EXERCISE 21.1

Find the sum of the following series to n terms:

1. $1^3 + 3^3 + 5^3 + 7^3 + \ldots\ldots$

Solution:

Let T_n be the nth term of the given series.

We have:

$T_n = [1 + (n - 1)2]^3$

$= (2n - 1)^3$

$= (2n)^3 – 3 (2n)^2. 1 + 3.1^2.2n-1^3$ [Since, $(a-b)^3 = a^3 – 3a^2b + 3ab^2 - b$]

$= 8n^3 – 12n^2 + 6n – 1$

Now, let S_n be the sum of n terms of the given series.

We have:

$$S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n} [2k-1]^3$$
$$= \sum_{k=1}^{n} [8k^3 - 1 - 6k(2k-1)]$$
$$= \sum_{k=1}^{n} [8k^3 - 1 - 12k^2 + 6k]$$
$$= \sum_{k=1}^{n} [8k^3 - 1 - 12k^2 + 6k]$$
$$= 8\sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} 1 - 12\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k$$
$$= \frac{8n^2(n+1)^2}{4} - n - \frac{12n(n+1)(2n+1)}{6} + \frac{6\,n(n+1)}{2}$$

Upon simplification we get,

$= 2n^2 (n + 1)^2 – n – 2n (n + 1) (2n + 1) + 3n (n + 1)$

$= n (n + 1) [2n (n + 1) – 2 (2n + 1) + 3] – n$

$= n (n + 1) [2n^2 – 2n + 1] – n$

$= n [2n^3 – 2n^2 + n + 2n^2 – 2n + 1 - 1]$

$= n [2n^3 - n]$

$= n^2 [2n^2 - 1]$

∴ The sum of the series is $n^2 [2n^2 - 1]$

2. $2^3 + 4^3 + 6^3 + 8^3 + \ldots\ldots\ldots$

Solution:

Let T_n be the nth term of the given series.

We have:

$T_n = (2n)^3$

$= 8n^3$

Now, let S_n be the sum of n terms of the given series.

We have:

$$S_n = \sum_{k=1}^{n} 8k^3$$

$$= 8\sum_{k=1}^{n} k^3$$

$$= 8\left[\frac{n(n+1)}{2}\right]^2$$

$$= 8 \times \frac{n^2(n+1)^2}{4}$$

$= 2n^2 (n + 1)^2$

$= 2\{n (n + 1)\}^2$

∴ The sum of the series is $2\{n (n + 1)\}^2$

3. 1.2.5 + 2.3.6 + 3.4.7 + ……..

Solution:

Let T_n be the nth term of the given series.

We have:

$T_n = n (n + 1) (n + 4)$

$= n (n^2 + 5n + 4)$

$= n^3 + 5n^2 + 4n$

Now, let S_n be the sum of n terms of the given series.

We have:

$$S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 5k^2 + \sum_{k=1}^{n} 4k$$

$$= \sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k$$

Upon simplification we get,

$$= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}$$

$$= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + 2n(n+1)$$

$$= \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4\right)$$

$$= \frac{n(n+1)}{2}\left(\frac{n^2+n}{2} + \frac{10n+5}{3} + 4\right)$$

$$= \frac{n(n+1)}{2}\left(\frac{3n^2+3n+20n+10+24}{6}\right)$$
$$= \frac{n}{12}(n+1)(3n^2+23n+34)$$

∴ The sum of the series is
$$= \frac{n}{12}(n+1)(3n^2+23n+34)$$

4. 1.2.4 + 2.3.7 + 3.4.10 + … to n terms.

Solution:

Let T_n be the nth term of the given series.

We have:

$T_n = n(n+1)(3n+1)$

$= n(3n^2 + 4n + 1)$

$= 3n^3 + 4n^2 + n$

Now, let S_n be the sum of n terms of the given series.

We have:

$$S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n} 3k^3 + \sum_{k=1}^{n} 4k^2 + \sum_{k=1}^{n} k$$
$$= 3\sum_{k=1}^{n} k^3 + 4\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

Upon simplification we get,

$$= \frac{3n^2(n+1)^2}{4} + \frac{4n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{3n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left(\frac{3n(n+1)}{2} + \frac{4(2n+1)}{3} + 1\right)$$
$$= \frac{n(n+1)}{2}\left(\frac{3n^2+3n}{2} + \frac{8n+4}{3} + 1\right)$$
$$= \frac{n(n+1)}{2}\left(\frac{9n^2+9n+16n+8+6}{6}\right)$$
$$= \frac{n}{12}(n+1)(9n^2+25n+14)$$

∴ The sum of the series is
$$= \frac{n}{12}(n+1)(9n^2+25n+14)$$

5. 1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + … to n terms

Solution:
Let T_n be the nth term of the given series.
We have:
$T_n = n(n+1)/2$
$= (n^2 + n)/2$
Now, let S_n be the sum of n terms of the given series.
We have:

$$S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n} \left(\frac{k^2+k}{2}\right)$$
$$= \frac{1}{2}\sum_{k=1}^{n} (k^2 + k)$$
$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$
$$= \frac{n(n+1)}{4}\left(\frac{2n+1}{3} + 1\right)$$
$$= \frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right)$$
$$= \frac{n(n+1)(2n+4)}{12}$$
$$= \frac{n(n+1)(n+2)}{6}$$

∴ The sum of the series is [n(n+1)(n+2)]/6