EXERCISE 5.5

1. $If\ A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $prove\ that\ A - A^T\ is\ a\ skew - symmetric\ matrix.$

Solution:

Given

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

Consider,

$$(A - A^T) = \left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^T\right)$$

$$= \left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2-2 & 3-4 \\ 4-3 & 5-5 \end{bmatrix}$$

$$(A - A^T) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \ldots (i)$$

$$-(A - A^T)^T = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^T$$

$$= -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$-(A - A^T) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \ldots (ii)$$

From (i) and (ii) we can see that

A skew-symmetric matrix is a square matrix whose transpose equal to its negative, that is,

$X = -X^T$

So, $A - A^T$ is a skew-symmetric.

2. $If\ A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, $show\ that\ A - A^T\ is\ a\ skew - symmetric\ matrix.$

Solution:

Given

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

Consider,

$(A - A^T) = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$ ... (i)

$-(A - A^T)^T = -\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}^T$

$= -\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$

$-(A - A^T) = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$ ... (ii)

From (i) and (ii) we can see that

A skew-symmetric matrix is a square matrix whose transpose equals its negative, that is, $X = -X^T$

So, $A - A^T$ is a skew-symmetric matrix.

$3. \, If \, the \, matrix \, A = \begin{bmatrix} 5 & 2 & x \\ y & z & -3 \\ 4 & t & -7 \end{bmatrix}, \, is \, a \, symmetric \, matrix \, matrix \, find \, x, y, z \, and \, t$

Solution:

Given,

$A = \begin{bmatrix} 5 & 2 & x \\ y & z & -3 \\ 4 & t & -7 \end{bmatrix}$ is a symmetric matrix.

We know that $A = [a_{ij}]_{m \times n}$ is a symmetric matrix if $a_{ij} = a_{ji}$

So,

$x = a_{13} = a_{31} = 4$

$y = a_{21} = a_{12} = 2$

$z = a_{22} = a_{22} = z$

$t = a_{32} = a_{23} = -3$

Hence, x = 4, y = 2, t = -3 and z can have any value.

4. Let $A = \begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & 3 \\ -2 & 5 & 8 \end{bmatrix}$**. Find matrices X and Y such that X + Y = A, where X is a symmetric and y is a skew-symmetric matrix.**

Solution:

Given, $A = \begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & 3 \\ -2 & 5 & 8 \end{bmatrix}$ Then $A^T = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 4 & 5 \\ 7 & 3 & 8 \end{bmatrix}$

$$X = \frac{1}{2}(A + A^T)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & 3 \\ -2 & 5 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -2 \\ 2 & 4 & 5 \\ 7 & 3 & 8 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 3+3 & 2+1 & 7-2 \\ 1+2 & 4+4 & 3+5 \\ -2+7 & 5+3 & 8+8 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 3 & 5 \\ 3 & 8 & 8 \\ 5 & 8 & 16 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & \frac{3}{2} & \frac{5}{2} \\ \frac{3}{2} & 4 & 4 \\ \frac{5}{2} & 4 & 8 \end{bmatrix}$$

Now,

$$Y = \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & 3 \\ -2 & 5 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 & -2 \\ 2 & 4 & 5 \\ 7 & 3 & 8 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 3-3 & 2-1 & 7+2 \\ 1-2 & 4-4 & 3-5 \\ -2-7 & 5-3 & 8-8 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 1 & 9 \\ -1 & 0 & -2 \\ -9 & 2 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & \frac{1}{2} & \frac{9}{2} \\ -\frac{1}{2} & 0 & -1 \\ -\frac{9}{2} & 1 & 0 \end{bmatrix}$$

Now,

$$X^T = \begin{bmatrix} 3 & \frac{3}{2} & \frac{5}{2} \\ \frac{3}{2} & 4 & 4 \\ \frac{5}{2} & 4 & 8 \end{bmatrix}^T = \begin{bmatrix} 3 & \frac{3}{2} & \frac{5}{2} \\ \frac{3}{2} & 4 & 4 \\ \frac{5}{2} & 4 & 8 \end{bmatrix} = X$$

$\Rightarrow$ X is a symmetric matrix.

Now,

$$-Y^T = -\begin{bmatrix} 0 & \frac{1}{2} & \frac{9}{2} \\ -\frac{1}{2} & 0 & -1 \\ -\frac{9}{2} & 1 & 0 \end{bmatrix}^T = -\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{9}{2} \\ \frac{1}{2} & 0 & 1 \\ \frac{9}{2} & -1 & 0 \end{bmatrix}$$

$$-Y^T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{9}{2} \\ -\frac{1}{2} & 0 & -1 \\ -\frac{9}{2} & 1 & 0 \end{bmatrix}$$

$-Y^T = Y$

Y is a skew symmetric matrix.

And,

$$X + Y = \begin{bmatrix} 3 & \frac{3}{2} & \frac{5}{2} \\ \frac{3}{2} & 4 & 4 \\ \frac{5}{2} & 4 & 8 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} & \frac{9}{2} \\ -\frac{1}{2} & 0 & -1 \\ -\frac{9}{2} & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0 & \frac{3}{2}+\frac{1}{2} & \frac{5}{2}+\frac{9}{2} \\ \frac{3}{2}-\frac{1}{2} & 4+0 & 4-1 \\ \frac{5}{2}-\frac{9}{2} & 4+1 & 8+0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & 3 \\ -2 & 5 & 8 \end{bmatrix} = A$$

Hence, X + Y = A