EXERCISE 19.17

Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{2x - x^2}} dx$

Solution:

Let $I = \int \frac{1}{\sqrt{2x-x^2}} dx$

The above equation can be written as

$$= \int \frac{1}{\sqrt{-(x^2 - 2x)}} dx$$

Now by adding and subtracting 1^2 to the denominator we get

$$= \int \frac{1}{\sqrt{-[x^2 - 2x(1) + 1^2 - 1^2]}} dx$$

On simplifying

$$= \int \frac{1}{\sqrt{-[(x-1)^2 - 1]}} dx$$

The above equation becomes

$$= \int \frac{1}{\sqrt{1 - (x-1)^2}} dx$$

Let (x - 1) = t and dx = dt

So, $I = \int \frac{1}{\sqrt{1-t^2}} dt$

$$= \sin^{-1} t + c \text{ [since } \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c \text{]}$$

$$I = \sin^{-1}(x - 1) + c$$

2. $\int \frac{1}{\sqrt{8+3x-x^2}}\,dx$

Solution:

The denominator of given question 8 + 3x – x^2 by adding and subtracting (9/4) can be written as

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

Therefore

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

The above equation can be written as

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

Substituting these values in given question we get

$$\int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Let x-3/2=t

dx = dt

$$\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+c$$

$$\left[\text{since} \int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+c\right]$$

$$= \sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+c$$

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+c$$

3. $$\int \frac{1}{\sqrt{5-4x-2x^2}}\,dx$$

Solution:

Let I = $\int \frac{1}{\sqrt{5-4x-2x^2}}dx$

Now taking out 2 as common from the denominator we get

$$= \int \frac{1}{\sqrt{-2\left[x^2+2x-\frac{5}{2}\right]}}dx$$

By adding and subtracting 1^2 to the denominator we get

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{-\left[x^2+2x+(1)^2-(1)^2-\frac{5}{2}\right]}}dx$$

By computing

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{-\left[(x+1)^2-\frac{7}{2}\right]}}dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\frac{7}{2}-(x+1)^2}}dx$$

Let (x + 1) = t

Differentiating both sides, we get, dx = dt

So, $I = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(\sqrt{\left(\frac{7}{2}\right)}\right)^2 - t^2}}dt$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t}{\sqrt{\frac{7}{2}}}\right) + c$$

[since $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$]

$$I = \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{2}{7}} \times (x + 1)\right) + c$$

4. $\int \frac{1}{\sqrt{3x^2 + 5x + 7}}dx$

Solution:

Let $I = \int \frac{1}{\sqrt{3x^2+5x+7}}dx$

Taking 1/√3 as common from the denominator we get

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2 + \frac{5}{3}x + \frac{7}{3}}}dx$$

Now by adding and subtracting $(5/6)^2$ to the denominator complete perfect square, we get

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2 + 2x\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2 + \frac{7}{3}}}dx$$

The above equation can be written as

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x + \frac{5}{6}\right)^2 + \frac{59}{36}}} dx$$

let $\left(x + \frac{5}{6}\right) = t$

dx = dt

$$I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{t^2 + \left(\frac{\sqrt{59}}{6}\right)^2}} dt$$

$$= \frac{1}{\sqrt{3}} \log \left| t + \sqrt{t^2 + \left(\frac{\sqrt{59}}{6}\right)} \right| + c \text{ [since } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log \left| x + \sqrt{x^2 + a^2} \right| + c$$

On simplification we get

$$I = \frac{1}{\sqrt{3}} \log \left| x + \frac{5}{6} + \sqrt{\left(x + \frac{5}{6}\right)^2 + \left(\frac{\sqrt{59}}{6}\right)^2} \right| + c$$

$$I = \frac{1}{\sqrt{3}} \log \left| x + \frac{5}{6} + \sqrt{x^2 + \frac{5x}{3} + \frac{7}{3}} \right| + c$$