EXERCISE 20.4

Question. 1
Solution:

From the question it is given that, $\int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} dx$

We know that, $\int_0^{2\pi} f(x)dx = \int_0^{2\pi} f(2\pi - x)dx$

Then,

$$\int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} dx = \int_0^{2\pi} \frac{e^{\sin(2\pi - x)}}{e^{\sin(2\pi - x)} + e^{-\sin(2\pi - x)}} dx$$

We also know that, sin (2π - x) = - sin x

So,

$$\int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} dx = \int_0^{2\pi} \frac{e^{-\sin x}}{e^{-\sin x} + e^{\sin x}} dx$$

Let us assume, I = $\int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} dx$ ... [equation (i)]

and also assume, I = $\int_0^{2\pi} \frac{e^{-\sin x}}{e^{-\sin x} + e^{\sin x}} dx$ ... [equation (ii)]

By adding equation (i) and equation (ii),

$$2I = \int_0^{2\pi} \frac{e^{-\sin x}}{e^{-\sin x} + e^{\sin x}} dx + \int_0^{2\pi} \frac{e^{\sin x}}{e^{-\sin x} + e^{\sin x}} dx$$

Then,

$$2I = \int_0^{2\pi} \frac{e^{-\sin x}}{e^{-\sin x} + e^{\sin x}} + \frac{e^{\sin x}}{e^{-\sin x} + e^{\sin x}} dx$$

$$2I = \int_0^{2\pi} dx$$

2I = 2π

Therefore, I = π

Question. 2
Solution:

From the question it is given that, $\int_{0}^{2\pi} \log(\sec x + \tan x)dx$

We know that, $\int_{0}^{2\pi} f(x)dx = \int_{0}^{2\pi} f(2\Pi - x)dx$

Then,

$$\int_{0}^{2\pi} \log(\sec x + \tan x)dx = \int_{0}^{2\pi} \log(\sec(2\Pi - x) + \tan(2\Pi - x))dx$$

So,

$$\int_{0}^{2\pi} \log(\sec x + \tan x)dx = \int_{0}^{2\pi} \log(\sec x - \tan x)dx$$

Let us assume, I = $\int_{0}^{2\pi} \log(\sec x + \tan x)dx$ … [equation (i)]

and also assume I = $\int_{0}^{2\pi} \log(\sec x - \tan x)dx$ … [equation (ii)]

By adding equation (i) and equation (ii), we get,

$$2I = \int_{0}^{2\pi} \log(\sec x + \tan x)dx + \int_{0}^{2\pi} \log(\sec x - \tan x)dx$$

Then,

$$2I = \int_{0}^{2\pi} \log(\sec x + \tan x) + \log(\sec x - \tan x)dx$$

We know that log (a) + log (b) = log (ab)

$$2I = \int_{0}^{2\pi} \log(\sec^2 x - \tan^2 x)dx$$

Also we know that, $\sec^2 x - \tan^2 x = 1$

$$2I = \int_{0}^{2\pi} \log(1)dx$$

2I = 0

I = 0

Question. 3

Solution:

From the question it is given that, $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$

We know that, $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(a + b - x)dx$

Then,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan(\frac{\Pi}{2}-x)}}{\sqrt{\tan(\frac{\Pi}{2}-x)}+\sqrt{\cot(\frac{\Pi}{2}-x)}} dx$$

Also we know that, Trignometric property,
tan ((π/2) - x) = cot x
cot ((π/2) - x) = tan x

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx$$

Let us assume, I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx$ ... [equation (i)]

So, assume I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx$ ... [equation (ii)]

By adding equation (i) and equation (ii), we get,

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx$$

Then,

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} + \frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 dx$$

Now applying limits, we get,
2I = π/3 – π/6
2I = (2π - π)/6
2I = π/6
I = π/12

Question. 4
Solution:

From the question it is given that, $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$

We know that, $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(a+b-x)dx$

Then,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin(\frac{\Pi}{2}-x)}}{\sqrt{\sin(\frac{\Pi}{2}-x)}+\sqrt{\cos(\frac{\Pi}{2}-x)}} dx$$

Also we know that, Trignometric property,

sin ((π/2) - x) = cos x

cos ((π/2) - x) = sin x

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

Let us assume, I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ ... [equation (i)]

Then, assume I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ ... [equation (ii)]

By adding equation (i) and equation (ii), we get,

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1dx$$

Now applying limits, we get,

2I = π/3 – π/6

2I = (2π - π)/6

2I = π/6

I = π/12

Question. 6

Solution:

From the question it is given that, $\int_{-a}^{a} \frac{1}{1+a^x} dx$

We know that, $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Then,

$$\int_{-a}^{a} \frac{1}{1+a^x} dx = \int_{-a}^{a} \frac{1}{1+a^{-x}} dx$$

Let us assume, I = $\int_{-a}^{a} \frac{1}{1+a^x} dx$ ... [equation (i)]

Also assume, I = $\int_{-a}^{a} \frac{1}{1+a^{-x}} dx$ ... [equation (ii)]

By adding equation (i) and equation (ii), we get,

$$2I = \int_{-a}^{a} \frac{1}{1+a^x} + \frac{1}{1+a^{-x}} dx$$

$$2I = \int_{-a}^{a} \frac{1}{1+a^x} + \frac{a^x}{1+a^x} dx$$

$$2I = \int_{-a}^{a} 1 dx$$

Now applying limits, we get,

2I = a – (-a)

2I = a + a

2I = 2a

I = a