\frac{2}{3}}} = \frac{V\sqrt[3]{(2V)}}{2V} = \frac{1}{2}\sqrt[3]{2V}$$

$$h = \frac{1}{2}\sqrt[3]{2V}$$

Therefore when the cost for the digging is minimum, when $x = \sqrt[3]{2V}$ and $h = \frac{1}{2}\sqrt[3]{2V}$

33. Question

A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum.

Answer

Given,

- Length of the wire is 36 cm.
- The wire is cut into 2 pieces.
- One piece is made to a square.
- Another piece made into a equilateral triangle.

Let us consider,

- The perimeter of the square is x.
- The perimeter of the equilateral triangle is (36-x).
- Side of the square is $\frac{x}{4}$
- Side of the triangle is $\frac{(36-x)}{3}$

Let the Sum of the Area of the square and triangle is

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{36-x}{3}\right)^2$$

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(12 - \frac{x}{3}\right)^2 = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)$$

$$A = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right) \text{--- (1)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function A(x) has a maximum/minimum at a point c then A'(c) = 0.</u>

Differentiating the equation (1) with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)\right]$$

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(0 + \frac{2x}{9} - 8\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right)$ ----- (2)

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right) = 0$$

$$\frac{2x}{16} = \frac{\sqrt{3}}{4}\left(8 - \frac{2x}{9}\right)$$

$$\frac{2x}{16} = 2\sqrt{3} - \frac{\sqrt{3}x}{18}$$

$$\frac{2x}{16} + \frac{\sqrt{3}x}{18} = 2\sqrt{3}$$

$$x\left(\frac{2(9) + \sqrt{3}(8)}{144}\right) = 2\sqrt{3}$$

$$x\left(\frac{18 + 8\sqrt{3}}{144}\right) = 2\sqrt{3}$$

$$x = 2\sqrt{3}\left(\frac{144}{18 + 8\sqrt{3}}\right) = \frac{144\sqrt{3}}{(9 + 4\sqrt{3})}$$

Now to check if this critical point will determine the minimum area, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right)\right]$$

$\frac{d^2A}{dx^2} = \frac{1}{8} + \frac{\sqrt{3}}{4}\left(\frac{2}{9}\right) = \frac{9+4\sqrt{3}}{72}$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2A}{dx^2}\right)_{x=\frac{144\sqrt{3}}{(9+4\sqrt{3})}} = \frac{9 + 4\sqrt{3}}{72}$$

As $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{144\sqrt{3}}{(9+4\sqrt{3})}} = \frac{9+4\sqrt{3}}{72} > 0$, so the function A is minimum at

$$x = \frac{144\sqrt{3}}{(9 + 4\sqrt{3})}$$

Now, the length of each piece is $x = \frac{144\sqrt{3}}{(9+4\sqrt{3})}$ cm and $36 - x = 36 - \frac{144\sqrt{3}}{(9+4\sqrt{3})} = \frac{324}{(9+4\sqrt{3})}$ cm

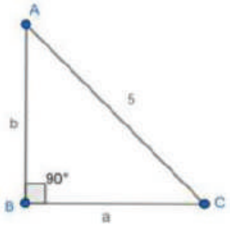
34. Question

Find the largest possible area of a right-angles triangle whose hypotenuse is 5 cm.

Answer

Given,

- The triangle is right angled triangle.
- Hypotenuse is 5cm.



Let us consider,

- The base of the triangle is ‘a’.
- The adjacent side is ‘b’.

Now $AC^2 = AB^2 + BC^2$

As AC = 5, AB = b and BC = a

$25 = a^2 + b^2$

$b^2 = 25 - a^2$ ---- (1)

Now, the area of the triangle is

$$A = \frac{1}{2}ab$$

Squaring on both sides

$$A^2 = \frac{1}{4}a^2b^2$$

Substituting (1) in the area formula

$Z = A^2 = \frac{1}{4}a^2(25 - a^2)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with a and then equating it to zero. <u>This is because if the function Z (x) has a maximum/minimum at a point c then Z‘(c) = 0.</u>

Differentiating the equation (2) with respect to a:

$$\frac{dZ}{da} = \frac{d}{da}\left[\frac{1}{4}a^2(25 - a^2)\right]$$

$$\frac{dZ}{da} = \frac{1}{4}[25(2a) - 4a^3]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dZ}{da} = \frac{25a}{2} - a^3$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{da} = \frac{25a}{2} - a^3 = 0$$

$$a\left(\frac{25}{2} - a^2\right) = 0$$

a=0 (or) $a = \frac{5}{\sqrt{2}}$

$$a = \frac{5}{\sqrt{2}}$$

[as a cannot be zero]

<u>Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with a:

$$\frac{d^2Z}{da^2} = \frac{d}{da}\left[\frac{25a}{2} - a^3\right]$$

$$\frac{d^2Z}{da^2} = \frac{25}{2} - 3a^2 \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2Z}{da^2}\right)_{a=\frac{5}{\sqrt{2}}} = \frac{25}{2} - 3\left(\frac{5}{\sqrt{2}}\right)^2 = \frac{25}{2} - \frac{(3)25}{2} = -25$$

As $\left(\frac{d^2Z}{da^2}\right)_{a=\frac{5}{\sqrt{2}}} = -25 < 0$, so the function A is maximum at $a = \frac{5}{\sqrt{2}}$

Substituting value of A in (1)

$$b^2 = 25 - \frac{25}{2} = \frac{25}{2}$$

$$b = \frac{5}{\sqrt{2}}$$

Now the maximum area is

$$A = \frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)\left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4}$$

$$\therefore A = \frac{25}{4}\,\text{cm}^2$$

Exercise 11G

1. Question

Show that the function $f(x) = 5x - 2$ is a strictly increasing function on R.

Answer

Domain of the function is R

Finding derivative f'(x)=5

Which is greater than 0

Mean strictly increasing in its domain i.e R

2. Question

Show the function $f(x) = -2x + 7$ is a strictly decreasing function on R.

Answer

Domain of the function is R

Finding derivative f'(x)=-2