EXERCISE 19.29

Evaluate the following integrals:

1. $\int (x+1)\sqrt{x^2-x+1}\,dx$

Solution:

Let us assume $x+1=\lambda\frac{d}{dx}(x^2-x+1)+\mu$

$\Rightarrow x+1=\lambda\left[\frac{d}{dx}(x^2)-\frac{d}{dx}(x)+\frac{d}{dx}(1)\right]+\mu$

We know $\frac{d}{dx}(x^n)=nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x+1=\lambda\,(2x^{2-1}-1+0)+\mu$

$\Rightarrow x+1=\lambda\,(2x-1)+\mu$

$\Rightarrow x+1=2\lambda x+\mu-\lambda$

Comparing the coefficient of x on both sides, we get

$2\lambda=1\Rightarrow\lambda=\frac{1}{2}$

Comparing the constant on both sides, we get

$\mu-\lambda=1$

$\Rightarrow \mu-\frac{1}{2}=1$

$\therefore \mu=\frac{3}{2}$

Hence, we have $x+1=\frac{1}{2}(2x-1)+\frac{3}{2}$

Substituting this value in I, we can write the integral as

$$I=\int\left[\frac{1}{2}(2x-1)+\frac{3}{2}\right]\sqrt{x^2-x+1}dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x-1)\sqrt{x^2-x+1} + \frac{3}{2}\sqrt{x^2-x+1}\right] dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x-1)\sqrt{x^2-x+1}dx + \int \frac{3}{2}\sqrt{x^2-x+1}dx$$

$$\Rightarrow I = \frac{1}{2}\int (2x-1)\sqrt{x^2-x+1}dx + \frac{3}{2}\int \sqrt{x^2-x+1}dx$$

Let $I_1 = \frac{1}{2}\int (2x-1)\sqrt{x^2-x+1}dx$

Now, put $x^2 - x + 1 = t$

$\Rightarrow$ (2x – 1) dx = dt (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{2}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{2}\int t^{\frac{1}{2}}dt$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3}(x^2 - x + 1)^{\frac{3}{2}} + c$$

Let $I_2 = \frac{3}{2}\int \sqrt{x^2-x+1}dx$

We can write $x^2 - x + 1 = x^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2}\int\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

We know that $\int\sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{\left(x - \frac{1}{2}\right)}{2}\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\ln\left|\left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{2x - 1}{4}\sqrt{x^2 - x + 1} + \frac{3}{8}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right|\right] + c$$

$$\therefore I_2 = \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{3}(x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right| + c$$

Thus,

$$\int(x + 1)\sqrt{x^2 - x + 1}\,dx = \frac{1}{3}(x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\ln\left|x - \frac{1}{2} + \sqrt{x^2 - x + 1}\right| + c$$

2. $\int (x+1)\sqrt{2x^2+3}\,dx$

Solution:

Let $I = \int (x+1)\sqrt{2x^2+3}dx$

Let us assume $x+1 = \lambda\frac{d}{dx}(2x^2+3)+\mu$

$\Rightarrow x+1 = \lambda\left[\frac{d}{dx}(2x^2)+\frac{d}{dx}(3)\right]+\mu$

$\Rightarrow x+1 = \lambda\left[2\frac{d}{dx}(x^2)+\frac{d}{dx}(3)\right]+\mu$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 1 = \lambda (2 \times 2x^{2-1} + 0) + \mu$

$\Rightarrow x + 1 = \lambda (4x) + \mu$

$\Rightarrow x + 1 = 4\lambda x + \mu$

Comparing the coefficient of x on both sides, we get

$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$

Comparing the constant on both sides, we get

$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$

Comparing the constant on both sides, we get

$\mu = 1$

Hence, we have $x+1 = \frac{1}{4}(4x)+1$

Substituting this value in I, we can write the integral as

$I = \int \left[\frac{1}{4}(4x)+1\right]\sqrt{2x^2+3}dx$

$\Rightarrow I = \int \left[\frac{1}{4}(4x)\sqrt{2x^2+3}+\sqrt{2x^2+3}\right]dx$

$$\Rightarrow I = \int \frac{1}{4}(4x)\sqrt{2x^2+3}dx + \int \sqrt{2x^2+3}dx$$

$$\Rightarrow I = \frac{1}{4}\int (4x)\sqrt{2x^2+3}dx + \int \sqrt{2x^2+3}dx$$

Let $I_1 = \frac{1}{4}\int (4x)\sqrt{2x^2+3}dx$

Now, put $2x^2 + 3 = t$

$\Rightarrow$ (4x) dx = dt (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{4}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{4}\int t^{\frac{1}{2}}dt$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{4}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{6} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + c$$

Let $I_2 = \int \sqrt{2x^2+3}dx$

We can write $2x^2 + 3 = 2\left(x^2 + \frac{3}{2}\right)$

$$\Rightarrow 2x^2 + 3 = 2\left[x^2 + \left(\sqrt{\frac{3}{2}}\right)^2\right]$$

Hence, we can write I_2 as

$$I_2 = \int \sqrt{2\left[x^2 + \left(\sqrt{\frac{3}{2}}\right)^2\right]}dx \Rightarrow I_2 = \sqrt{2}\int \sqrt{x^2 + \left(\sqrt{\frac{3}{2}}\right)^2}\,dx$$

We know that $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$

$$\Rightarrow I_2 = \sqrt{2}\left[\frac{x}{2}\sqrt{x^2 + \left(\sqrt{\frac{3}{2}}\right)^2} + \frac{\left(\sqrt{\frac{3}{2}}\right)^2}{2}\ln\left|x + \sqrt{x^2 + \left(\sqrt{\frac{3}{2}}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = \sqrt{2}\left[\frac{x}{2}\sqrt{x^2 + \frac{3}{2}} + \frac{3}{4}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right|\right] + c$$

$$\Rightarrow I_2 = \sqrt{2}\left[\frac{x}{2\sqrt{2}}\sqrt{2x^2 + 3} + \frac{3}{2 \times 2}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right|\right] + c$$

$$\therefore I_2 = \frac{x}{2}\sqrt{2x^2 + 3} + \frac{3}{2\sqrt{2}}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{6}(2x^2 + 3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2 + 3} + \frac{3}{2\sqrt{2}}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right| + c$$

Thus,

$$\int (x + 1)\sqrt{2x^2 + 3}dx = \frac{1}{6}(2x^2 + 3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2 + 3} + \frac{3}{2\sqrt{2}}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right| + c$$

3. $\int (2x - 5)\sqrt{2 + 3x - x^2}\,dx$

Solution:

Let $I = \int (2x - 5)\sqrt{2 + 3x - x^2}dx$

Let us assume $2x - 5 = \lambda \frac{d}{dx}(2 + 3x - x^2) + \mu$

$\Rightarrow 2x - 5 = \lambda\left[\frac{d}{dx}(2) + \frac{d}{dx}(3x) - \frac{d}{dx}(x^2)\right] + \mu$

$\Rightarrow 2x - 5 = \lambda\left[\frac{d}{dx}(2) + 3\frac{d}{dx}(x) - \frac{d}{dx}(x^2)\right] + \mu$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow 2x - 5 = \lambda (0 + 3 - 2x^{2-1}) + \mu$

$\Rightarrow 2x - 5 = \lambda (3 - 2x) + \mu$

$\Rightarrow 2x - 5 = -2\lambda x + 3\lambda + \mu$

Comparing the coefficient of x on both sides, we get

$-2\lambda = 2 \Rightarrow \lambda = -1$

Comparing the constant on both sides, we get

$3\lambda + \mu = -5$

$\Rightarrow 3(-1) + \mu = -5$

$\Rightarrow -3 + \mu = -5$

$\therefore \mu = -2$

Hence, we have $2x - 5 = -(3 - 2x) - 2$

Substituting this value in I, we can write the integral as

$I = \int [-(3 - 2x) - 2]\sqrt{2 + 3x - x^2}dx$

$\Rightarrow I = \int \left[-(3 - 2x)\sqrt{2 + 3x - x^2} - 2\sqrt{2 + 3x - x^2}\right]dx$

$$\Rightarrow I = -\int (3-2x)\sqrt{2+3x-x^2}dx - \int 2\sqrt{2+3x-x^2}dx$$

$$\Rightarrow I = -\int (3-2x)\sqrt{2+3x-x^2}dx - 2\int \sqrt{2+3x-x^2}dx$$

Let $I_1 = -\int (3-2x)\sqrt{2+3x-x^2}dx$

Now, put $2 + 3x - x^2 = t$

$\Rightarrow (3 - 2x)\, dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = -\int \sqrt{t}dt$$

$$\Rightarrow I_1 = -\int t^{\frac{1}{2}}dt$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = -\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow I_1 = -\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{2}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}} + c$$

Let $I_2 = -2\int \sqrt{2+3x-x^2}dx$

We can write $2 + 3x - x^2 = -(x^2 - 3x - 2)$

$$\Rightarrow 2+3x-x^2=-\left[x^2-2(x)\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-2\right]$$

$$\Rightarrow 2+3x-x^2=-\left[\left(x-\frac{3}{2}\right)^2-\frac{9}{4}-2\right]$$

$$\Rightarrow 2+3x-x^2=-\left[\left(x-\frac{3}{2}\right)^2-\frac{17}{4}\right]$$

$$\Rightarrow 2+3x-x^2=\frac{17}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow 2+3x-x^2=\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2=-2\int\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$$

We have $\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

$$\Rightarrow I_2=-2\left[\frac{\left(x-\frac{3}{2}\right)}{2}\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}+\frac{\left(\frac{\sqrt{17}}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{17}}{2}}\right)\right]+c$$

$$\Rightarrow I_2=-2\left[\frac{2x-3}{4}\sqrt{2+3x-x^2}+\frac{17}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)\right]+c$$

$$\therefore I_2=-\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c$$

Substituting I_1 and I_2 in I, we get

$$I=-\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}}-\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c$$

Thus,

$\int(2x-5)\sqrt{2+3x-x^2}dx = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}} - \frac{1}{2}(2x-3)\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$

4. $\int (x+2)\sqrt{x^2+x+1}\,dx$

Solution:

Let $I = \int(x+2)\sqrt{x^2+x+1}dx$

Let us assume $x+2 = \lambda\frac{d}{dx}(x^2+x+1)+\mu$

$\Rightarrow x+2 = \lambda\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(x)+\frac{d}{dx}(1)\right]+\mu$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow x + 2 = \lambda\,(2x^{2-1} + 1 + 0) + \mu$

$\Rightarrow x + 2 = \lambda\,(2x + 1) + \mu$

$\Rightarrow x + 2 = 2\lambda x + \lambda + \mu$

Comparing the coefficient of x on both sides, we get

$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

Comparing the constant on both sides, we get

$\lambda + \mu = 2$

$\Rightarrow \frac{1}{2} + \mu = 2$

$\therefore \mu = \frac{3}{2}$

Hence, we have $x+2 = \frac{1}{2}(2x+1)+\frac{3}{2}$

Substituting this value in I, we can write the integral as

$$I = \int \left[\frac{1}{2}(2x+1) + \frac{3}{2}\right]\sqrt{x^2+x+1}dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x+1)\sqrt{x^2+x+1} + \frac{3}{2}\sqrt{x^2+x+1}\right]dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x+1)\sqrt{x^2+x+1}dx + \int \frac{3}{2}\sqrt{x^2+x+1}dx$$

$$\Rightarrow I = \frac{1}{2}\int (2x+1)\sqrt{x^2+x+1}dx + \frac{3}{2}\int \sqrt{x^2+x+1}dx$$

Let $I_1 = \frac{1}{2}\int (2x+1)\sqrt{x^2+x+1}dx$

Now, put $x^2 + x + 1 = t$

$\Rightarrow (2x + 1)\, dx = dt$ (Differentiating both sides)

Substituting this value in I_1, we can write

$$I_1 = \frac{1}{2}\int \sqrt{t}dt$$

$$\Rightarrow I_1 = \frac{1}{2}\int t^{\frac{1}{2}}dt$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3}(x^2+x+1)^{\frac{3}{2}} + c$$

Let $I_2 = \frac{3}{2}\int \sqrt{x^2 + x + 1}dx$

We can write $x^2 + x + 1 = x^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2}\int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

We know that $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{\left(x + \frac{1}{2}\right)}{2}\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\ln\left|\left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right|\right] + c$$

$$\Rightarrow I_2 = \frac{3}{2}\left[\frac{2x + 1}{4}\sqrt{x^2 + x + 1} + \frac{3}{8}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right|\right] + c$$

$$\therefore I_2 = \frac{3}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{9}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$$

Substituting I_1 and I_2 in I, we get

$$I = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{9}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$$

Thus,

$$\int (x + 2)\sqrt{x^2 + x + 1}dx = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{9}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$$