EXERCISE 3.1

Define a function as a set of ordered pairs. Solution:

Let A and B be two non-empty sets. A relation from A to B, i.e., a subset of A×B, is called a function (or a mapping) from A to B, if

(i) for each a ∈ A there exists b ∈ B such that (a, b) ∈ f

(ii) (a, b) ∈ f and (a, c) ∈ f ⇒ b = c

2. Define a function as a correspondence between two sets.

Solution:

Let A and B be two non-empty sets. Then a function 'f' from set A to B is a rule or method or correspondence which associates elements of set A to elements of set B such that:

(i) all elements of set A are associated to elements in set B.

(ii) an element of set A is associated to a unique element in set B.

3. What is the fundamental difference between a relation and a function? Is every relation a function?

Solution:

Let 'f' be a function and R be a relation defined from set X to set Y.

The domain of the relation R might be a subset of the set X, but the domain of the function f must be equal to X. This is because each element of the domain of a function must have an element associated with it, whereas this is not necessary for a relation.

In relation, one element of X might be associated with one or more elements of Y, while it must be associated with only one element of Y in a function.

Thus, not every relation is a function. However, every function is necessarily a relation.

4. Let A = {–2, –1, 0, 1, 2} and f: A → Z be a function defined by $f(x) = x^2 - 2x - 3$. Find:

(i) range of f i.e. f (A)

(ii) pre-images of 6, –3 and 5

Solution:

Given:

A = {–2, –1, 0, 1, 2}

f : A → Z such that $f(x) = x^2 - 2x - 3$

(i) Range of f i.e. f (A)
A is the domain of the function f. Hence, range is the set of elements f(x) for all x ∈ A.
Substituting x = –2 in f(x), we get
$f(-2) = (-2)^2 - 2(-2) - 3$
$= 4 + 4 - 3$
$= 5$
Substituting x = –1 in f(x), we get
$f(-1) = (-1)^2 - 2(-1) - 3$
$= 1 + 2 - 3$
$= 0$
Substituting x = 0 in f(x), we get
$f(0) = (0)^2 - 2(0) - 3$
$= 0 - 0 - 3$
$= -3$
Substituting x = 1 in f(x), we get
$f(1) = 1^2 - 2(1) - 3$
$= 1 - 2 - 3$
$= -4$
Substituting x = 2 in f(x), we get
$f(2) = 2^2 - 2(2) - 3$
$= 4 - 4 - 3$
$= -3$
Thus, the range of f is {-4, -3, 0, 5}.

(ii) pre-images of 6, –3 and 5
Let x be the pre-image of 6 ⇒ f(x) = 6
$x^2 - 2x - 3 = 6$
$x^2 - 2x - 9 = 0$
$x = [-(-2) \pm \sqrt{((-2)^2 - 4(1)(-9))}] / 2(1)$
$= [2 \pm \sqrt{(4+36)}] / 2$
$= [2 \pm \sqrt{40}] / 2$
$= 1 \pm \sqrt{10}$
However, $1 \pm \sqrt{10} \notin A$
Thus, there exists no pre-image of 6.

Now, let x be the pre-image of –3 ⇒ f(x) = –3
$x^2 - 2x - 3 = -3$
$x^2 - 2x = 0$
$x(x - 2) = 0$

$x = 0$ or 2
Clearly, both 0 and 2 are elements of A.
Thus, 0 and 2 are the pre-images of –3.

Now, let x be the pre-image of 5 ⇒ f(x) = 5
$x^2 - 2x - 3 = 5$
$x^2 - 2x - 8 = 0$
$x^2 - 4x + 2x - 8 = 0$
$x(x - 4) + 2(x - 4) = 0$
$(x + 2)(x - 4) = 0$
$x = -2$ or 4
However, $4 \notin A$ but $-2 \in A$
Thus, –2 is the pre-images of 5.
∴ Ø, {0, 2}, -2 are the pre-images of 6, -3, 5

5. If a function f: R → R be defined by

$$f(x) = \begin{cases} 3x - 2, x < 0 \\ 1, x = 0 \\ 4x + 1, x > 0 \end{cases}$$

Find: f (1), f (–1), f (0), f (2).
Solution:
Given:
Let us find f (1), f (–1), f (0) and f (2).
When x > 0, f (x) = 4x + 1
Substituting x = 1 in the above equation, we get
f (1) = 4(1) + 1
= 4 + 1
= 5

When x < 0, f(x) = 3x – 2
Substituting x = –1 in the above equation, we get
f (–1) = 3(–1) – 2
= –3 – 2
= –5

When x = 0, f(x) = 1
Substituting x = 0 in the above equation, we get
f (0) = 1

When $x > 0$, $f(x) = 4x + 1$
Substituting $x = 2$ in the above equation, we get
$f(2) = 4(2) + 1$
$= 8 + 1$
$= 9$
$\therefore f(1) = 5, f(-1) = -5, f(0) = 1$ and $f(2) = 9$.

6. A function f: R → R is defined by $f(x) = x^2$. Determine
(i) range of f
(ii) {x: f(x) = 4}
(iii) {y: f(y) = –1}
Solution:
Given:
$f : R \to R$ and $f(x) = x^2$.
(i) range of f
Domain of f = R (set of real numbers)
We know that the square of a real number is always positive or equal to zero.
$\therefore$ range of f = $R^+ \cup \{0\}$

(ii) {x: f(x) = 4}
Given:
$f(x) = 4$
we know, $x^2 = 4$
$x^2 - 4 = 0$
$(x - 2)(x + 2) = 0$
$\therefore x = \pm 2$
$\therefore \{x: f(x) = 4\} = \{-2, 2\}$

(iii) {y: f(y) = –1}
Given:
$f(y) = -1$
$y^2 = -1$
However, the domain of f is R, and for every real number y, the value of y^2 is non-negative.
Hence, there exists no real y for which $y^2 = -1$.
$\therefore \{y: f(y) = -1\} = \emptyset$

7. Let f: $R^+ \to R$, where R^+ is the set of all positive real numbers, be such that f(x) =

$\log_e x$. Determine
(i) the image set of the domain of f
(ii) {x: f (x) = –2}
(iii) whether f (xy) = f (x) + f (y) holds.
Solution:
Given f: $R^+ \rightarrow R$ and $f(x) = \log_e x$.
(i) the image set of the domain of f
Domain of f = R^+ (set of positive real numbers)
We know the value of logarithm to the base e (natural logarithm) can take all possible real values.
∴ The image set of f = R

(ii) {x: f(x) = –2}
Given f(x) = –2
$\log_e x = -2$
∴ $x = e^{-2}$ [since, $\log_b a = c \Rightarrow a = b^c$]
∴ {x: f(x) = –2} = $\{e^{-2}\}$

(iii) Whether f (xy) = f (x) + f (y) holds.
We have $f(x) = \log_e x \Rightarrow f(y) = \log_e y$
Now, let us consider f (xy)
F (xy) = $\log_e (xy)$
f (xy) = $\log_e (x \times y)$ [since, $\log_b (a \times c) = \log_b a + \log_b c$]
f (xy) = $\log_e x + \log_e y$
f (xy) = f (x) + f (y)
∴ the equation f (xy) = f (x) + f (y) holds.

8. Write the following relations as sets of ordered pairs and find which of them are functions:
(i) {(x, y): y = 3x, x ∈ {1, 2, 3}, y ∈ {3, 6, 9, 12}}
(ii) {(x, y): y > x + 1, x = 1, 2 and y = 2, 4, 6}
(iii) {(x, y): x + y = 3, x, y ∈ {0, 1, 2, 3}}
Solution:
(i) {(x, y): y = 3x, x ∈ {1, 2, 3}, y ∈ {3, 6, 9, 12}}
When x = 1, y = 3(1) = 3
When x = 2, y = 3(2) = 6
When x = 3, y = 3(3) = 9
∴ R = {(1, 3), (2, 6), (3, 9)}
Hence, the given relation R is a function.

(ii) {(x, y): y > x + 1, x = 1, 2 and y = 2, 4, 6}
When x = 1, y > 1 + 1 or y > 2 ⇒ y = {4, 6}
When x = 2, y > 2 + 1 or y > 3 ⇒ y = {4, 6}
∴ R = {(1, 4), (1, 6), (2, 4), (2, 6)}
Hence, the given relation R is not a function.

(iii) {(x, y): x + y = 3, x, y ∈ {0, 1, 2, 3}}
When x = 0, 0 + y = 3 ⇒ y = 3
When x = 1, 1 + y = 3 ⇒ y = 2
When x = 2, 2 + y = 3 ⇒ y = 1
When x = 3, 3 + y = 3 ⇒ y = 0
∴ R = {(0, 3), (1, 2), (2, 1), (3, 0)}
Hence, the given relation R is a function.

9. Let f: R → R and g: C → C be two functions defined as f(x) = x^2 and g(x) = x^2. Are they equal functions?
Solution:
Given:
f: R → R ∈ f(x) = x^2 and g : R → R ∈ g(x) = x^2
f is defined from R to R, the domain of f = R.
g is defined from C to C, the domain of g = C.
Two functions are equal only when the domain and codomain of both the functions are equal.
In this case, the domain of f ≠ domain of g.
∴ f and g are not equal functions.