NCERT Solutions for Class-XII Math

Chapter-4
Exercise- Miscellaneous

NCERT Math Class 12

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

1. $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$

$= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \cos\theta\sin\theta + x\cos^2\theta$

$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x = -x^3$, which is independent of θ.

2. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

2. RHS = $\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$

LHS = $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Multiplying row 1 with a, row 2 with b and row 3 with c

$[R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3]$

LHS = $\dfrac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$

$$= \frac{1}{abc} \times abc \times \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$ [Taking common abc from C_3]

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$$ [Applying $C_1 \leftrightarrow C_3$]

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$ [Applying $C_2 \leftrightarrow C_3$]

= RHS

Since, LHS = RHS

∴ the given result is proved.

3. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos a \end{vmatrix}$

3. $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos a \end{vmatrix}$

$= -\sin\alpha\,(-\sin a \sin^2\beta - \sin\alpha\cos^2\beta) - 0(\cos\alpha\cos\beta\sin\alpha\sin\beta - \cos\alpha\sin\beta\sin\alpha\cos\beta) + \cos\alpha\,(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta)$ [Expending along C_3]

$= \sin^2\alpha\,(\sin^2\beta + \cos^2\beta) + \cos^2\alpha\,(\cos^2\beta + \sin^2\beta)$

$= \sin^2\alpha + \sin^2\alpha = 1$

4. If a, b and c are real numbers and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$=0, show that either a + b + c = 0 or a = b = c.

4. Given,

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying Elementary transformations, we get

$R_1 \rightarrow R_1 + R_2 + R_3$ we have,

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$\Delta = 2\ (a + b + c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Now applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$$\Delta = 2\ (a + b + c) \begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1

$\Delta = 2\ (a + b + c)\ [1\ \{(b-c)(c-b) - (b-a)(c-a)\} - 0 + 0]$

$\Delta = 2\ (a + b + c)\ [- b^2 - c^2 + 2bc - bc + ba + ac - a^2]$

$\Delta = 2\ (a + b + c)\ [ab + bc + ca - a^2 - b^2 - c^2]$

Given that $\Delta = 0$

$\therefore 2\ (a + b + c)\ [ab + bc + ca - a^2 - b^2 - c^2] = 0$

$\Rightarrow$ Either $a + b + c = 0$, or $ab + bc + ca - a^2 - b^2 - c^2 = 0$

Now

$ab + bc + ca - a^2 - b^2 - c^2 = 0$

Multiplying both sides by -2

$\Rightarrow - 2ab - 2bc - 2ca + 2a^2 + 2b^2 + 2c^2 = 0$

$\Rightarrow a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2 = 0$

$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

Since, $(a - b)^2$, $(b - c)^2$, $(c - a)^2$ are non-negative

$\therefore (a - b)^2 = (b - c)^2 = (c - a)^2$

$\Rightarrow (a - b) = (b - c) = (c - a)$

$\Rightarrow a = b = c$

Hence, if $\Delta = 0$, then either $a + b + c = 0$ or $a = b = c$

5. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

5. Given that: $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0,$

$\Rightarrow \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x=a & x \\ x & x & x+a \end{vmatrix} = 0$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$ [Taking (3x + a) as common from R_1]

$\Rightarrow (3x+a)\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$ [Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$]

$\Rightarrow (3x + a)1\ [a^2 - 0] = 0$ [Expending along R_1]

$\Rightarrow a^2(3x + a) = 0 \quad \Rightarrow (3x + a) = 0$ [Q $a \neq 0$]

$\Rightarrow x = -\frac{a}{3}$

6. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

6. Given,

$\text{LHS} = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

$\text{RHS} = 4a^2b^2c^2$

$\text{LHS} = \Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Taking out common factors a, b and c from C_1, C_2 and C_3, we have

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying Elementary Transformations

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

$R_2 \to R_2 + R_1$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

$R_3 \to R_3 + R_2$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix}$$

$$\Delta = 2ab^2c\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

$C_2 \to C_2 - C_1$

$$\Delta = 2ab^2c\begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3 we get,

$\Delta = 2ab^2c\,[a(c-a) + a(a+c)]$

$= 2ab^2c\,[ac - a^2 + a^2 + ac]$

$= 2ab^2c\,(2ac)$

$= 4a^2b^2c^2$

Δ = RHS

$\therefore$ LHS = RHS

Hence, Proved

7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

7. Hence $B=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$

Therefore, $|B| = 1(3-0) - 2(-1-0) - 2(2-0) = 1 \neq 0 \Rightarrow B^{-1}$ exists.

$A_{11} = 3$ $\quad A_{12} = 1$ $\quad A_{13} = 2$

$A_{21} = 2$ $\quad A_{22} = 1$ $\quad A_{23} = 2$

$A_{31} = 6$ $\quad A_{32} = 2$ $\quad A_{33} = 5$

$$B^{-1}=\frac{1}{|B|}adj\ B=\frac{1}{1}\begin{bmatrix}B_{11} & B_{21} & B_{31}\\ B_{12} & B_{22} & B_{32}\\ B_{13} & B_{23} & B_{33}\end{bmatrix}=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$$

We know that: $(AB)^{-1} = B^{-1}A^{-1}$, therefore

$$(AB)^{-1}=B^{-1}A^{-1}=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}$$

$$=\begin{bmatrix}9-30+30 & -3+12-12 & 3-10+12\\ 3-15+10 & -1+6-4 & 1-5+4\\ 6-30+25 & -2+12-10 & 2-10+10\end{bmatrix}=\begin{bmatrix}9 & -3 & 5\\ -2 & 1 & 0\\ 1 & 0 & 2\end{bmatrix}$$

8. Let $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$, verify that

(i) $(\text{adj } A)^{-1} = \text{adj}(A^{-1})$

(ii) $(A^{-1})^{-1} = A$

8. $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$

$|A| = 1(3\times 5 - 1\times 1) - (-2)((-2)\times 5 - 1\times 1) + 1((-2)\times 1 - 3\times 1)$

$|A| = (15-1) + 2(-10-1) + (-2-3)$

$|A| = 14 - 22 - 5 = -13$

To find the inverse of a matrix we need to find the Adjoint of that matrix

For finding the adjoint of the matrix we need to find its cofactors

Let A_{ij} denote the cofactors of Matrix A

Minor of an element $a_{ij} = M_{ij}$

$a_{11} = 1$, Minor of element $a_{11} = M_{11} = \begin{vmatrix}3 & 1\\ 1 & 5\end{vmatrix} = (3\times 5) - (1\times 1) = 14$

$a_{12} = -2$, Minor of element $a_{12} = M_{12} = \begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = (-2 \times 5) - (1 \times 1) = -11$

$a_{13} = 1$, Minor of element $a_{13} = M_{13} = \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = (-2 \times 1) - (3 \times 1) = -5$

$a_{21} = -2$, Minor of element $a_{21} = M_{21} = \begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = ((-2) \times 5) - (1 \times 1) = -11$

$a_{22} = 3$, Minor of element $a_{22} = M_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = (1 \times 5) - (1 \times 1) = 4$

$a_{23} = 1$, Minor of element $a_{23} = M_{23} = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = (1 \times 1) - ((-2) \times 1) = 3$

$a_{31} = 1$, Minor of element $a_{31} = M_{31} = \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = (-2 \times 1) - (3 \times 1) = -5$

$a_{32} = 1$, Minor of element $a_{32} = M_{32} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = (1 \times 1) - (1 \times (-2)) = 3$

$a_{33} = 5$, Minor of element $a_{33} = M_{33} = \begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix} = (1 \times 3) - ((-2) \times (-2)) = -1$

Cofactor of an element $a_{ij} = A_{ij}$

$A_{11} = (-1)^{1+1} \times 14 = 1 \times 14 = 14$

$A_{12} = (-1)^{1+2} \times (-11) = (-1) \times (-11) = 11$

$A_{13} = (-1)^{1+3} \times (-5) = 1 \times (-5) = -5$

$A_{21} = (-1)^{2+1} \times (-11) = (-1) \times (-11) = 11$

$A_{22} = (-1)^{2+2} \times 4 = 1 \times 4 = 4$

$A_{23} = (-1)^{2+3} \times 3 = (-1) \times 3 = -3$

$A_{31} = (-1)^{3+1} \times (-5) = 1 \times (-5) = -5$

$A_{32} = (-1)^{3+2} \times 3 = (-1) \times 3 = -3$

$A_{33} = (-1)^{3+3} \times (-1) = 1 \times (-1) = -1$

$$\text{Adj } A = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}' = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$A^{-1} = (\text{Adj } A)/|A|$

$$A^{-1} = -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

(i)

$|Adj\ A| = 14(-4 - 9) - 11\ (-11 - 15) - 5\ (-33 + 20)$

$= 14 \times (-13) - 11 \times (-26) - 5\ (-13)$

$= -\ 182 + 286 + 65 = 169$

Similarly Finding the Adj (Adj A) as found above

$$\text{Adj (Adj A)} = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$[Adj\ A]^{-1} = Adj\ (Adj\ A)/|Adj\ A|$

$$= \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

Similarly Finding the Adj (A^{-1}) as found above

$$\text{Adj}(A^{-1}) = \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

Hence, $[Adj\ A]^{-1} = Adj\ (A^{-1})$

(ii) To find $(A^{-1})^{-1}$ we have to find out $Adj(A^{-1})$

$$A^{-1} = -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$|A^{-1}| = (-1/13)^3\ [14\ (4 \times (-1) - (-3) \times (-3)) - 11\ (11 \times (-1) - (-3) \times (-5)) + (-5)\ (11 \times (-3) - 4 \times (-5))]$

$|A| = (-1/13)^3\ [14\ (-4 - 9) - 11\ (-11 - 15) - 5\ (-33 + 20)]$

$|A| = (-1/13)^3\ [14 \times (-13) - 11 \times (-26) - 5 \times (-13)]$

$|A| = (-1/13)^3 \times 169 = -1/13$

Cofactor of an element $a_{ij} = A_{ij}$

$A_{11} = (-1)^{1+1} \times (-1/13) = 1 \times (-1/13) = -1/13$

$A_{12} = (-1)^{1+2} \times (-2/13) = (-1) \times (-2/13) = 2/13$

$A_{13} = (-1)^{1+3} \times (-1/13) = 1 \times (-1/13) = -1/13$

$A_{21} = (-1)^{2+1} \times (-2/13) = (-1) \times (-2/13) = 2/13$

$A_{22} = (-1)^{2+2} \times (3/13) = 1 \times (-3/13) = -3/13$

$A_{23} = (-1)^{2+3} \times (1/13) = (-1) \times (1/13) = -1/13$

$A_{31} = (-1)^{3+1} \times (-1/13) = 1 \times (-1/13) = -1/13$

$A_{32} = (-1)^{3+2} \times (1/13) = (-1) \times 1/13 = -1/13$

$A_{33} = (-1)^{3+3} \times (-5/13) = 1 \times (-5/13) = -5/13$

$$\text{Adj }(A^{-1}) = \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix}' = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$(A^{-1})^{-1} = \text{Adj}(A^{-1})/|A^{-1}|$

$$(A^{-1})^{-1} = \frac{\frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}}{\frac{-1}{13}} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A$$

$\therefore (A^{-1})^{-1} = A$

9. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$.

9. Given that: $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

$$= \begin{vmatrix} 2(x+y) & y & x+y \\ 2(x+y) & x+y & x \\ 2(x+y) & x & y \end{vmatrix} \qquad \text{[applying } C_1 \to C_1 + C_2 + C_3\text{]}$$

$$= 2(x+y)\begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix}$$ [Taking 2(x + y) as common from C_1]

$$= 2(x+y)\begin{vmatrix} 0 & -x & y \\ 0 & y & x-y \\ 1 & y & y+k \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$]

$= 2(x + y)\{(-x)(x - y) - y.y\}$ [Expending along C_1]

$=2(x + y)(-x^2 + xy - y^2)$

$=-2(x + y)(x^2 - xy + y^2) = -2(x^3 + y^3)$

10. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x-y & y \\ 1 & x & x+y \end{vmatrix}$.

10. Given that: $\begin{vmatrix} 1 & x & y \\ 1 & x-y & y \\ 1 & x & x+y \end{vmatrix}$

$$= \begin{vmatrix} 0 & -y & 0 \\ 0 & y & -x \\ 1 & x & x+y \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$]

$=\{(-y)(-x) - y \,.\, 0\}$ [Expending along C_1]

$= xy$

Using properties of determinants in exercises 11 to 15, prove that:

11. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$

11. Let $\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$

Applying Row Transformations

$R_2 \to R_2 - R_1$

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta-\alpha & \beta^2-\alpha^2 & \alpha-\beta \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$$

$R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta-\alpha & \beta^2-\alpha^2 & \alpha-\beta \\ \gamma-\alpha & \gamma^2-\alpha^2 & \alpha-\gamma \end{vmatrix}$$

Taking $(\beta - \alpha)(\gamma - \alpha)$ from R_2 and R_3 respectively

$$\Delta = (\beta - \alpha)\ (\gamma - \alpha) \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\gamma & -1 \\ 1 & \gamma+\alpha & -1 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we have

$$\Delta = (\beta - \alpha)\ (\gamma - \alpha) \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\gamma & -1 \\ 0 & \gamma-\beta & 0 \end{vmatrix}$$

Expanding along R_3, we have

$\Delta = (\beta - \alpha)\ (\gamma - \alpha)\ [0\ (\alpha^2 \times (-1) - (\beta + \gamma) \times (\beta + \gamma) - (\gamma - \beta)((-1) \times \alpha - 1 \times (\beta + \gamma) + 0\ (\alpha \times (\beta + \gamma) - 1 \times \alpha^2)$

$\Delta = (\beta - \alpha)\ (\gamma - \alpha)\ [0 - (\gamma - \beta)(\ - \alpha - \beta - \gamma) + 0]$

$\Delta = (\beta - \alpha)\ (\gamma - \alpha)\ (\gamma - \beta)\ (\alpha + \beta + \gamma)$

$\Delta = (\alpha - \beta)\ (\beta - \gamma)\ (\gamma - \alpha)\ (\alpha + \beta + \gamma)$

12. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$

12. Let $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

Applying Elementary Row Transformations

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y-x & y^2-x^2 & p\left(y^3-x^3\right) \\ z-x & z^2-x^2 & p\left(z^3-x^3\right) \end{vmatrix}$$

Taking $(y - x)$ and $(z - x)$ common from R_2 and R_3 respectively

$$\Delta = (y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 1 & z+x & p\left(z^2+x^2+xz\right) \end{vmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$

$$\Delta = (y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 0 & z-y & p(z-y)(x+y+z) \end{vmatrix}$$

Taking (z – y) common from R_3

$$\Delta = (y-x)(z-x)(z-y)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 0 & 1 & p(x+y+z) \end{vmatrix}$$

Expanding along R_3, we have

$\Delta = (x-y)(y-z)(z-x)[0 - 1\{x \times p(y^2+x^2+xy) - 1 \times (1+px^3)\} + p(x+y+z)\{x \times (y+x) - 1 \times x^2\}$

$\Delta = (x-y)(y-z)(z-x)(-px^3 - pxy^2 - px^2y + 1 + px^3 + px^2y + pxy^2 + pxyz)$

$\Delta = (x-y)(y-z)(z-x)(1+pxyz)$

Hence, the given result is proved.

13. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

13. $\text{LHS} = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

$= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$ [applying $C_1 \rightarrow C_1 + C_2 + C_3$]

$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$ [Taking (a = b + c) as common from C_1]

$= (a+b+c)\begin{vmatrix} 0 & -a-2b & -a+b \\ 0 & 2b+c & -b-2c \\ 1 & -c+b & 3c \end{vmatrix}$ [Applying $R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$]

= (a + b + c){(-a – 2b)(-b – 2c) – (2b + c)(a + b)} [Expending along C_1]

$= (a + b + c)(ab + 2ac + 2b^2 + 4bc - (-2ab + 2b^2 - ac + bc))$

$= (a + b + c)(3ab + 3bc + 3ca) = 3(a + b + c)(ab + bc + ca) = \text{RHS}$

14. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

14. $\text{LHS} = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$ [Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$]

$= 1\{1.\ (7 + 3P) - (3)(2 + P)\}$ [Expending along C_1]

$= 7 + 3p - 6 - 3p = 1 = \text{RHS}$

15. $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

15. $\text{LHS} = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

$= \begin{vmatrix} \sin\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta\cos\delta - \sin\beta\sin\delta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta & \cos(\gamma+\delta) \end{vmatrix}$ [Applying $C_2 \to \text{COS}\ \delta\ C_2 - \sin\delta\ C_1$]

$= \begin{vmatrix} \sin\alpha & \cos(\alpha+\delta) & \cos(\alpha+\delta) \\ \sin\beta & \cos(\beta+\delta) & \cos(\beta+\delta) \\ \sin\gamma & \cos(\gamma+\delta) & \cos(\gamma+\delta) \end{vmatrix}$

$= 0 = \text{RHS}$ [$\because C_2 = C_3$]

16. Solve the system of equations:

$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$

$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$

$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

16. Given System of equations are

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}+\frac{20}{z}=2$$

Let $\frac{1}{x}=p;\frac{1}{y}=q;\frac{1}{z}=r$

∴ Given system of equation becomes

2p + 3q + 10r = 4

4p – 6q + 5r = 1

6p + 9q – 20r = 2

The given System of Equations can be written in the form of AX = B

Here $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$, $X = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

Now we need to find |A|

$\therefore |A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2 \times (120 - 45) - 3 \times (-80 - 30) + 10 \times (36 + 36)$

= 150 + 330 + 720

= 1200

Since, $|A| \neq 0$

∴ A is non-singular. So, it's inverse exists

Cofactor of an element $a_{ij} = A_{ij}$

$A_{11} = (-1)^{1+1} \times 75 = 1 \times 75 = 75$

$A_{12} = (-1)^{1+2} \times (-110) = (-1) \times (-110) = 110$

$A_{13} = (-1)^{1+3} \times 72 = 1 \times (72) = 72$

$A_{21} = (-1)^{2+1} \times (-150) = (-1) \times (-150) = 150$

$A_{22} = (-1)^{2+2} \times (-100) = 1 \times (-100) = -100$

$A_{23} = (-1)^{2+3} \times 0 = (-1) \times 0 = 0$

$A_{31} = (-1)^{3+1} \times 75 = 1 \times 75 = 75$

$A_{32} = (-1)^{3+2} \times (-30) = (-1) \times (-30) = 30$

$A_{33} = (-1)^{3+3} \times (-24) = 1 \times (-24) = -24$

$$\text{Adj } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}' = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$A^{-1} = (\text{Adj } A)/|A|$

$$A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Now,

Since, AX = B

$\therefore X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75\times4+150\times1+75\times2 \\ 110\times4-100\times1+30\times2 \\ 72\times4+0\times1-24\times2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \frac{600}{1200} \\ \frac{400}{1200} \\ \frac{240}{1200} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$

$\therefore$ p = ½, q = 1/3, r = 1/5

$\therefore$ x = 1/p = 2 ; y = 1/q = 3 ; z = 1/r = 5

So, x = 2 ; y = 3 ; z = 5

17. if a, b, c, are in A.P, then the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is:

(A) 0

(B) 1

(C) x
(D) 2x

17. Given $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

$= \begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 2(2b-a-c) \\ x+4 & x+5 & x+2c \end{vmatrix}$ [Applying $R_2 \to 2R_2 - (R_1 + R_3)$]

$= \begin{vmatrix} x+2 & x+3 & x+2a \\ 0 & 0 & 0 \\ x+4 & x+5 & x+2c \end{vmatrix}$ [$\because$ a, b, c are in AP, therefore 2b = a + c]

= 0

Hence, the option (A) is correct.

18. If x, y, z are nonzero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is:

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(B) $xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

(D) $\frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. A. $\frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

$|A| = x \times (y \times z) = xyz$

Since, $|A| \neq 0$

A^{-1} exists

To find the inverse of a matrix we need to find the Adjoint of that matrix

For finding the adjoint of the matrix we need to find its cofactors

Let A_{ij} denote the cofactors of Matrix A

Minor of an element $a_{ij} = M_{ij}$

$a_{11} = x$, Minor of element $a_{11} = M_{11} = \begin{vmatrix} y & 0 \\ 0 & z \end{vmatrix} = (y \times z) - (0 \times 0) = yz$

$a_{12} = 0$, Minor of element $a_{12} = M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = (0 \times z) - (0 \times 0) = 0$

$a_{13} = 0$, Minor of element $a_{13} = M_{13} = \begin{vmatrix} 0 & y \\ 0 & 0 \end{vmatrix} = (0 \times 0) - (0 \times y) = 0$

$a_{21} = 0$, Minor of element $a_{21} = M_{21} = \begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = (0 \times z) - (0 \times 0) = 0$

$a_{22} = y$, Minor of element $a_{22} = M_{22} = \begin{vmatrix} x & 0 \\ 0 & z \end{vmatrix} = (x \times z) - (0 \times 0) = xz$

$a_{23} = 0$, Minor of element $a_{23} = M_{23} = \begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = (x \times 0) - (0 \times 0) = 0$

$a_{31} = 0$, Minor of element $a_{31} = M_{31} = \begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = (z \times 0) - (0 \times 0) = 0$

$a_{32} = 0$, Minor of element $a_{32} = M_{32} = \begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = (x \times 0) - (0 \times 0) = 0$

$a_{33} = z$, Minor of element $a_{33} = M_{33} = \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} = (x \times y) - (0 \times 0) = xy$

Cofactor of an element a_{ij}, $A_{ij} = (-1)^{i+j} \times M_{ij}$

$A_{11} = (-1)^{1+1} \times M_{11} = 1 \times yz = yz$

$A_{12} = (-1)^{1+2} \times M_{12} = (-1) \times 0 = 0$

$A_{13} = (-1)^{1+3} \times M_{13} = 1 \times 0 = 0$

$A_{21} = (-1)^{2+1} \times M_{21} = (-1) \times 0 = 0$

$A_{22} = (-1)^{2+2} \times M_{22} = 1 \times xz = xz$

$A_{23} = (-1)^{2+3} \times M_{23} = (-1) \times 0 = 0$

$A_{31} = (-1)^{3+1} \times M_{31} = 1 \times 0 = 0$

$A_{32} = (-1)^{3+2} \times M_{32} = (-1) \times 0 = 0$

$A_{33} = (-1)^{3+3} \times M_{33} = 1 \times xy = xy$

$$\text{Adj } A = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}' = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$A^{-1} = \text{adj } A / |A|$

$$A^{-1} = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix} / xyz$$

$$A^{-1} = \begin{bmatrix} yz/xyz & 0 & 0 \\ 0 & xz/xyz & 0 \\ 0 & 0 & xy/xyz \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

The correct answer is A

19. Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$ then:

(A) det(A) = 0

(B) det(A) $\in (2, \infty)$

(C) det(A) $\in (2, 4)$

(D) det(A) $\in [2, 4]$

19. $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$= 1(1 + \sin^2\theta) + \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$ [Expending along C_1]

$= 2(1 + \sin^2\theta)$

Now, given that: $0 \le \theta \le 2\pi$

$\Rightarrow 0 \le \sin\theta \le 1$

$\Rightarrow 0 \le \sin^2\theta \le 1$

$\Rightarrow 1 \le 1 + \sin^2\theta \le 2$

$\Rightarrow 2 \le 2(1 + \sin^2\theta \le 4$

$\Rightarrow$ det(A) $\in [2, 4]$

Hence, the option (D) is correct.