

11. Applications of Derivatives

Exercise 11A

1. Question

The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.

Answer

Let the side of the square be a

$$\text{Rate of change of side} = \frac{da}{dt} = 0.2 \text{ cm/s}$$

$$\text{Perimeter of the square} = 4a$$

$$\text{Rate of change of perimeter} = 4 \frac{da}{dt} = 4 \times 0.2$$

$$\frac{dP}{dt} = 0.8 \text{ cm/s}$$

2. Question

The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Answer

Let the radius of the circle be r

$$\frac{dr}{dt} = 0.7 \text{ cm/s}$$

$$\text{Circumference of the circle} = 2\pi r$$

$$\text{Rate of change of circumference} = 2\pi \frac{dr}{dt}$$

$$= 2 \times 3.14 \times 0.7$$

$$\frac{dC}{dt} = 4.4 \text{ cm/s}$$

3. Question

The radius of a circle is increasing uniformly at the rate of 0.3 centimetre per second. At what rate is the area increasing when the radius is 10 cm?

(Take $\pi = 3.14$.)

Answer

Let the radius of the circle be r

$$\frac{dr}{dt} = 0.3 \text{ cm/s}$$

$$\text{Area of the circle} = \pi r^2$$

$$\text{Rate of change of Area} = 2\pi r \frac{dr}{dt}$$

$$= 2 \times 3.14 \times 10 \times 0.3$$

$$\frac{dA}{dt} = 18.84 \text{ cm}^2/\text{s}$$

4. Question

The side of a square sheet of metal is increasing at 3 centimetres per minute. At what rate is the area increasing when the side is 10 cm long?

Answer

Let the side of the square be a

$$\text{Rate of change of side} = \frac{da}{dt} = 3 \text{ cm/s}$$

$$\text{Area of the square} = a^2$$

$$\text{Rate of change of Area} = 2a \frac{da}{dt} = 2 \times 10 \times 3$$

$$\frac{dA}{dt} = 60 \text{ cm}^2/\text{s}$$

5. Question

The radius of a circular soap bubble is increasing at the rate of 0.2 cm/s. Find the rate of increase of its surface area when the radius is 7 cm.

Answer

Soap bubble will be in the shape of a sphere

Let the radius of the soap bubble be r

$$\frac{dr}{dt} = 0.2 \text{ cm/s}$$

$$\text{Surface area of the soap bubble} = 4\pi r^2$$

$$\text{Rate of change of Surface area} = 8\pi r \frac{dr}{dt}$$

$$= 8 \times 3.14 \times 7 \times 0.2$$

$$\frac{dS}{dt} = 35.2 \text{ cm}^2/\text{s}$$



6. Question

The radius of an air bubble is increasing at the rate of 0.5 centimetre per second. At what rate is the volume of the bubble increasing when the radius is 1 centimetre?

Answer

Soap bubble will be in the shape of a sphere

Let the radius of the soap bubble be r

$$\frac{dr}{dt} = 0.5 \text{ cm/s}$$

$$\text{Volume of the soap bubble} = \frac{4}{3}\pi r^3$$

$$\text{Rate of change of Volume} = 4\pi r^2 \frac{dr}{dt}$$

$$= 4 \times 3.14 \times 1^2 \times 0.5$$

$$\frac{dV}{dt} = 6.28 \text{ cm}^3/\text{s}$$

7. Question

The volume of a spherical balloon is increasing at the rate of 25 cubic centimetres per second. Find the rate of change of its surface at the instant when its radius is 5 cm.

Answer

Let the radius of the balloon be r

Let the volume of the spherical balloon be V

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$25 \text{ cm}^3/\text{s} = 4 \times \pi \times 5^2 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi}$$

Surface area of the bubble $= 4\pi r^2$

$$\text{Rate of change of Surface area} = 8\pi r \frac{dr}{dt}$$

$$= 8 \times \pi \times 5 \times \frac{1}{4\pi}$$

$$\frac{dS}{dt} = 10 \text{ cm}^2/\text{s}$$

8. Question

A balloon which always remains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.

Answer

When we pump a balloon its volume changes.

Let the radius of the balloon be r

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$900 \text{ cm}^3/\text{s} = 4 \times \pi \times 15^2 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{900}{4 \times 3.14 \times 225}$$

$$\frac{dr}{dt} = 0.32 \text{ cm/s}$$

9. Question

The bottom of a rectangular swimming tank is 25 m by 40 m. Water is pumped into the tank at the rate of 500 cubic metres per minute. Find the rate at which the level of water in the tank is rising.

Answer

Let the volume of the water tank be V

$$V = l \times b \times h$$

$$V = 25 \times 40 \times h$$

$$\frac{dV}{dt} = 1000 \times \frac{dh}{dt}$$

$$500 = 1000 \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.5 \text{ m/min}$$

10. Question

A stone is dropped into a quiet lake and waves move in circles at a speed of 3.5 cm per second. At the instant when the radius of the circular wave is 7.5 cm. how fast is the enclosed area increasing? (Take $\pi = 22/7$.)

Answer

Let the radius of the circle be r

$$\frac{dr}{dt} = 3.5 \text{ cm/s}$$

Area of the circle = πr^2

$$\text{Rate of change of Area} = 2\pi r \frac{dr}{dt}$$

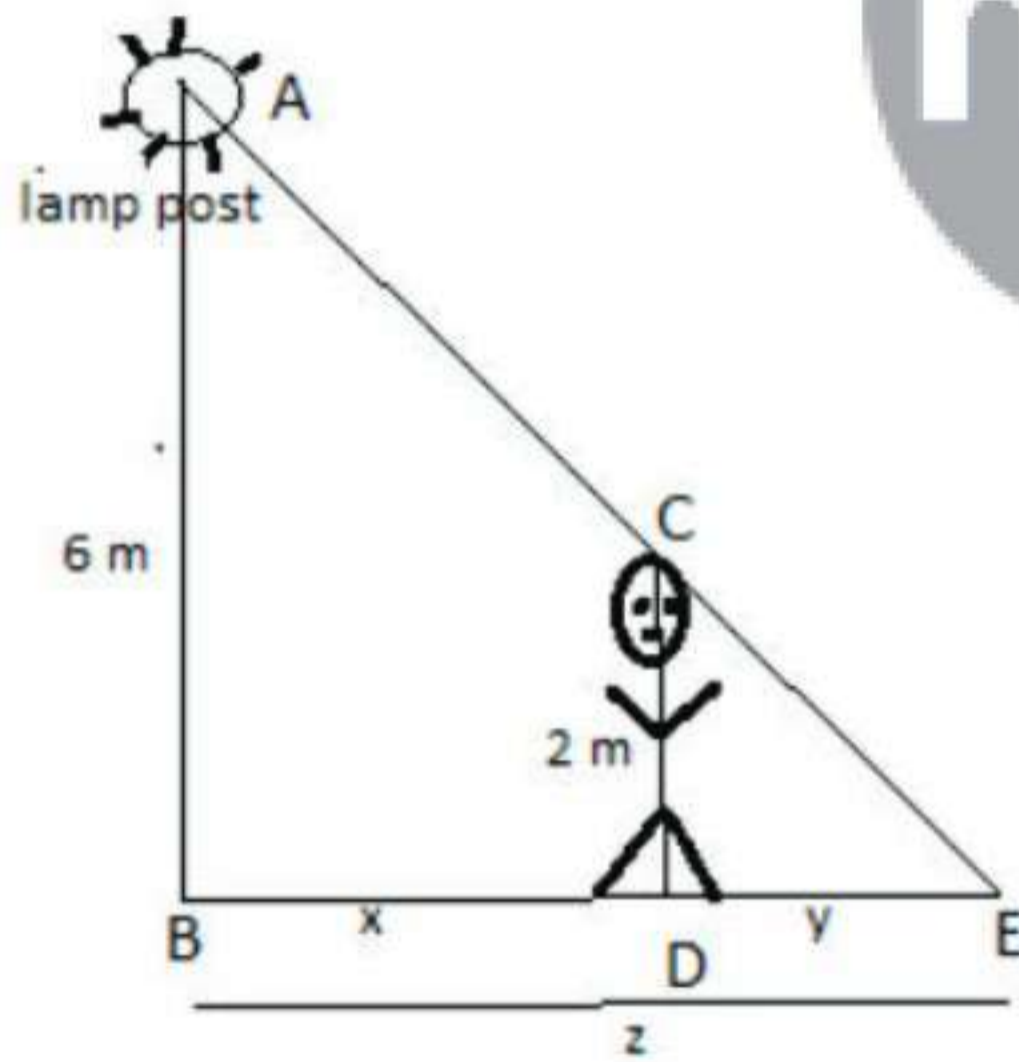
$$= 2 \times 3.14 \times 7.5 \times 3.5$$

$$= 165 \text{ cm}^2/\text{s}$$

11. Question

A 2-m tall man walks at a uniform speed of a uniform speed of 5 km per hour away from a 6-metre-high lamp post. Find the rate at which the length of his shadow increases.

Answer



ABE and CDE are similar triangles.

So,

$$\frac{AB}{BE} = \frac{CD}{DE}$$

$$\frac{0.006}{x+y} = \frac{0.002}{y}$$

$$6y = 2(x+y)$$

$$6 \frac{dy}{dt} = 2 \left(\frac{dx}{dt} + \frac{dy}{dt} \right)$$

$$6 \frac{dy}{dt} = 2 \left(5 + \frac{dy}{dt} \right)$$

$$6 \frac{dy}{dt} = 10 + 2 \frac{dy}{dt}$$

$$4 \frac{dy}{dt} = 10$$

$$\frac{dy}{dt} = 2.5 \text{ km/h}$$

12. Question

An inverted cone has a depth of 40 cm and a base of radius 5 cm. Water is poured into it at a rate of 1.5 cubic centimetres per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.

Answer

Let the volume of the cone be V

$$\frac{dV}{dt} = 1.5 \text{ cm}^3/\text{s}$$

$$V = \frac{1}{3} \pi r^2 h$$

$$V = \frac{1}{3} \pi 5^2 h$$

$$V = \frac{25}{3} \pi h$$

$$\frac{dV}{dt} = \frac{25}{3} \pi \frac{dh}{dt}$$

$$\frac{15}{10} = \frac{25}{3} \pi \frac{dh}{dt}$$

13. Question

Sand is pouring from a pipe at the rate of $18 \text{ cm}^3/\text{s}$. The falling sand forms a cone on the ground in such a way that the height of the cone is one-sixth of the radius of the base. How fast is the height of the sand cone increasing when its height is 3 cm?

Answer

$$h = \frac{1}{6} r$$

$$V = \frac{1}{3} \pi r^2 h$$

$$V = \frac{1}{3} \pi (6h)^2 h$$

$$V = 12\pi h^3$$

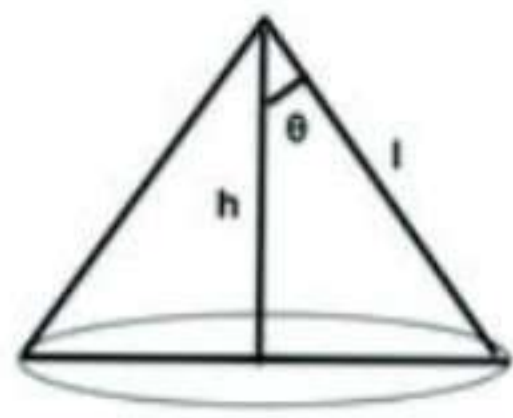
$$\frac{dV}{dt} = 36\pi h^2 \frac{dh}{dt}$$

$$18 = 36 \times 9 \times \pi \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{18\pi} \text{ cm/s}$$

14. Question

Water is dripping through a tiny hole at the vertex in the bottom of a conical funnel at a uniform rate of $4 \text{ cm}^3/\text{s}$. When the slant height of the water is 3 cm, find the rate of decrease of the slant height of the water, given that the vertical angle of the funnel is 120° .

Answer

Let the volume of the cone be V

$$\frac{dV}{dt} = 4\text{cm}^3/\text{s}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\cos Q = \frac{h}{l} = \cos 120 = \cos(180 - 60) = -\frac{1}{2}$$

$$\sin Q = \frac{r}{l} = \sin 120 = \sin(180 - 60) = \sin 60 = \frac{\sqrt{3}}{2}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{\sqrt{3}}{2}l\right)^2 \left(-\frac{1}{2}l\right)$$

$$V = -\frac{3}{24}\pi l^3$$

$$\frac{dV}{dt} = -\frac{9}{24}\pi l^2 \frac{dl}{dt}$$

$$4 = -\frac{3}{8}\pi l^2 \frac{dl}{dt}$$

$$-\frac{32}{27\pi} \text{ cm/s} = \frac{dl}{dt}$$

**15. Question**

Oil is leaking at the rate of 15 mL/s from a vertically kept cylindrical drum containing oil. If the radius of the drum is 7 cm and its height is 60 cm, find the rate at which the level of the oil is changing when the oil level is 18 cm.

Answer

$$\frac{dV}{dt} = 15 \text{ mL/s}$$

$$\frac{d}{dt}(\pi r^2 h) = 15$$

$$\frac{d}{dt}(\pi 7^2 h) = 15$$

$$49\pi \frac{dh}{dt} = 15$$

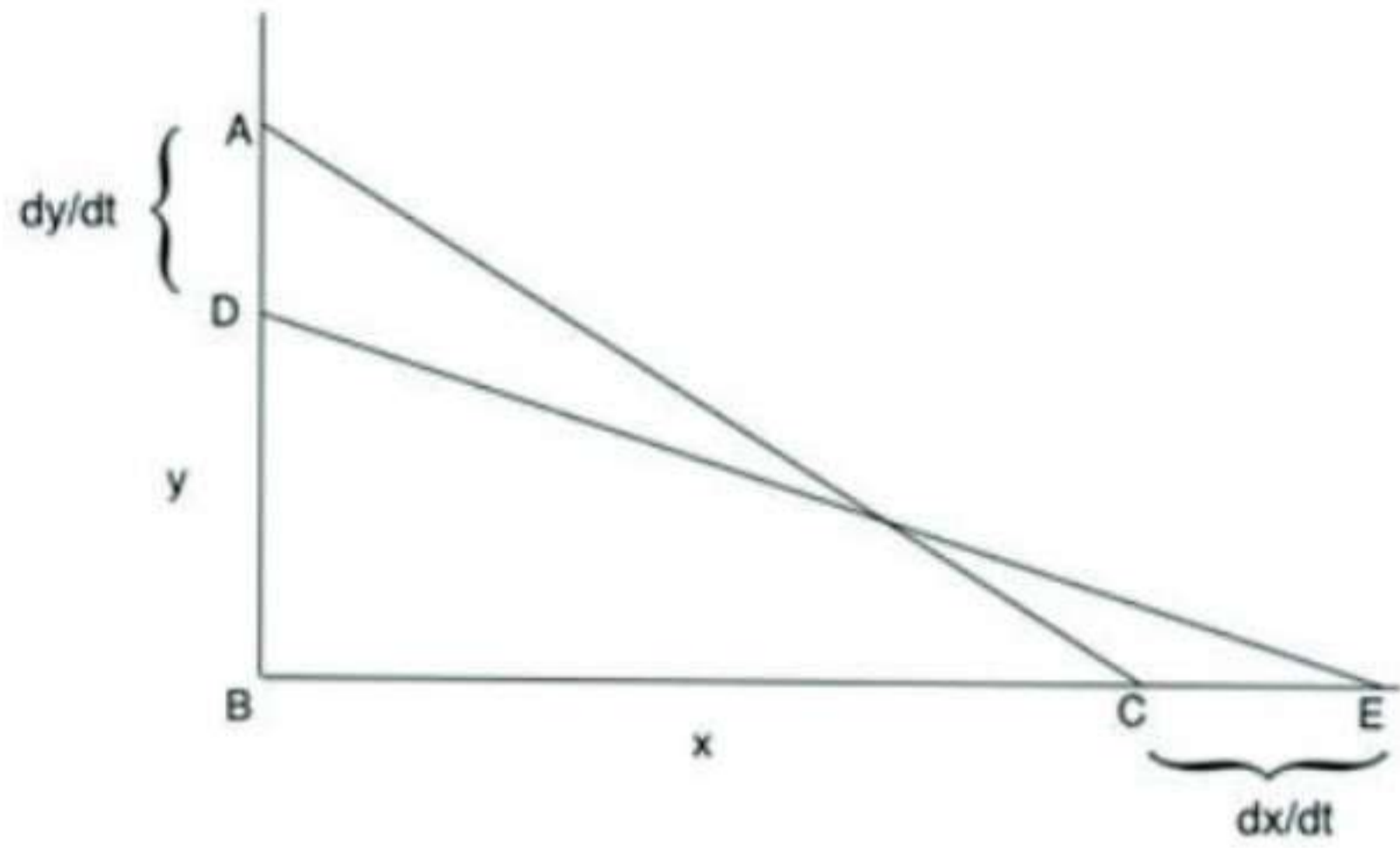
$$\frac{dh}{dt} = \frac{15}{49\pi}$$

16. Question

A 13-m long ladder is leaning against a wall. The bottom of the ladder is pulled along the ground, away from

the wall, at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5 m away from the wall?

Answer



Let the original ladder be AC and the pulled ladder be DE

Let AB=y and BC=x

Applying Pythagoras Theorem in ABC

$$x^2 + y^2 = 13^2 \dots(1)$$

$$5^2 + y^2 = 13^2$$

$$y = 12\text{cm}$$

Differentiating both sides of eqn (1) wrt to t

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

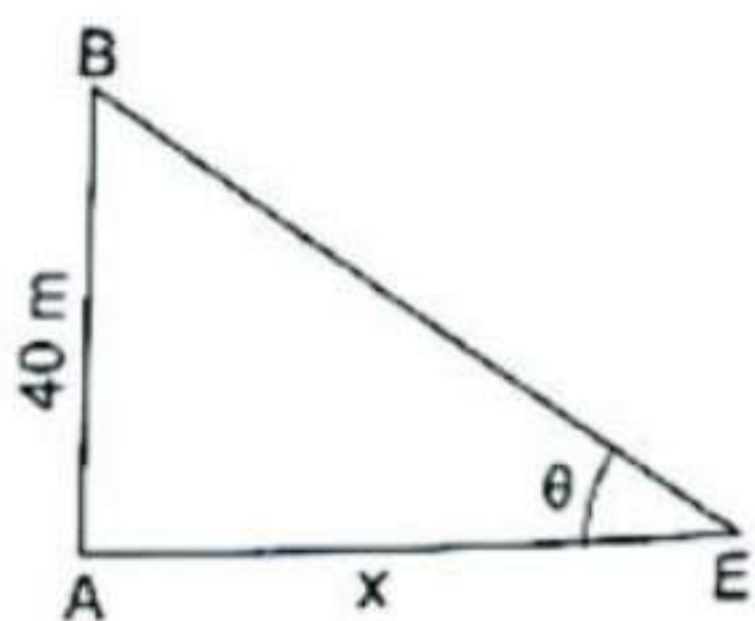
$$5.2 + 12 \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{10}{12} = -\frac{5}{6} \text{ cm/s}$$

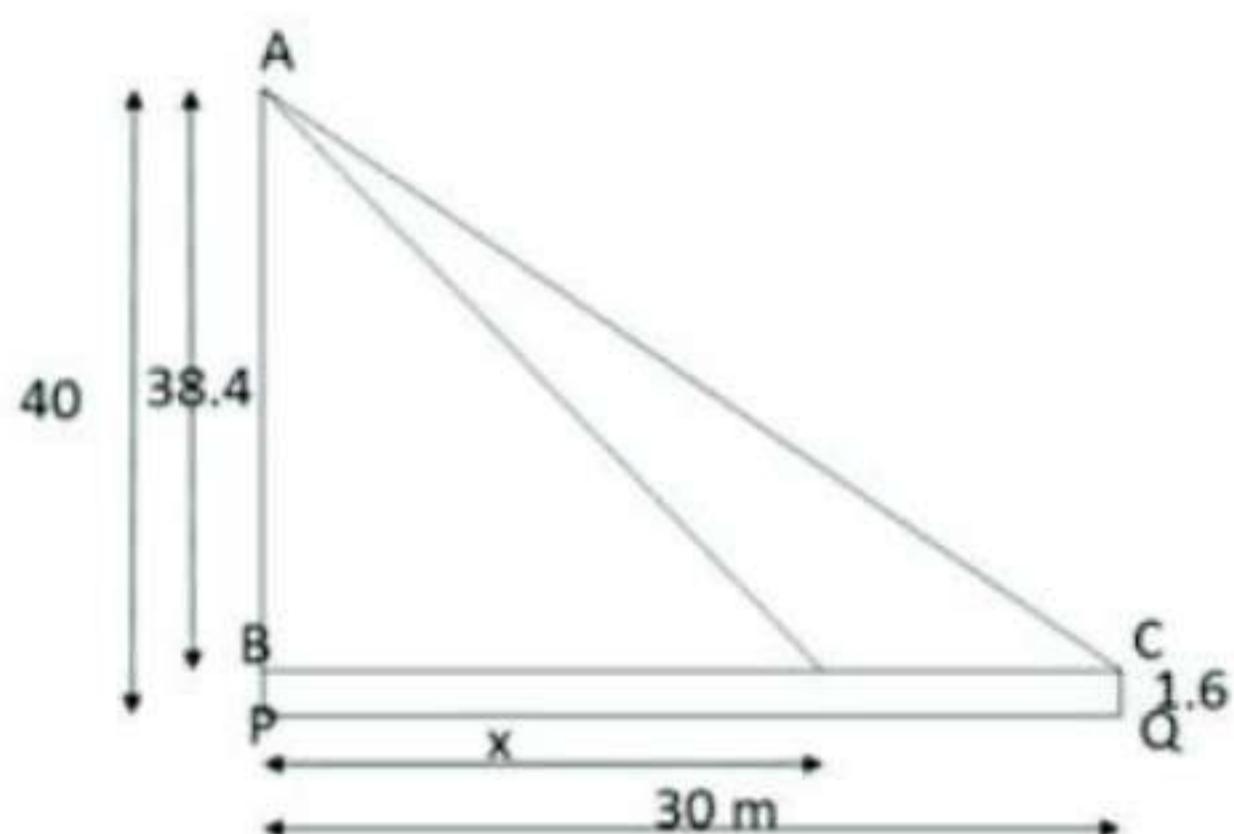


17. Question

A man is moving away from a 40-m high tower at a speed of 2 m/s. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 metres from the foot of the tower. Assume that the eye level of the man is 1.6 m from the ground.



Answer



$$\frac{dx}{dt} = -2 \text{ cm/s}$$

$$\tan Q = \frac{38.4}{x}$$

$$Q = \tan^{-1} \frac{38.4}{x}$$

$$\frac{dQ}{dt} = \frac{1}{1 + \frac{38.4^2}{x^2}} \left(-\frac{1}{x^2} \right) \cdot 38.4$$

$$\frac{dQ}{dt} = \frac{x^2}{x^2 + 1474.56} \left(-\frac{1}{x^2} \right) \cdot 38.4$$

$$\frac{dQ}{dt} = -\frac{1}{30^2 + 1474.56} \cdot 38.4 \cdot \frac{dx}{dt}$$

$$\frac{dQ}{dt} = -\frac{1}{30^2 + 1474.56} \cdot 38.4 \times 2$$

$$\frac{dQ}{dt} = -0.032 \text{ radian/second}$$

18. Question

Find an angle x which increases twice as fast as its sine.

Answer

ATQ,

$$\frac{dx}{dt} = 2 \frac{d}{dt} (\sin x)$$

$$\frac{dx}{dt} = 2 \cos x \frac{dx}{dt}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$



19. Question

The radius of a balloon is increasing at the rate of 10 m/s. At what rate is the surface area of the balloon increasing when the radius is 15 cm?

Answer

$$\frac{dr}{dt} = 10 \text{ m/s}$$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi \cdot 15 \cdot 10$$

$$\frac{dS}{dt} = 1200\pi \text{ cm}^2/\text{s}$$

20. Question

An edge of a variable cube is increasing at the rate of 5 cm/s. How fast is the volume of the cube increasing

when the edge is 10 cm long?

Answer

$$\frac{da}{dt} = 5 \text{ cm/s}$$

$$V = a^3$$

$$\frac{dV}{dt} = 3a^2 \frac{da}{dt}$$

$$\frac{dV}{dt} = 3 \cdot 10^2 \cdot 5$$

$$\frac{dV}{dt} = 1500 \text{ cm}^3/\text{s}$$

21. Question

The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area is increasing when the side is 10 cm.

Answer

$$\frac{da}{dt} = 2 \text{ cm/s}$$

$$A = \frac{\sqrt{3}}{4} a^2$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} 2a \frac{da}{dt}$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} \cdot 10 \cdot 2$$

$$\frac{dA}{dt} = 10\sqrt{3} \text{ cm}^2/\text{s}$$



Exercise 11B

1. Question

Using differentials, find the approximate values of:

find the approximate values of $\sqrt{37}$.

Answer

Let $y = \sqrt{x}$.

Let $x = 36$ and $\Delta x = 1$.

As $y = \sqrt{x}$.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \frac{1}{2\sqrt{x}} \cdot \Delta x$$

$$\Rightarrow \Delta y = \frac{1}{2\sqrt{36}} \cdot 1$$

$$\Rightarrow \Delta y = \frac{1}{12}$$

$$\therefore \Delta y = 0.08$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore 0.08 = \sqrt{36+1} - \sqrt{36}$$

$$\Rightarrow 0.08 = \sqrt{37} - 6$$

$$\Rightarrow \sqrt{37} = 6.08$$

2. Question

Using differentials, find the approximate values of:

Find the approximate values of $\sqrt[3]{29}$.

Answer

$$\text{Let } y = \sqrt[3]{x}.$$

$$\text{Let } x = 27 \text{ and } \Delta x = 2.$$

$$\text{As } y = \sqrt[3]{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \frac{1}{3}x^{-\frac{2}{3}} \cdot \Delta x$$

$$\Rightarrow \Delta y = \frac{1}{3}27^{-\frac{2}{3}} \cdot 2$$

$$\Rightarrow \Delta y = \frac{2}{27}$$

$$\therefore \Delta y = 0.074$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore 0.074 = \sqrt[3]{27+2} - \sqrt[3]{27}$$

$$\Rightarrow 0.074 = \sqrt[3]{29} - 3$$

$$\Rightarrow \sqrt[3]{29} = 3.074$$

3. Question

Using differentials, find the approximate values of:

Find the approximate values of $\sqrt{27}$

Answer

$$\text{Let } y = \sqrt{x}.$$

$$\text{Let } x = 25 \text{ and } \Delta x = 2.$$

$$\text{As } y = \sqrt{x}.$$



$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \frac{1}{2\sqrt{x}} \cdot \Delta x$$

$$\Rightarrow \Delta y = \frac{1}{2\sqrt{25}} \cdot 2$$

$$\Rightarrow \Delta y = \frac{1}{5}$$

$$\therefore \Delta y = 0.2$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore 0.2 = \sqrt{25+2} - \sqrt{25}$$

$$\Rightarrow 0.2 = \sqrt{27} - 5$$

$$\Rightarrow \sqrt{27} = 5.2$$

4. Question

Using differentials, find the approximate values of:

Find the approximate values of $\sqrt{0.24}$

Answer

Let $y = \sqrt{x}$.

Let $x = 0.25$ and $\Delta x = -0.01$.

As $y = \sqrt{x}$.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \frac{1}{2\sqrt{x}} \cdot \Delta x$$

$$\Rightarrow \Delta y = \frac{1}{2\sqrt{0.25}} \cdot (-0.01)$$

$$\Rightarrow \Delta y = -0.01$$

$$\therefore \Delta y = -0.01$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore -0.01 = \sqrt{0.25 - 0.01} - \sqrt{0.25}$$

$$\Rightarrow -0.01 = \sqrt{0.24} - 0.5$$

$$\Rightarrow \sqrt{0.24} = 0.49$$

5. Question

Using differentials, find the approximate values of:



Find the approximate values of $\sqrt{49.5}$

Answer

Let $y = \sqrt{x}$.

Let $x = 19$ and $\Delta x = 0.5$.

As $y = \sqrt{x}$.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \frac{1}{2\sqrt{x}} \cdot \Delta x$$

$$\Rightarrow \Delta y = \frac{1}{2\sqrt{49}} \cdot 0.5$$

$$\Rightarrow \Delta y = \frac{0.5}{14}$$

$$\therefore \Delta y = 0.0357$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore 0.0357 = \sqrt{49 + 0.5} - \sqrt{49}$$

$$\Rightarrow 0.0357 = \sqrt{49.5} - 7$$

$$\Rightarrow \sqrt{49.5} = 7.0357.$$



6. Question

Using differentials, find the approximate values of:

Find the approximate values of $\sqrt[4]{15}$

Answer

Let $y = \sqrt[4]{x}$.

Let $x = 16$ and $\Delta x = 1$.

As $y = \sqrt[4]{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4} x^{-\frac{3}{4}}$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \frac{1}{4} x^{-\frac{3}{4}} \cdot \Delta x$$

$$\Rightarrow \Delta y = \frac{1}{4} 16^{-\frac{3}{4}} \cdot (-1)$$

$$\Rightarrow \Delta y = \frac{-1}{32}$$

$$\therefore \Delta y = -0.03125$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore -0.03125 = \sqrt[4]{16-1} - \sqrt[4]{16}$$

$$\Rightarrow -0.03125 = \sqrt[4]{15} - 2$$

$$\Rightarrow \sqrt[4]{15} = 1.96875$$

7. Question

find the approximate values of $\frac{1}{(2.002)^2}$

Answer

$$\text{Let } y = \frac{1}{x^2}$$

$$\text{Let } x = 2 \text{ and } \Delta x = 0.002.$$

$$\text{As } y = \frac{1}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{x^3}$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \frac{-2}{x^3} \cdot \Delta x$$

$$\Rightarrow \Delta y = \frac{-2}{8} \cdot (0.002)$$

$$\Rightarrow \Delta y = \frac{-0.5}{1000}$$

$$\therefore \Delta y = -0.0005$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore -0.0005 = \frac{1}{(2.002)^2} - \frac{1}{2^2}$$

$$\Rightarrow -0.0005 = \frac{1}{(2.002)^2} - 0.25$$

$$\Rightarrow \frac{1}{(2.002)^2} = 0.2495$$

8. Question

find the approximate values of $\log_e 10.02$, given that $\log_e 10 = 2.3026$

Answer

$$\text{Let } y = \log_e x$$

$$\text{Let } x = 10 \text{ and } \Delta x = 0.02.$$

$$\text{As } y = \log_e x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \frac{1}{x} \cdot \Delta x$$



$$\Rightarrow \Delta y = \frac{1}{10} \cdot (0.02)$$

$$\Rightarrow \Delta y = \frac{0.02}{10}$$

$$\therefore \Delta y = 0.002$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore 0.002 = \log_e(10+0.02) - \log_e(10)$$

$$\Rightarrow 0.002 = \log_e(10.02) - 2.3026$$

$$\Rightarrow \log_e(10.02) = 2.3046.$$

9. Question

find the approximate values of $\log_{10}(4.04)$, it being given that $\log_{10}4 = 0.6021$ and $\log_{10}e = 0.4343$

Answer

$$\text{Let } y = \log_{10} x$$

$$\therefore y = \frac{\log_e x}{\log_e 10}$$

$$\therefore y = 0.4343 \log_e x$$

$$\text{Let } x = 4 \text{ and } \Delta x = 0.04.$$

$$\text{As } y = 0.4343 \log_e x$$

$$\Rightarrow \frac{dy}{dx} = \frac{0.4343}{x}$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \frac{0.4343}{x} \cdot \Delta x$$

$$\Rightarrow \Delta y = \frac{0.4343}{4} \cdot (0.04)$$

$$\Rightarrow \Delta y = \frac{0.017372}{4}$$

$$\therefore \Delta y = 0.004343$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore 0.004343 = \log_e(4+0.04) - \log_e(4)$$

$$\Rightarrow 0.004343 = \log_e(4.04) - 0.6021$$

$$\Rightarrow \log_e(4.04) = 0.606443.$$

10. Question

find the approximate values of $\cos 61^\circ$, it being given that $\sin 60^\circ = 0.86603$ and $1^\circ = 0.01745$ radian

Answer

$$\text{Let } y = \cos x$$

$$\text{Let } x = 60^\circ \text{ and } \Delta x = 1^\circ.$$

$$\text{As } y = \cos x$$



$$\Rightarrow \frac{dy}{dx} = \sin x$$

We, know

$$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$$

$$\therefore \Delta y = \sin x \cdot \Delta x$$

$$\Rightarrow \Delta y = \sin(60)(0.01745)$$

$$\Rightarrow \Delta y = (0.86603)(0.01745)$$

$$\therefore \Delta y = 0.01511$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\therefore 0.01511 = \cos(60+1) - \cos(60)$$

$$\Rightarrow 0.01511 = \cos 61^\circ - 0.5$$

$$\Rightarrow \cos 61^\circ = 0.48489$$

11. Question

If $y = \sin x$ and x changes from $\frac{\pi}{2}$ to $\frac{22}{14}$, what is the approximate change in y ?

Answer

Given x is $\pi/2$

Value of π is $22/7$

$22/14$ is $\pi/2$

Hence there will be no change.



12. Question

A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate, if the radius of the plate before heating is 10 cm.

Answer

Let the radius of the plate 10cm.

Radius increases by 2% by heating

$$\therefore \text{After increment} = \frac{2}{100} \times 10 = 0.2$$

Change in radius $dr = 0.2$

$$\therefore \text{New radius} = 10 + 0.2 = 10.2 \text{ cm}$$

Area of circular plate = $A = \pi r^2$

$$\therefore \text{Change in Area} = \frac{dA}{dr}$$

$$\Rightarrow \frac{dA}{dr} = 2\pi r dr$$

$$\Rightarrow \frac{dA}{dr} = 2 \times \pi \times 10 \times 0.2$$

$$\Rightarrow \frac{dA}{dr} = 4\pi \text{ cm}^2$$

13. Question

If the length of a simple pendulum is decreased by 2%, find the percentage decrease in its period T, where

$$T = 2\pi \sqrt{\frac{l}{g}}$$

Answer

The formula for time period -

$$\therefore T = 2\pi \sqrt{\frac{l}{g}}$$

Here $2, \pi, g$ have no dimensions. So we can eliminate them.

$$\text{Now } \frac{\Delta T}{T} = \frac{1}{2} \times \frac{\Delta L}{L}$$

By representing in percentage error

$$\Rightarrow \frac{\Delta T}{T} \times 100\% = \frac{1}{2} \times \frac{\Delta L}{L} \times 100\%$$

$$\Rightarrow \frac{\Delta T}{T} \times 100\% = \frac{1}{2} \times \frac{\Delta L}{L} \times 100\%$$

$$\Rightarrow \frac{\Delta T}{T} \% = \frac{1}{2} \times 2\%$$

$$\Rightarrow \frac{\Delta T}{T} \% = 1\%$$

Hence the time period becomes 1 %.

14. Question

The pressure p and the volume V of a gas are connected by the relation, $pV^{1/4} = k$, where k is a constant. Find the percentage increase in the pressure, corresponding to a diminution of 0.5% in the volume.

Answer

$$\text{Given: } pV^{1/4} = k$$

%decrease in the volume = 1/2%

$$\therefore \frac{\Delta V}{V} \times 100 = \frac{-1}{2}$$

$$pV^{1/4} = k$$

taking log on both sides

$$\log[pV^{1/4}] = \log a$$

$$\log P + 1.4\log V = \log a$$

Differentiating both the sides we get

$$\Rightarrow \frac{1}{P} dP + \frac{1.4}{V} dV = 0$$

$$\Rightarrow \frac{dP}{P} + 1.4 \frac{dV}{V} = 0$$

Multiplying both sides by 100.

$$\Rightarrow \frac{dP}{P} \times 100 + 1.4 \times \frac{dV}{V} \times 100 = 0$$

$$\Rightarrow \frac{dP}{P} \times 100 + 1.4 \left(\frac{-1}{2} \right) = 0$$

$$\Rightarrow \frac{dP}{P} \times 100 = 0.7$$

%error in P = 0.7%.

15. Question

The radius of a sphere shrinks from 10 cm to 9.8 cm. Find approximately the decrease in (i) volume, and (ii) surface area.

Answer

Decrease in radius = $dr = 10 - 9.8$

$$\therefore dr = 0.2$$

Volume of the sphere is given by = $V = \frac{4}{3}\pi r^3$

Change in volume = $dV = 4\pi r^2 dr$

$$\therefore dV = 4\pi(10)^2 \times 0.2$$

$$\Rightarrow dV = 80\pi \text{ cm}^3$$

Surface area of the sphere is given by = $A = 4\pi r^2$

Change in volume = $dA = 8\pi r dr$

$$\therefore dA = 8\pi \times 10 \times 0.2$$

$$\therefore dA = 16\pi.$$

16. Question

If there is an error of 0.1% in the measurement of the radius of a sphere, find approximately the percentage error in the calculation of the volume of the sphere.

Answer

Volume of the sphere is given by = $V = \frac{4}{3}\pi r^3$

Change in volume = $dV = 4\pi r^2 dr$

Given: $\Delta r = 0.1$

$$\Rightarrow \Delta r \cdot \frac{dV}{dr} = 4\pi r^2 \Delta r$$

$$\Rightarrow \Delta V = 4\pi r^2 \Delta r$$

Percentage error

$$\Rightarrow \frac{\Delta V}{V} = \frac{4\pi r^2}{\frac{4\pi r^3}{3}} \times 0.1$$

$$= 0.3\%$$

17. Question

Show that the relative error in the volume of a sphere, due to an error in measuring the diameter, is three times the relative error in the diameter.

Answer

Let d be the diameter r be the radius and V be the volume of Sphere

Volume of the sphere is given by = $V = \frac{4}{3}\pi r^3$

$$\Rightarrow V = \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 = \frac{1}{6}\pi D^3$$

Let Δd be the error in d and the corresponding error in V be ΔV .

$$\therefore \Delta V = \frac{dV}{dd} \Delta d = \frac{1}{2}\pi d^2 \Delta D$$

$$\therefore \frac{\Delta V}{V} = \frac{\frac{1}{2}\pi d^2 \Delta D}{\frac{1}{6}\pi D^3} = 3 \frac{\Delta D}{D}$$

Hence Proved

Exercise 11C

1. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x^2 \text{ on } [-1, 1]$$

Answer

Condition (1):

Since, $f(x)=x^2$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x)=x^2$ is continuous on $[-1, 1]$.

Condition (2):

Here, $f'(x)=2x$ which exist in $[-1, 1]$.

So, $f(x)=x^2$ is differentiable on $(-1, 1)$.

Condition (3):

Here, $f(-1)=(-1)^2=1$

And $f(1)=1^2=1$

i.e. $f(-1)=f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (-1, 1)$ such that $f'(c)=0$

i.e. $2c=0$

i.e. $c=0$

Value of $c=0 \in (-1, 1)$

Thus, Rolle's theorem is satisfied.

2. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x^2 - x - 12 \text{ in } [-3, 4]$$

Answer

Condition (1):

Since, $f(x)=x^2-x-12$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x)=x^2-x-12$ is continuous on $[-3, 4]$.

Condition (2):

Here, $f'(x)=2x-1$ which exist in $[-3, 4]$.

So, $f(x)=x^2-x-12$ is differentiable on $(-3, 4)$.

Condition (3):

Here, $f(-3)=(-3)^2-3-12=0$



$$\text{And } f(4) = 4^2 - 4 - 12 = 0$$

$$\text{i.e. } f(-3) = f(4)$$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (-3, 4)$ such that $f'(c) = 0$

$$\text{i.e. } 2c - 1 = 0$$

$$\text{i.e. } c = \frac{1}{2}$$

$$\text{Value of } c = \frac{1}{2} \in (-3, 4)$$

Thus, Rolle's theorem is satisfied.

3. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x^2 - 5x + 6 \text{ in } [2, 3]$$

Answer

Condition (1):

Since, $f(x) = x^2 - 5x + 6$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = x^2 - 5x + 6$ is continuous on $[2, 3]$.

Condition (2):

Here, $f'(x) = 2x - 5$ which exist in $[2, 3]$.

So, $f(x) = x^2 - 5x + 6$ is differentiable on $(2, 3)$.

Condition (3):

$$\text{Here, } f(2) = 2^2 - 5 \times 2 + 6 = 0$$

$$\text{And } f(3) = 3^2 - 5 \times 3 + 6 = 0$$

$$\text{i.e. } f(2) = f(3)$$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (2, 3)$ such that $f'(c) = 0$

$$\text{i.e. } 2c - 5 = 0$$

$$\text{i.e. } c = \frac{5}{2}$$

$$\text{Value of } c = \frac{5}{2} \in (2, 3)$$

Thus, Rolle's theorem is satisfied.

4. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x^2 - 5x + 6 \text{ in } [-3, 6]$$

Answer

Condition (1):

Since, $f(x) = x^2 - 5x + 6$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.



$\Rightarrow f(x) = x^2 - 5x + 6$ is continuous on $[-3, 6]$.

Condition (2):

Here, $f'(x) = 2x - 5$ which exist in $[-3, 6]$.

So, $f(x) = x^2 - 5x + 6$ is differentiable on $(-3, 6)$.

Condition (3):

Here, $f(-3) = (-3)^2 - 5 \times (-3) + 6 = 30$

And $f(6) = 6^2 - 5 \times 6 + 6 = 12$

i.e. $f(-3) \neq f(6)$

Conditions (3) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.

5. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x^2 - 4x + 3 \text{ in } [1, 3]$$

Answer

Condition (1):

Since, $f(x) = x^2 - 4x + 3$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = x^2 - 4x + 3$ is continuous on $[1, 3]$.

Condition (2):

Here, $f'(x) = 2x - 4$ which exist in $[1, 3]$.

So, $f(x) = x^2 - 4x + 3$ is differentiable on $(1, 3)$.

Condition (3):

Here, $f(1) = (1)^2 - 4(1) + 3 = 0$

And $f(3) = (3)^2 - 4(3) + 3 = 0$

i.e. $f(1) = f(3)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (1, 3)$ such that $f'(c) = 0$

i.e. $2c - 4 = 0$

i.e. $c = 2$

Value of $c = 2 \in (1, 3)$

Thus, Rolle's theorem is satisfied.

6. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x(x - 4)^2 \text{ in } [0, 4]$$

Answer

Condition (1):



Since, $f(x)=x(x-4)^2$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x)=x(x-4)^2$ is continuous on $[0,4]$.

Condition (2):

Here, $f'(x)=(x-4)^2+2x(x-4)$ which exist in $[0,4]$.

So, $f(x)=x(x-4)^2$ is differentiable on $(0,4)$.

Condition (3):

Here, $f(0)=0(0-4)^2=0$

And $f(4)=4(4-4)^2=0$

i.e. $f(0)=f(4)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (0,4)$ such that $f'(c)=0$

i.e. $(c-4)^2+2c(c-4)=0$

i.e. $(c-4)(3c-4)=0$

i.e. $c=4$ or $c=3/4$

Value of $c = \frac{3}{4} \in (0,4)$

Thus, Rolle's theorem is satisfied.

7. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x^3 - 7x^2 + 16x - 12 \text{ in } [2,3]$$

Answer

Condition (1):

Since, $f(x)=x^3-7x^2+16x-12$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x)=x^3-7x^2+16x-12$ is continuous on $[2,3]$.

Condition (2):

Here, $f'(x)=3x^2-14x+16$ which exist in $[2,3]$.

So, $f(x)=x^3-7x^2+16x-12$ is differentiable on $(2,3)$.

Condition (3):

Here, $f(2)=2^3-7(2)^2+16(2)-12=0$

And $f(3)=3^3-7(3)^2+16(3)-12=0$

i.e. $f(2)=f(3)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (2,3)$ such that $f'(c)=0$

i.e. $3c^2-14c+16=0$

i.e. $(c-2)(3c-7)=0$

i.e. $c=2$ or $c=7/3$



Value of $c = \frac{7}{3} \in (2,3)$

Thus, Rolle's theorem is satisfied.

8. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x^3 + 3x^2 - 24x - 80 \text{ in } [-4, 5]$$

Answer

Condition (1):

Since, $f(x) = x^3 + 3x^2 - 24x - 80$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = x^3 + 3x^2 - 24x - 80$ is continuous on $[-4, 5]$.

Condition (2):

Here, $f'(x) = 3x^2 + 6x - 24$ which exist in $[-4, 5]$.

So, $f(x) = x^3 + 3x^2 - 24x - 80$ is differentiable on $(-4, 5)$.

Condition (3):

$$\text{Here, } f(-4) = (-4)^3 + 3(-4)^2 - 24(-4) - 80 = 0$$

$$\text{And } f(5) = (5)^3 + 3(5)^2 - 24(5) - 80 = 0$$

$$\text{i.e. } f(-4) = f(5)$$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (-4, 5)$ such that $f'(c) = 0$

$$\text{i.e. } 3c^2 + 6c - 24 = 0$$

$$\text{i.e. } c = -4 \text{ or } c = 2$$

Value of $c = 2 \in (-4, 5)$

Thus, Rolle's theorem is satisfied.

9. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = (x-1)(x-2)(x-3) \text{ in } [1, 3]$$

Answer

Condition (1):

Since, $f(x) = (x-1)(x-2)(x-3)$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = (x-1)(x-2)(x-3)$ is continuous on $[1, 3]$.

Condition (2):

Here, $f'(x) = (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$ which exist in $[1, 3]$.

So, $f(x) = (x-1)(x-2)(x-3)$ is differentiable on $(1, 3)$.

Condition (3):

$$\text{Here, } f(1) = (1-1)(1-2)(1-3) = 0$$

$$\text{And } f(3) = (3-1)(3-2)(3-3) = 0$$

$$\text{i.e. } f(1) = f(3)$$



Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (1,3)$ such that $f'(c)=0$

$$\text{i.e. } (c-2)(c-3) + (c-1)(c-3) + (c-1)(c-2) = 0$$

$$\text{i.e. } (c-3)(2c-3) + (c-1)(c-2) = 0$$

$$\text{i.e. } (2c^2 - 9c + 9) + (c^2 - 3c + 2) = 0$$

$$\text{i.e. } 3c^2 - 12c + 11 = 0$$

$$\text{i.e. } c = \frac{12 \pm \sqrt{12}}{6}$$

$$\text{i.e. } c = 2.58 \text{ or } c = 1.42$$

Value of $c = 1.42 \in (1,3)$ and $c = 2.58 \in (1,3)$

Thus, Rolle's theorem is satisfied.

10. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = (x-1)(x-2)^2 \text{ in } [1,2]$$

Answer

Condition (1):

Since, $f(x) = (x-1)(x-2)^2$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = (x-1)(x-2)^2$ is continuous on $[1,2]$.

Condition (2):

Here, $f'(x) = (x-2)^2 + 2(x-1)(x-2)$ which exist in $[1,2]$.

So, $f(x) = (x-1)(x-2)^2$ is differentiable on $(1,2)$.

Condition (3):

$$\text{Here, } f(1) = (1-1)(1-2)^2 = 0$$

$$\text{And } f(2) = (2-1)(2-2)^2 = 0$$

$$\text{i.e. } f(1) = f(2)$$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (1,2)$ such that $f'(c)=0$

$$\text{i.e. } (c-2)^2 + 2(c-1)(c-2) = 0$$

$$(3c-4)(c-2) = 0$$

$$\text{i.e. } c = 2 \text{ or } c = \frac{4}{3}$$

Value of $c = \frac{4}{3} = 1.33 \in (1,2)$

Thus, Rolle's theorem is satisfied.

11. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = (x-2)^4(x-3)^3 \text{ in } [2,3]$$

Answer

Condition (1):

Since, $f(x) = (x-2)^4(x-3)^3$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = (x-2)^4(x-3)^3$ is continuous on $[2,3]$.

Condition (2):

Here, $f'(x) = 4(x-2)^3(x-3)^3 + 3(x-2)^4(x-3)^2$ which exist in $[2,3]$.

So, $f(x) = (x-2)^4(x-3)^3$ is differentiable on $(2,3)$.

Condition (3):

Here, $f(2) = (2-2)^4(2-3)^3 = 0$

And $f(3) = (3-2)^4(3-3)^3 = 0$

i.e. $f(2) = f(3)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (2,3)$ such that $f'(c) = 0$

i.e. $4(c-2)^3(c-3)^3 + 3(c-2)^4(c-3)^2 = 0$

$(c-2)^3(c-3)^2(7c-18) = 0$

i.e. $c = 2$ or $c = 3$ or $c = 18 \div 7$

Value of $c = \frac{18}{7} = 2.57 \in (2,3)$

Thus, Rolle's theorem is satisfied.

12. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = \sqrt{1-x^2} \text{ in } [-1,1]$$

Answer

Condition (1):

Since, $f(x) = \sqrt{1-x^2}$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = \sqrt{1-x^2}$ is continuous on $[-1,1]$.

Condition (2):

Here, $f'(x) = -\frac{x}{\sqrt{1-x^2}}$ which exist in $[-1,1]$.

So, $f(x) = \sqrt{1-x^2}$ is differentiable on $(-1,1)$.

Condition (3):

Here, $f(-1) = \sqrt{1-(-1)^2} = 0$

And $f(1) = \sqrt{1-1^2} = 0$

i.e. $f(-1) = f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (-1,1)$ such that $f'(c) = 0$

i.e. $-\frac{c}{\sqrt{1-c^2}} = 0$



i.e. $c=0$

Value of $c=0 \in (-1,1)$

Thus, Rolle's theorem is satisfied.

3. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = \cos x \text{ in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Answer

Condition (1):

Since, $f(x)=\cos x$ is a trigonometric function and we know every trigonometric function is continuous.

$\Rightarrow f(x)=\cos x$ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Condition (2):

Here, $f'(x)=-\sin x$ which exist in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

So, $f(x)=\cos x$ is differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Condition (3):

Here, $f\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

And $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

i.e. $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$



Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $f'(c)=0$

i.e. $-\sin c=0$

i.e. $c=0$

Value of $c = 0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, Rolle's theorem is satisfied.

14. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = \cos 2x \text{ in } [0, \pi]$$

Answer

Condition (1):

Since, $f(x)=\cos 2x$ is a trigonometric function and we know every trigonometric function is continuous.

$\Rightarrow f(x)=\cos 2x$ is continuous on $[0, \pi]$.

Condition (2):

Here, $f'(x) = -2\sin 2x$ which exist in $[0, \pi]$.

So, $f(x)=\cos 2x$ is differentiable on $(0, \pi)$.

Condition (3):

$$\text{Here, } f(0) = \cos 0 = 1$$

$$\text{And } f(\pi) = \cos 2\pi = 1$$

$$\text{i.e. } f(0) = f(\pi)$$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (0, \pi)$ such that $f'(c) = 0$

$$\text{i.e. } -2\sin 2c = 0$$

$$\text{i.e. } 2c = \pi$$

$$\text{i.e. } c = \frac{\pi}{2}$$

$$\text{Value of } c = \frac{\pi}{2} \in (0, \pi)$$

Thus, Rolle's theorem is satisfied.

15. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = \sin 3x \text{ in } [0, \pi]$$

Answer

Condition (1):

Since, $f(x) = \sin 3x$ is a trigonometric function and we know every trigonometric function is continuous.

$\Rightarrow f(x) = \sin 3x$ is continuous on $[0, \pi]$.

Condition (2):

Here, $f'(x) = 3\cos 3x$ which exist in $[0, \pi]$.

So, $f(x) = \sin 3x$ is differentiable on $(0, \pi)$.

Condition (3):

$$\text{Here, } f(0) = \sin 0 = 0$$

$$\text{And } f(\pi) = \sin 3\pi = 0$$

$$\text{i.e. } f(0) = f(\pi)$$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (0, \pi)$ such that $f'(c) = 0$

$$\text{i.e. } 3\cos 3c = 0$$

$$\text{i.e. } 3c = \frac{\pi}{2}$$

$$\text{i.e. } c = \frac{\pi}{6}$$

$$\text{Value of } c = \frac{\pi}{6} \in (0, \pi)$$

Thus, Rolle's theorem is satisfied.

16. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = \sin x + \cos x \text{ in } \left[0, \frac{\pi}{2}\right]$$

Answer

Condition (1):

Since, $f(x) = \sin x + \cos x$ is a trigonometric function and we know every trigonometric function is continuous.

$\Rightarrow f(x) = \sin x + \cos x$ is continuous on $\left[0, \frac{\pi}{2}\right]$.

Condition (2):

Here, $f'(x) = \cos x - \sin x$ which exist in $\left[0, \frac{\pi}{2}\right]$.

So, $f(x) = \sin x + \cos x$ is differentiable on $\left(0, \frac{\pi}{2}\right)$

Condition (3):

Here, $f(0) = \sin 0 + \cos 0 = 1$

And $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) = 1$

i.e. $f(0) = f\left(\frac{\pi}{2}\right)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$

i.e. $\cos c - \sin c = 0$

i.e. $c = \frac{\pi}{4}$

Value of $c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Thus, Rolle's theorem is satisfied.



17. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = e^{-x} \sin x \text{ in } [0, \pi]$$

Answer

Condition (1):

Since, $f(x) = e^{-x} \sin x$ is a combination of exponential and trigonometric function which is continuous.

$\Rightarrow f(x) = e^{-x} \sin x$ is continuous on $[0, \pi]$.

Condition (2):

Here, $f'(x) = e^{-x} (\cos x - \sin x)$ which exist in $[0, \pi]$.

So, $f(x) = e^{-x} \sin x$ is differentiable on $(0, \pi)$

Condition (3):

Here, $f(0) = e^{-0} \sin 0 = 0$

And $f(\pi) = e^{-\pi} \sin \pi = 0$

i.e. $f(0) = f(\pi)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (0, \pi)$ such that $f'(c) = 0$

$$\text{i.e. } e^{-c} (\cos c - \sin c) = 0$$

$$\text{i.e. } \cos c - \sin c = 0$$

$$\text{i.e. } c = \frac{\pi}{4}$$

$$\text{Value of } c = \frac{\pi}{4} \in (0, \pi)$$

Thus, Rolle's theorem is satisfied.

18. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = e^{-x} (\sin x - \cos x) \text{ in } \left[\frac{\pi}{4}, \frac{5\pi}{4} \right]$$

Answer

Condition (1):

Since, $f(x) = e^{-x} (\sin x - \cos x)$ is a combination of exponential and trigonometric function which is continuous.

$$\Rightarrow f(x) = e^{-x} (\sin x - \cos x) \text{ is continuous on } \left[\frac{\pi}{4}, \frac{5\pi}{4} \right].$$

Condition (2):

$$\text{Here, } f'(x) = e^{-x} (\sin x + \cos x) - e^{-x} (\sin x - \cos x)$$

$$= e^{-x} \cos x \text{ which exist in } \left[\frac{\pi}{4}, \frac{5\pi}{4} \right].$$

$$\text{So, } f(x) = e^{-x} (\sin x - \cos x) \text{ is differentiable on } \left(\frac{\pi}{4}, \frac{5\pi}{4} \right)$$

Condition (3):

$$\text{Here, } f\left(\frac{\pi}{4}\right) = e^{-\frac{\pi}{4}} \left(\sin \frac{\pi}{4} - \cos \frac{\pi}{4}\right) = 0$$

$$\text{And } f\left(\frac{5\pi}{4}\right) = e^{-\frac{5\pi}{4}} \left(\sin \frac{5\pi}{4} - \cos \frac{5\pi}{4}\right) = 0$$

$$\text{i.e. } f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$$

Conditions of Rolle's theorem are satisfied.

$$\text{Hence, there exist at least one } c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \text{ such that } f'(c) = 0$$

$$\text{i.e. } e^{-c} \cos c = 0$$

$$\text{i.e. } \cos c = 0$$

$$\text{i.e. } c = \frac{\pi}{2}$$

$$\text{Value of } c = \frac{\pi}{2} \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

Thus, Rolle's theorem is satisfied.

19. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = \sin x - \sin 2x \text{ in } [0, 2\pi]$$

Answer

Condition (1):

Since, $f(x) = \sin x - \sin 2x$ is a trigonometric function and we know every trigonometric function is continuous.

$\Rightarrow f(x) = \sin x - \sin 2x$ is continuous on $[0, 2\pi]$.

Condition (2):

Here, $f'(x) = \cos x - 2\cos 2x$ which exist in $[0, 2\pi]$.

So, $f(x) = \sin x - \sin 2x$ is differentiable on $(0, 2\pi)$

Condition (3):

Here, $f(0) = \sin 0 - \sin 0 = 0$

And $f(2\pi) = \sin(2\pi) - \sin(4\pi) = 0$

i.e. $f(0) = f(2\pi)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (0, 2\pi)$ such that $f'(c) = 0$

i.e. $\cos x - 2\cos 2x = 0$

i.e. $\cos x - 4\cos^2 x + 2 = 0$

i.e. $4\cos^2 x - \cos x - 2 = 0$

i.e. $\cos x = \frac{1 \pm \sqrt{33}}{8}$

i.e. $c = 32^\circ 32'$ or $c = 126^\circ 23'$

Value of $c = 32^\circ 32' \in (0, 2\pi)$

Thus, Rolle's theorem is satisfied.



20. Question

Verify Rolle's theorem for each of the following functions:

$$f(x) = x(x+2)e^x \text{ in } [-2, 0]$$

Answer

Condition (1):

Since, $f(x) = x(x+2)e^x$ is a combination of exponential and polynomial function which is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = x(x+2)e^x$ is continuous on $[-2, 0]$.

Condition (2):

Here, $f'(x) = (x^2 + 4x + 2)e^x$ which exist in $[-2, 0]$.

So, $f(x) = x(x+2)e^x$ is differentiable on $(-2, 0)$.

Condition (3):

Here, $f(-2) = (-2)(-2+2)e^{-2} = 0$

And $f(0) = 0(0+2)e^0 = 0$

i.e. $f(-2) = f(0)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (-2, 0)$ such that $f'(c) = 0$

i.e. $(c^2 + 4c + 2)e^c = 0$

$$\text{i.e. } (c+\sqrt{2})^2=0$$

$$\text{i.e. } c=-\sqrt{2}$$

Value of $c=-\sqrt{2} \in (-2,0)$

Thus, Rolle's theorem is satisfied.

21. Question

Verify Rolle's theorem for each of the following functions:

Show that $f(x) = x(x-5)^2$ satisfies Rolle's theorem on $[0, 5]$ and that the value of c is $(5/3)$

Answer

Condition (1):

Since, $f(x)=x(x-5)^2$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x)=x(x-5)^2$ is continuous on $[0,5]$.

Condition (2):

Here, $f'(x) = (x-5)^2 + 2x(x-5)$ which exist in $[0,5]$.

So, $f(x)=x(x-5)^2$ is differentiable on $(0,5)$.

Condition (3):

$$\text{Here, } f(0) = 0(0-5)^2 = 0$$

$$\text{And } f(5) = 5(5-5)^2 = 0$$

$$\text{i.e. } f(0) = f(5)$$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (0,5)$ such that $f'(c) = 0$

$$\text{i.e. } (c-5)^2 + 2c(c-5) = 0$$

$$\text{i.e. } (c-5)(3c-5) = 0$$

$$\text{i.e. } c = \frac{5}{3} \text{ or } c = 5$$

$$\text{Value of } c = \frac{5}{3} \in (0,5)$$

Thus, Rolle's theorem is satisfied.

22. Question

Discuss the applicability for Rolle's theorem, when:

$$f(x) = (x-1)(2x-3), \text{ where } 1 \leq x \leq 3$$

Answer

Condition (1):

Since, $f(x)=(x-1)(2x-3)$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = (x-1)(2x-3)$ is continuous on $[1,3]$.

Condition (2):

Here, $f'(x) = (2x-3) + 2(x-1)$ which exist in $[1,3]$.

So, $f(x) = (x-1)(2x-3)$ is differentiable on $(1,3)$.



Condition (3):

$$\text{Here, } f(1) = (1-1)(2(1)-3) = 0$$

$$\text{And } f(5) = (3-1)(2(3)-3) = 6$$

i.e. $f(1) \neq f(3)$

Condition (3) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.

23. Question

Discuss the applicability for Rolle's theorem, when:

$$f(x) = x^{1/2} \text{ on } [-1, 1]$$

Answer

Condition (1):

Since, $f(x) = x^{1/2}$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = x^{1/2}$ is continuous on $[-1, 1]$.

Condition (2):

Here, $f'(x) = \frac{1}{2x^{1/2}}$ which does not exist at $x=0$ in $[-1, 1]$.

$f(x) = x^{1/2}$ is not differentiable on $(-1, 1)$.

Condition (2) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.



24. Question

Discuss the applicability for Rolle's theorem, when:

$$f(x) = 2 + (x-1)^{2/3} \text{ on } [0, 2]$$

Answer

Condition (1):

Since, $f(x) = 2 + (x-1)^{2/3}$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow f(x) = 2 + (x-1)^{2/3}$ is continuous on $[0, 2]$.

Condition (2):

Here, $f'(x) = \frac{2}{3(x-1)^{1/3}}$ which does not exist at $x=1$ in $[0, 2]$.

$f(x) = 2 + (x-1)^{2/3}$ is not differentiable on $(0, 2)$.

Condition (2) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.

25. Question

Discuss the applicability for Rolle's theorem, when:

$$f(x) = \cos \frac{1}{x} \text{ on } [-1, 1]$$

Answer

Condition (1):

Since, $f(x) = \cos \frac{1}{x}$ which is discontinuous at $x=0$

$\Rightarrow f(x) = \cos \frac{1}{x}$ is not continuous on $[-1,1]$.

Condition (1) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.

26. Question

Discuss the applicability for Rolle's theorem, when:

$f(x) = [x]$ on $[-1,1]$, where $[x]$ denotes the greatest integer not exceeding x

Answer

Condition (1):

Since, $f(x)=[x]$ which is discontinuous at $x=0$

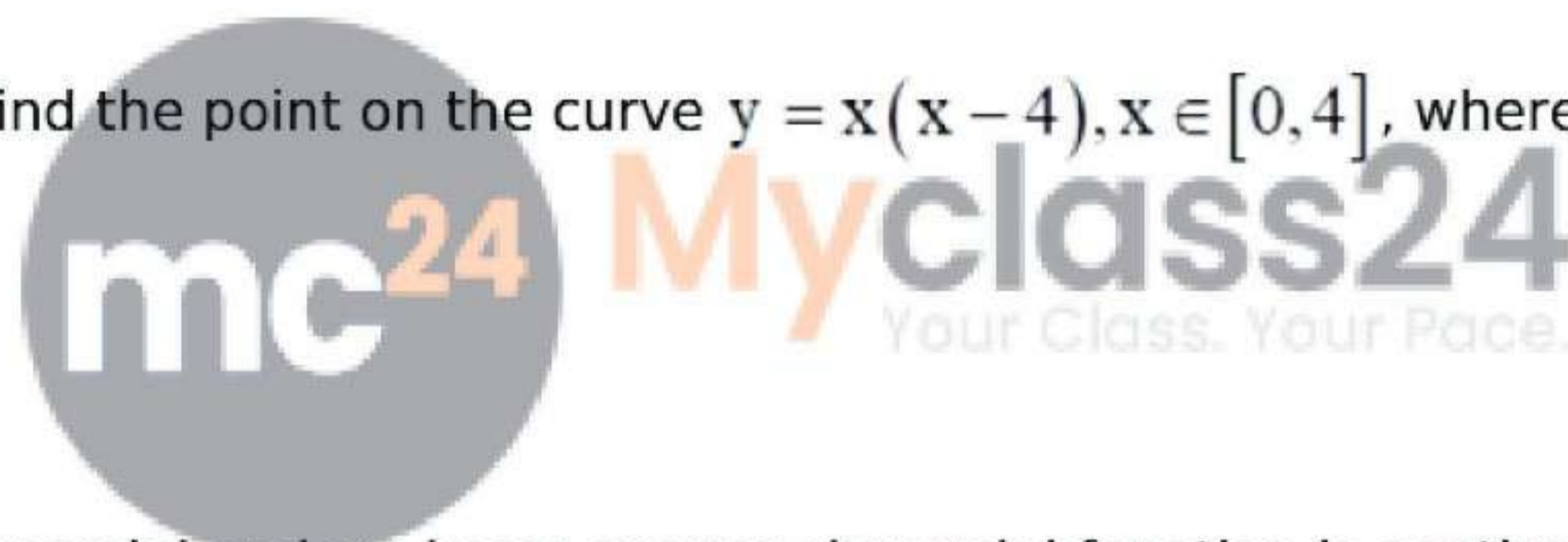
$\Rightarrow f(x)=[x]$ is not continuous on $[-1,1]$.

Condition (1) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.

27. Question

Using Rolle's theorem, find the point on the curve $y = x(x - 4)$, $x \in [0,4]$, where the tangent is parallel to the x-axis.

**Answer**

Condition (1):

Since, $y=x(x-4)$ is a polynomial and we know every polynomial function is continuous for all $x \in \mathbb{R}$.

$\Rightarrow y = x(x-4)$ is continuous on $[0,4]$.

Condition (2):

Here, $y' = (x-4)+x$ which exist in $[0,4]$.

So, $y = x(x-4)$ is differentiable on $(0,4)$.

Condition (3):

Here, $y(0)=0(0-4)=0$

And $y(4)= 4(4-4)=0$

i.e. $y(0)=y(4)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (0,4)$ such that $y'(c)=0$

i.e. $(c-4)+c=0$

i.e. $2c-4=0$

i.e. $c=2$

Value of $c=2 \in (0,4)$

So, $y(c)=y(2)=2(2-4)=-4$

By geometric interpretation, (2,-4) is a point on a curve $y=x(x-4)$, where tangent is parallel to x-axis.

Exercise 11D

1. Question

Verify Lagrange's mean-value theorem for the following function:

$$f(x) = x^2 + 2x + 3 \text{ on } [4, 6]$$

Answer

Given:

Since the $f(x)$ is a polynomial function,

It is continuous as well as differentiable in the interval $[4, 6]$.

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{(36 + 12 + 3) - (16 + 8 + 3)}{6 - 4} \end{aligned}$$

$$= \frac{24}{2}$$

$$= 12$$

$$\Rightarrow f'(c) = 2c + 2$$

$$\Rightarrow 2c + 2 = 12$$

$$\Rightarrow c = 5$$

2. Question

Verify Lagrange's mean-value theorem for the following function:

$$f(x) = x^2 + x - 1 \text{ on } [0, 4]$$

Answer

Given:

Since the $f(x)$ is a polynomial function,

It is continuous as well as differentiable in the interval $[0, 4]$.

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{(16 + 4 - 1) - (0 + 0 - 1)}{4 - 0} \end{aligned}$$

$$= 5$$

$$\Rightarrow f'(c) = 2c + 1$$

$$\Rightarrow 2c + 1 = 5$$

$$\Rightarrow c = 2$$

3. Question

Verify Lagrange's mean-value theorem for the following function:

$$f(x) = 2x^2 - 3x + 1 \text{ on } [1, 3]$$

Answer



Given:

Since the $f(x)$ is a polynomial function,

It is continuous as well as differentiable in the interval $[1,3]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
$$= \frac{(18 - 9 + 1) - (2 - 3 + 1)}{3 - 1}$$

$$= 5$$

$$\Rightarrow f'(c) = 4c - 3$$

$$\Rightarrow 4c - 3 = 5$$

$$\Rightarrow c = 2$$

4. Question

Verify Lagrange's mean-value theorem for the following function:

$$f(x) = x^3 + x^2 - 6x \text{ on } [-1, 4]$$

Answer

Given:

Since the $f(x)$ is a polynomial function,

It is continuous as well as differentiable in the interval $[-1,4]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
$$= \frac{(64 + 16 - 24) - (-1 + 1 + 6)}{4 + 1}$$

$$= \frac{50}{5}$$

$$= 10$$

$$f'(c) = 3c^2 + 2c - 6$$

$$\Rightarrow 3c^2 + 2c - 6 = 10$$

$$\Rightarrow 3c^2 + 2c - 16 = 0$$

$$\Rightarrow 3c^2 - 6c + 8c - 16 = 0$$

$$\Rightarrow 3c(c - 2) + 8(c - 2) = 0$$

$$\Rightarrow (3c + 8)(c - 2) = 0$$

$$c = 2, \frac{-8}{3}$$

5. Question

Verify Lagrange's mean-value theorem for the following function:

$$f(x) = (x - 4)(x - 6)(x - 8) \text{ on } [4, 6]$$

Answer

Given:

$$f(x) = x^3 - 18x^2 + 104x - 192$$

Since the $f(x)$ is a polynomial function,

It is continuous as well as differentiable in the interval $[4,6]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow f'(c) = \frac{(216 - 648 + 624 - 192) - (64 - 288 + 416 - 192)}{6 - 2}$$

$$= 0$$

$$\Rightarrow f'(c) = 3c^2 - 36c + 104$$

$$= 3c^2 - 36c + 10$$

$$= 0$$

$$\Rightarrow c = \frac{36 \pm \sqrt{1296 - 1248}}{6}$$

$$\Rightarrow c = \frac{36 \pm \sqrt{48}}{6}$$

$$\Rightarrow c = 6 \pm \frac{2}{3}\sqrt{3}$$

6. Question

Verify Lagrange's mean-value theorem for the following function:

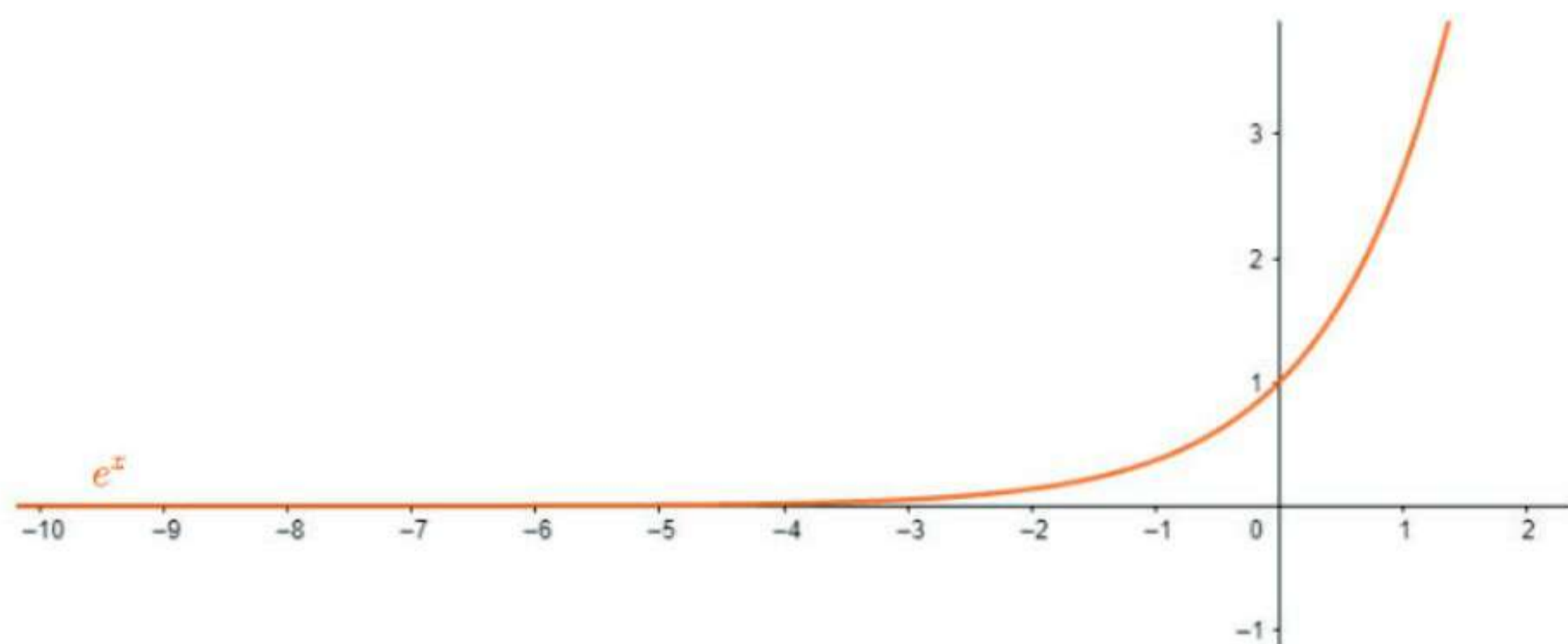
$$f(x) = e^x \text{ on } [0,1]$$



Answer

Given:

Since $f(x)$ is continuous as well as differentiable in the interval $[0,1]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{e - 1}{1}$$

$$\Rightarrow f'(c) = e^c$$

$$\Rightarrow e^c = e - 1$$

$$\Rightarrow \log_e e^c = \log_e(e - 1)$$

$$\Rightarrow c = \log_e(e - 1)$$

7. Question

Verify Lagrange's mean-value theorem for the following function:

$$f(x) = x^{\frac{2}{3}} \text{ on } [0,1]$$

Answer

Given:

Since the $f(x)$ is a polynomial function,

It is continuous as well as differentiable in the interval $[0,1]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{1 - 0}{1 - 0}$$

$$= 1$$

$$f'(c) = \frac{2}{3}c^{\frac{1}{3}}$$

$$\Rightarrow \frac{2}{3}c^{\frac{1}{3}} = 1$$

$$\Rightarrow c^{\frac{-1}{3}} = \frac{3}{2}$$

$$\Rightarrow c^{\frac{1}{3}} = \frac{2}{3}$$

$$\Rightarrow c = \frac{8}{27}$$



8. Question

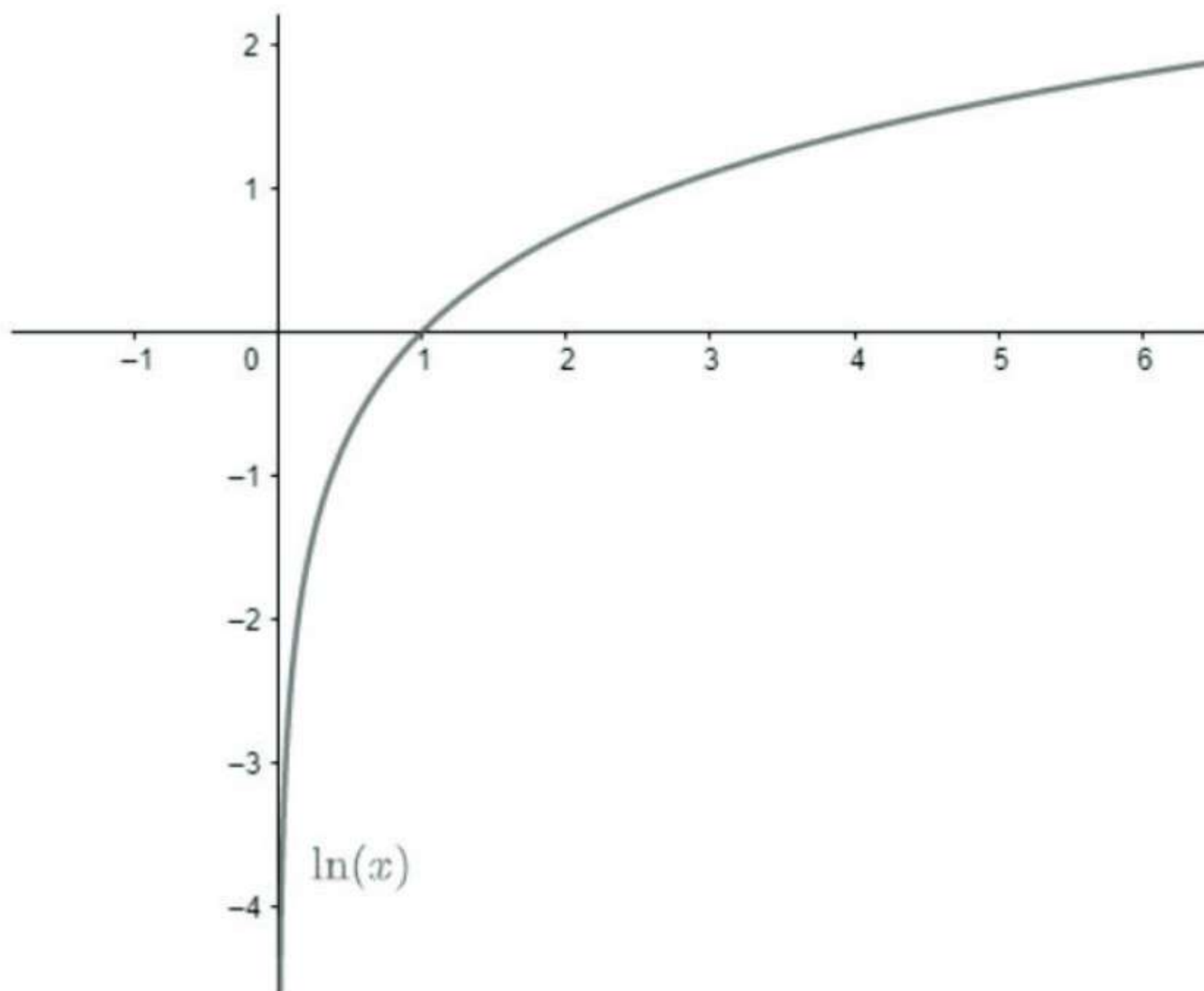
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = \log x \text{ on } [1, e]$$

Answer

Given:

Since $\log x$ is a continuous as well as differentiable function in the interval $[1, e]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\log e - \log 1}{e - 1}$$

$$= \frac{1}{e - 1}$$

$$f'(c) = \frac{1}{c}$$

$$\Rightarrow \frac{1}{e - 1} = \frac{1}{c}$$

$$c = e - 1$$



9. Question

Verify Lagrange's mean-value theorem for the following function:

$$f(x) = \tan^{-1} x \text{ on } [0, 1]$$

Answer

Given:

Since $\tan^{-1} x$ is a continuous as well as differentiable function in the interval $[0, 1]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\tan^{-1} 1 - \tan^{-1} 0}{1 - 0}$$

$$= \frac{\pi}{4}$$

$$f'(c) = \frac{1}{1 + c^2}$$

$$\Rightarrow \frac{1}{1 + c^2} = \frac{\pi}{4}$$

$$\Rightarrow 1 + c^2 = \frac{4}{\pi}$$

$$\Rightarrow c = \sqrt{\frac{4}{\pi} - 1}$$

10. Question

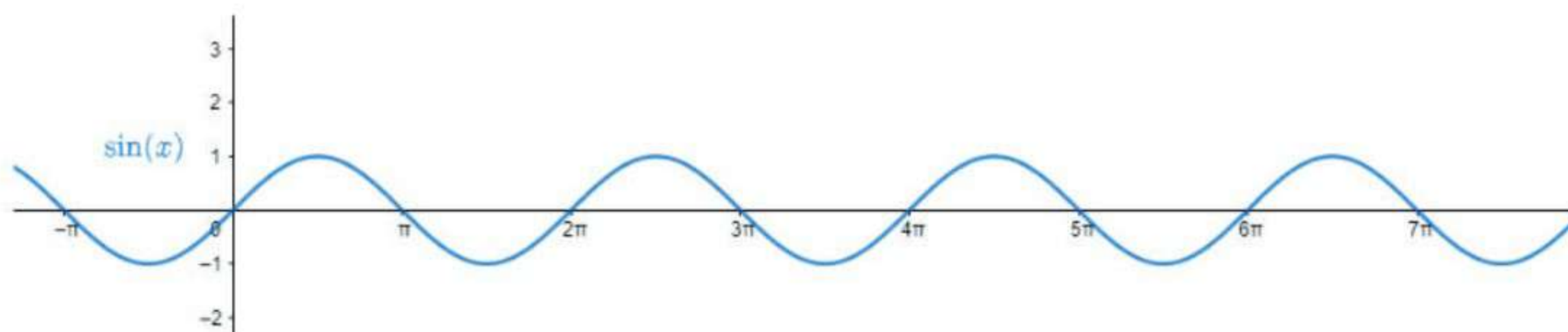
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = \sin x \text{ on } \left[\frac{\pi}{2}, \frac{5\pi}{2} \right]$$

Answer

Given:

Since $\sin x$ is a continuous as well as differentiable function in the interval $\left[\frac{\pi}{2}, \frac{5\pi}{2} \right]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sin \frac{5\pi}{2} - \sin \frac{\pi}{2}}{\frac{5\pi}{2} - \frac{\pi}{2}}$$

$$= 0$$

$$f'(c) = \cos x$$

$$\cos x = 0$$

$$x = \frac{n\pi}{2}, n \in \{1, 2, 3, 4, 5\}$$



11. Question

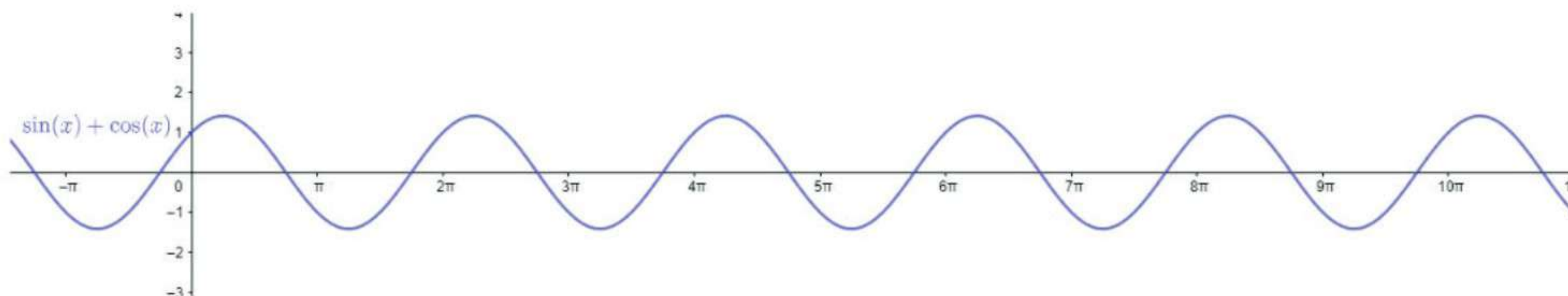
Verify Lagrange's mean-value theorem for the following function:

$$f(x) = (\sin x + \cos x) \text{ on } \left[0, \frac{\pi}{2} \right]$$

Answer

Given:

Since $(\sin x + \cos x)$ is a continuous as well as differentiable function in the interval $\left[0, \frac{\pi}{2} \right]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sin \frac{\pi}{2} + \cos \frac{\pi}{2} - \sin 0 - \cos 0}{\frac{\pi}{2} - 0}$$

$$= 0$$

$$f'(c) = \cos x - \sin x$$

$$\Rightarrow \cos x - \sin x = 0$$

$$\Rightarrow \cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} = 0$$

$$\Rightarrow \cos \left(x + \frac{\pi}{4} \right) = 0$$

$$\Rightarrow \left(x + \frac{\pi}{4} \right) = \cos^{-1} 0$$

$$\Rightarrow \left(x + \frac{\pi}{4} \right) = \frac{\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{4}$$

12. Question

Show that Lagrange's mean-value theorem is not applicable to $f(x) = |x|$ on $[-1, 1]$.

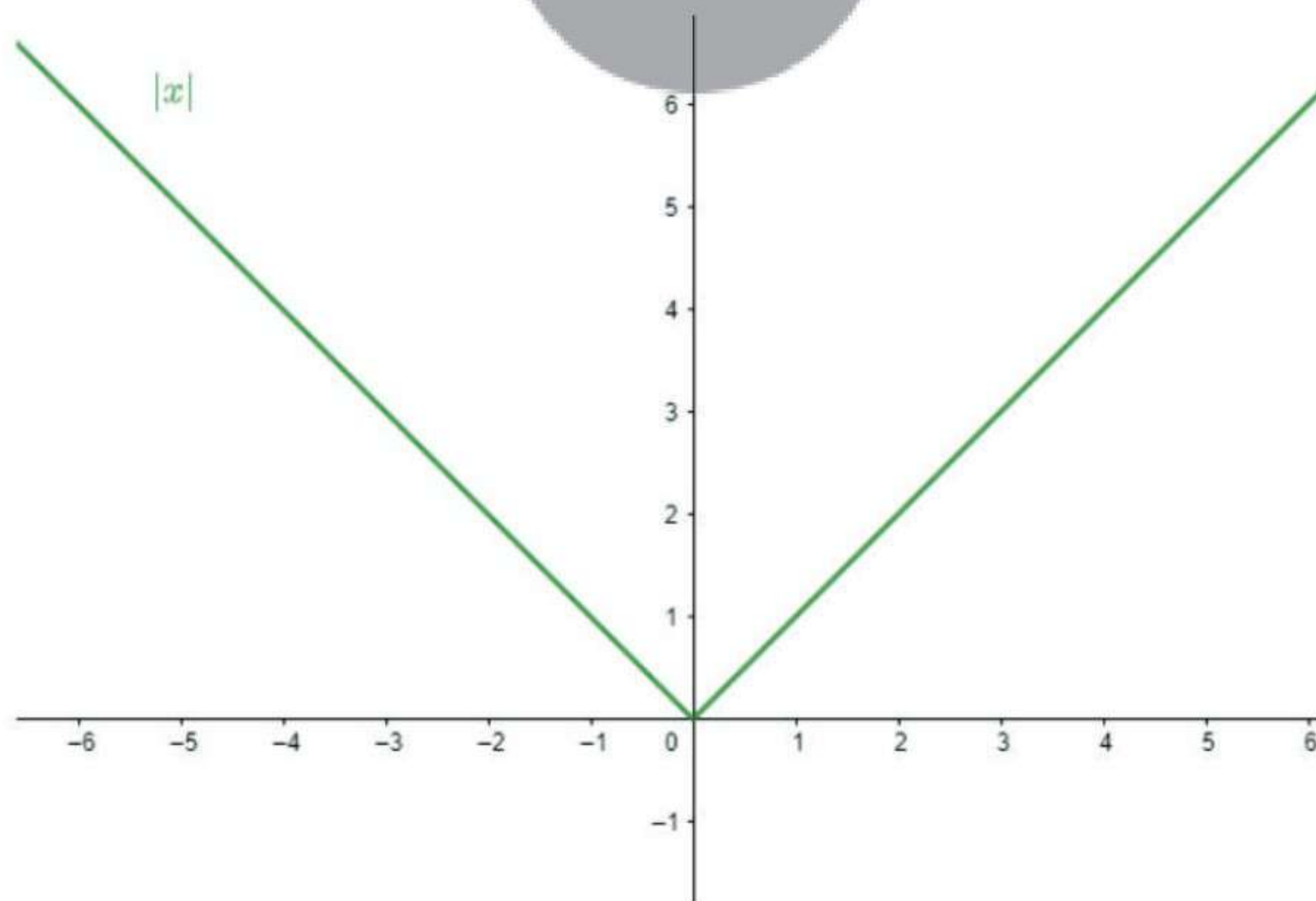
Answer

Given:

Since $f(x)$ is continuous in the interval $[-1, 1]$.

But is non differentiable at $x=0$ due to sharp corner.

So LMVT is not applicable to $f(x) = |x|$



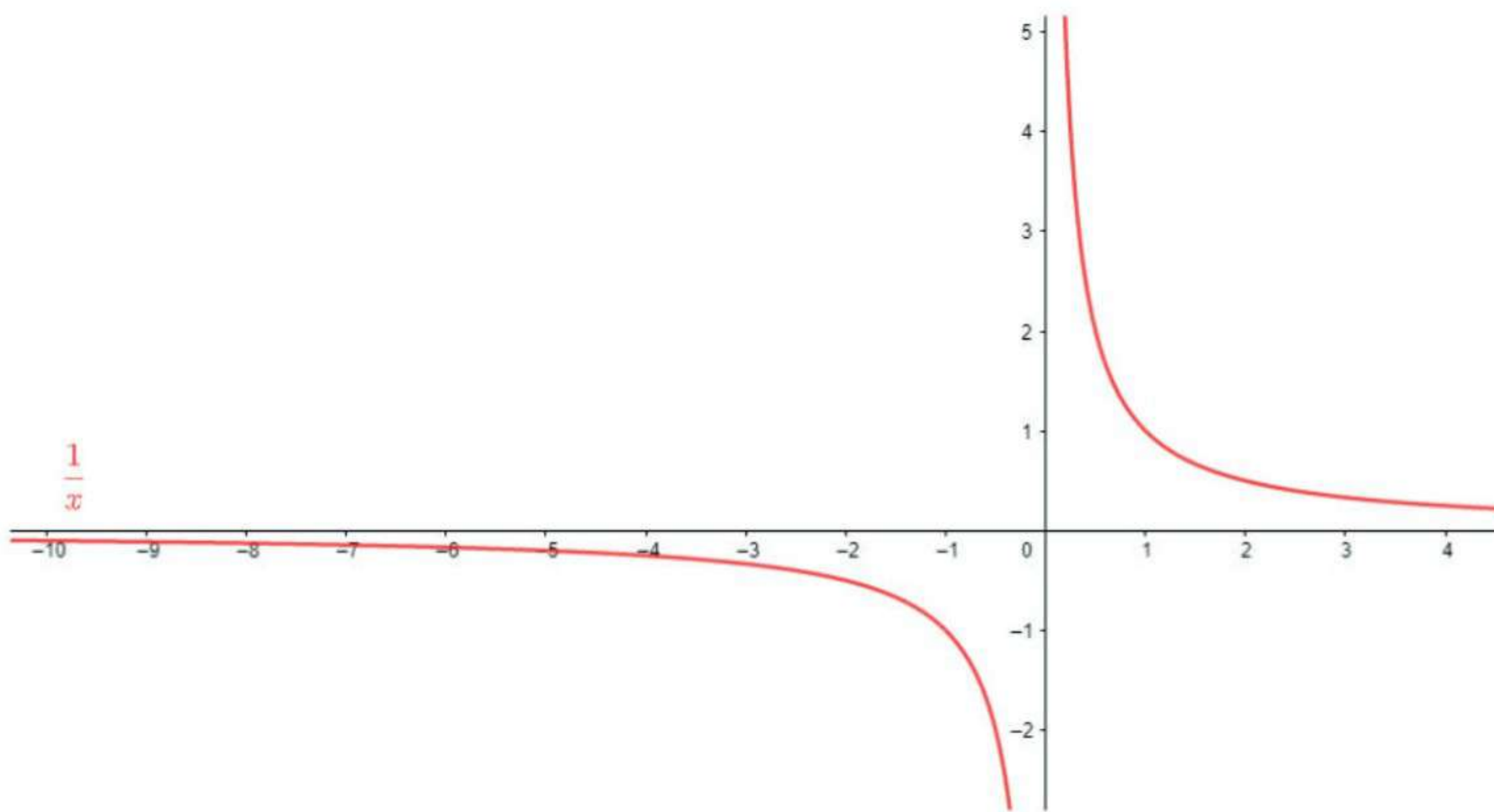
13. Question

Show that Lagrange's mean-value theorem is not applicable to $f(x) = \frac{1}{x}$ on $[-1, 1]$

Answer

Given:

Since the graph is discontinuous at $x=0$ as shown in the graph.



So LMVT is not applicable to the above function.

14 A. Question

Find 'c' of Lagrange's mean-value theorem for

$$f(x) = (x^3 - 3x^2 + 2x) \text{ on } \left[0, \frac{1}{2}\right]$$

Answer

Given:

Since the $f(x)$ is a polynomial function,

It is continuous as well as differentiable in the interval $\left[0, \frac{1}{2}\right]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\frac{1}{8} - \frac{3}{4} + 1 - 0}{\frac{1}{2} - 0}$$

$$= \frac{3}{4}$$

$$f'(c) = 3x^2 - 6x + 2$$

$$3x^2 - 6x + 2 = \frac{3}{4}$$

$$12x^2 - 24x + 8 = 3$$

$$12x^2 - 24x + 5 = 0$$

$$x = \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$x = 1 \pm \sqrt{\frac{336}{576}}$$

$$x = 1 \pm \sqrt{\frac{7}{12}}$$

14 B. Question