Exercise 4(D)

1. If x + 2y + 3z = 0 and $x^3 + 4y^3 + 9z^3 = 18xyz$; evaluate:

$$\frac{(x+2y)^2}{xy}+\frac{(2y+3z)^2}{yz}+\frac{(3z+x)^2}{zx}$$

Solution:

Given, $x^3 + 4y^3 + 9z^3 = 18xyz$ and x + 2y + 3z = 0

So,

x + 2y = - 3z, 2y + 3z = -x and 3z + x = -2y

Now,

$$\frac{(x+2y)^2}{xy}+\frac{(2y+3z)^2}{yz}+\frac{(3x+x)^2}{zx}=\frac{(-3z)^2}{xy}+\frac{(-x)^2}{yz}+\frac{(-2y)^2}{zx}$$

$$=\frac{9z^2}{xy}+\frac{x^2}{yz}+\frac{4y^2}{zx}$$

$$=\frac{x^3+4y^3+9z^3}{xyz}$$

Given that $x^3 + 4y^3 + 9z^3 = 18xyz$

$$\therefore \frac{(x+2y)^2}{xy}+\frac{(2y+3z)^2}{yz}+\frac{(3z+x)^2}{zx}=\frac{18xyz}{xyz}=18$$

2. If a + 1/a = m and a ≠ 0; find in terms of 'm'; the value of:

(i) a – 1/a

(ii) $a^2 – 1/a^2$

Solution:

(i) Given, a + 1/a = m

On squaring on both sides, we get

$(a + 1/a)^2 = m^2$

$a^2 + 1/a^2 + 2 = m^2$

$a^2 + 1/a^2 = m^2 – 2$ … (1)

Now, consider the expansion

$(a – 1/a)^2 = a^2 + 1/a^2 – 2$

$= m^2 – 2 – 2$ … [From (1)]

$= m^2 – 4$

So,

$(a – 1/a) = \pm\sqrt{(m^2 – 4)}$ … (2)

(ii) We know that,
$a^2 – 1/a^2 = (a – 1/a) (a + 1/a)$
$= m [\pm\sqrt{(m^2 – 4)}]$
$= \pm m\sqrt{(m^2 – 4)}$

3. In the expansion of $(2x^2 - 8) (x - 4)^2$; find the value of
(i) coefficient of x^3
(ii) coefficient of x^2
(iii) constant term
Solution:

We have, $(2x^2 - 8) (x - 4)^2$
$= (2x^2 - 8) (x^2 – 2 \times 4 \times x + 4^2)$
$= (2x^2 - 8) (x^2 – 8x + 16)$
$= 2x^2(x^2 – 8x + 16) – 8(x^2 – 8x + 16)$
$= 4x^4 – 16x^3 + 32x^2 – 8x^2 + 64x – 128$
$= 4x^4 – 16x^3 + 24x^2 + 64x – 128$
Now,
(i) coefficient of $x^3 = -16$
(ii) coefficient of $x^2 = 24$
(iii) constant term = -128

4. If x > 0 and $x^2 + 1/9x^2 = 25/36$. Find: $x^3 + 1/27x^3$
Solution:

Given, $x^2 + 1/9x^2 = 25/36$ … (1)
Now, consider the expansion
$(x + 1/3x)^2 = x^2 + (1/3x)^2 + (2 \times x \times 1/3x)$
$= (x^2 + 1/9x^2) + 2/3$
$= 25/36 + 2/3$ … [From (1)]
$= 49/36$
So,
$(x + 1/3x) = \pm\sqrt{(49/36)}$
$= \pm 7/6$ … (2)

Now, consider the expansion
$(x + 1/3x)^3 = x^3 + (1/3x)^3 + 3(x + 1/3x)$
$(7/6)^3 = x^3 + (1/3x)^3 + 3(7/6)$ …… [From (2)]
$343/216 = x^3 + 1/27x^3 + 21/6$
$x^3 + 1/27x^3 = 343/216 – 21/6$
$= (343 - 252)/216$
$= 91/216$
Thus, $x^3 + 1/27x^3 = 91/216$

5. If $2(x^2 + 1) = 5x$, find:
(i) x – 1/x

(ii) $x^3 – 1/x^3$
Solution:

(i) Given, $2(x^2 + 1) = 5x$
$x^2 + 1 = 5x/2$
On dividing by x on both sides, we have
$(x^2 + 1)/x = 5/2$
$\Rightarrow (x + 1/x) = 5/2$ … (1)

Now, consider the expansion of $(x + 1/x)^2$
$(x + 1/x)^2 = x^2 + 1/x^2 + 2$
$(5/2)^2 = x^2 + 1/x^2 + 2$ … [From (1)]
$x^2 + 1/x^2 = 25/4 – 2$
$= (25 - 8)/4$
$= 17/4$ … (2)
Now,
$(x – 1/x)^2 = x^2 + 1/x^2 – 2$
$= 17/4 – 2$ … [From (2)]
$= (17 - 8)/4$
$= 9/4$
So,
$x – 1/x = \sqrt{9/4}$
Thus,
(i) $x – 1/x = \pm 3/2$ … (3)

Next, we know that
$(x^3 – 1/x^3) = (x – 1/x)^3 + 3(x – 1/x)$
$= (\pm 3/2)^3 + 3(\pm 3/2)$ … [From (3)]
$= \pm 27/8 \pm 9/2$
$= \pm (27 + 36)/8$
$= \pm 63/8$
(ii) Thus, $x^3 – 1/x^3 = \pm 63/8$

6. If $a^2 + b^2 = 34$ and $ab = 12$; find:
(i) $3(a + b)^2 + 5(a - b)^2$
(ii) $7(a - b)^2 - 2(a + b)^2$
Solution:

We have, $a^2 + b^2 = 34$ and $ab= 12$
We know that,
$(a + b)^2 = (a^2 + b^2) + 2ab$
$= 34 + 2 \times 12$
$= 34 + 24$
$= 58$

Also, we know that

$(a - b)^2 = (a^2 + b^2) - 2ab$
$= 34 - 2 \times 12$
$= 34 - 24$
$= 10$

(i) $3(a + b)^2 + 5(a - b)^2$
$= 3 \times 58 + 5 \times 10$
$= 174 + 50$
$= 224$

(ii) $7(a - b)^2 - 2(a + b)^2$
$= 7 \times 10 - 2 \times 58$
$= 70 - 116$
$= -46$

7. If $3x - 4/x = 4$ and $x \neq 0$; find: $27x^3 - 64/x^3$.
Solution:

Given, $3x - 4/x = 4$
Now, let's consider the expansion of $(3x - 4/x)^3$
$(3x - 4/x)^3 = 27x^3 - 64/x^3 - 3 \times 3x \times 4/x(3x - 4/x)$
$(4)^3 = 27x^3 - 64/x^3 - 36(3x - 4/x)$
$64 = 27x^3 - 64/x^3 - 36(4)$
$64 = 27x^3 - 64/x^3 - 144$
$27x^3 - 64/x^3 = 144 + 64$
Hence,
$27x^3 - 64/x^3 = 208$

8. If $x^2 + 1/x^2 = 7$ and $x \neq 0$; find the value of: $7x^3 + 8x - 7/x^3 - 8/x$.
Solution:

Given, $x^2 + 1/x^2 = 7$
On subtracting 2 from both sides, we get
$x^2 + 1/x^2 - 2 = 7 - 2$
$(x - 1/x)^2 = 5$
$x - 1/x = \pm\sqrt{5}$ … (1)
Now, consider
$(x - 1/x)^3 = x^3 - 1/x^3 - 3(x - 1/x)$
$(\pm\sqrt{5})^3 = x^3 - 1/x^3 - 3(\pm\sqrt{5})$
$x^3 - 1/x^3 = (\pm\sqrt{5})^3 + 3(\pm\sqrt{5})$ … (2)

Taking,
$7x^3 + 8x - 7/x^3 - 8/x$
$= 7x^3 - 7/x^3 + 8x - 8/x$
$= 7(x^3 - 1/x^3) + 8(x - 1/x)$
$= 7[(\pm\sqrt{5})^3 + 3(\pm\sqrt{5})] + 8(\pm\sqrt{5})$

$= \pm 35\sqrt{5} \pm 21\sqrt{5} \pm 8\sqrt{5}$
$= \pm 64\sqrt{5}$

9. If $x = 1/(x - 5)$ and $x \neq 5$, find $x^2 - 1/x^2$.
Solution:

Given, $x = 1/(x - 5)$
By cross multiplying, we have
$x (x - 5) = 1$
$x^2 - 5x = 1$
$x^2 - 1 = 5x$
Dividing both sides by x,
$(x^2 - 1)/x = 5$
$(x - 1/x) = 5$ … (1)
Now,
$(x - 1/x)^2 = 5^2$
$x^2 + 1/x^2 - 2 = 25$
$x^2 + 1/x^2 = 25 + 2$
$= 27$ … (2)
Considering the expansion $(x + 1/x)^2$
$(x + 1/x)^2 = x^2 + 1/x^2 + 2$
$(x + 1/x)^2 = 27 + 2$ … [From (1)]
$(x + 1/x)^2 = 29$
$x + 1/x = \pm\sqrt{29}$ … (3)
We know that,
$x^2 - 1/x^2 = (x + 1/x)(x - 1/x)$
$= (\pm\sqrt{29})(5)$ … [From (3)]
$= \pm 5\sqrt{29}$

10. If $x = 1/(5 - x)$ and $x \neq 5$; find $x^3 + 1/x^3$.
Solution:

Given, $x = 1/(5 - x)$
By cross multiplying, we have
$x (5 - x) = 1$
$x^2 - 5x = -1$
$x^2 + 1 = 5x$
Dividing both sides by x,
$(x^2 + 1)/x = 5$
$x + 1/x = 5$ … (1)
Now,
$(x + 1/x)^3 = x^3 + 1/x^3 + 3(x + 1/x)$
$x^3 + 1/x^3 = (x + 1/x)^3 - 3(x + 1/x)$
$= 5^3 - 3(5)$
$= 125 - 15$
$= 110$

Thus, $x^3 + 1/x^3 = 110$

11. If 3a + 5b + 4c = 0,
Show that: $27a^3 + 125b^3 + 64c^3 = 180abc$
Solution:

Given, $3a + 5b + 4c = 0$
$\Rightarrow 3a + 5b = -4c$
On cubing on both sides, we have
$(3a + 5b)^3 = (-4c)^3$
$(3a)^3 + (5b)^3 + 3 \times 3a \times 5b\ (3a + 5b) = -64c^3$
$27a^3 + 125b^3 + 45ab\ (-4c) = -64c^3$
$27a^3 + 125b^3 - 180abc = -64c^3$
$27a^3 + 125b^3 + 64c^3 = 180abc$
- Hence Proved.

12. The sum of two numbers is 7 and the sum of their cubes is 133, find the sum of their square.
Solution:

Let's assume a and b to be the two numbers
So, $a + b = 7$ and $a^3 + b^3 = 133$
We know that,
$(a + b)^3 = a^3 + b^3 + 3ab\ (a + b)$
$(7)^3 = 133 + 3ab\ (7)$
$343 = 133 + 21ab$
$21ab = 343 - 133$
$= 210$
$\Rightarrow ab = 21$
Now,
$a^2 + b^2 = (a + b)^2 - 2ab$
$= 7^2 - 2 \times 10$
$= 49 - 20$
$= 29$

13. In each of the following, find the value of 'a':
(i) $4x^2 + ax + 9 = (2x + 3)^2$
(ii) $4x^2 + ax + 9 = (2x - 3)^2$
(iii) $9x^2 + (7a - 5)x + 25 = (3x + 5)^2$
Solution:

(i) $4x^2 + ax + 9 = (2x + 3)^2 = 4x^2 + 12x + 9$
On comparing coefficients of x terms, we get
$ax = 12x$
So,
$a = 12$

(ii) $4x^2 + ax + 9 = (2x - 3)^2 = 4x^2 + 12x + 9$
On comparing coefficients of x terms, we get
$ax = -12x$
So,
$a = -12$

(iii) $9x^2 + (7a - 5)x + 25 = (3x + 5)^2 = 9x^2 + 30x + 25$
On comparing coefficients of x terms, we get
$(7a - 5)x = 30x$
$7a - 5 = 30$
$7a = 35$
$\Rightarrow a = 5$

14. If $(x^2 + 1)/x = 3\ 1/3$ and $x > 1$; find
(i) $x – 1/x$
(ii) $x^3 – 1/x^3$
Solution:

Given,
$(x^2 + 1)/x = 3\ 1/3 = 10/3$
$x + 1/x = 10/3$
On squaring on both sides, we get
$(x + 1/x)^2 = (10/3)^2$
$x^2 + 1/x^2 + 2 = 100/9$
$x^2 + 1/x^2 = 100/9 – 2$
$= (100 - 18)/9$
$= 82/9$
Now,
$(x - 1/x)^2 = x^2 + 1/x^2 – 2$
$= 82/9 – 2$
$= (82 - 18)/9$
$= 64/9$
$x – 1/x = \sqrt{(64/9)}$
$= \pm 8/3$
On cubing both sides, we get
$(x – 1/x)^3 = (8/3)^3$
$x^3 – 1/x^3 – 3(x – 1/x) = 512/27$
$x^3 – 1/x^3 = 3(x – 1/x) + 512/27$
$= 3(8/3) + 512/27$
$= 24/3 + 512/27$
$= (216 + 512)/27$
$= 728/27$
Therefore, $x^3 – 1/x^3 = 728/27$

15. The difference between two positive numbers is 4 and the difference between their

cubes is 316.
Find:
(i) Their product
(ii) The sum of their squares
Solution:

Given, difference between two positive numbers is 4
And, the difference between their cubes is 316
Let's assume the positive numbers to be a and b
So,
$a - b = 4$
$a^3 - b^3 = 316$
On cubing both sides, we have
$(a - b)^3 = 64$
$a^3 - b^3 - 3ab(a - b) = 64$
Also,
Given: $a^3 - b^3 = 316$
So,
$316 - 64 = 3ab(4)$
$252 = 12ab$
So,
$ab = 21$
Thus, the product of numbers is 21

Now,
On squaring both sides, we get
$(a - b)^2 = 16$
$a^2 + b^2 - 2ab = 16$
$a^2 + b^2 = 16 + 42 = 58$
Thus, sum of their squares is 58.