

EXERCISE 19.24

Evaluate the following integrals:

$$1. \int \frac{1}{1 - \cot x} dx$$

Solution:

$$\text{Let, } I = \int \frac{1}{1 - \cot x} dx$$

To solve such integrals involving trigonometric terms in numerator and

denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{1}{1 - \cot x} dx = \int \frac{1}{1 - \frac{\cos x}{\sin x}} dx = \int \frac{\sin x}{\sin x - \cos x} dx$$

As I matches with the form described above, so we will take the steps as described.

$$\therefore \sin x = A \frac{d}{dx} (\sin x - \cos x) + B(\sin x - \cos x) + C$$

$$\Rightarrow \sin x = A(\cos x + \sin x) + B(\sin x - \cos x) + C \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \sin x = \sin x (B + A) + \cos x (A - B) + C$$

Comparing both sides we have:

$$C = 0$$

$$A - B = 0 \Rightarrow A = B$$

$$B + A = 1 \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$$

$$\therefore A = B = \frac{1}{2}$$

Thus I can be expressed as:

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x)}{\sin x - \cos x} dx + \int \frac{\frac{1}{2}(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$\therefore \text{Let } I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{\sin x - \cos x} dx \text{ and } I_2 = \frac{1}{2} \int \frac{(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{\sin x - \cos x} dx$$

$$\text{Let, } u = \sin x - \cos x \Rightarrow du = (\cos x + \sin x) dx$$

So, I_1 reduces to:

$$I_1 = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log|u| + C_1$$

$$\therefore I_1 = \frac{1}{2} \log|\sin x - \cos x| + C_1 \dots \text{Equation 2}$$

$$\text{As, } I_2 = \frac{1}{2} \int \frac{(\sin x - \cos x)}{\sin x - \cos x} dx = \frac{1}{2} \int dx$$

$$\therefore I_2 = \frac{x}{2} + C_2 \dots \text{Equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{1}{2} \log|\sin x - \cos x| + C_1 + \frac{x}{2} + C_2$$

$$\therefore I = \frac{1}{2} \log|\sin x - \cos x| + \frac{x}{2} + C$$

$$2. \int \frac{1}{1 - \tan x} dx$$

Solution:

$$\text{Let, } I = \int \frac{1}{1-\tan x} dx$$

To solve such integrals involving trigonometric terms in numerator and

denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{1}{1-\tan x} dx = \int \frac{1}{1-\frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{\cos x - \sin x} dx$$

As I matches with the form described above, so we will take the steps as described.

$$\therefore \cos x = A \frac{d}{dx} (\cos x - \sin x) + B(\cos x - \sin x) + C$$

$$\Rightarrow \cos x = A(-\sin x - \cos x) + B(\cos x - \sin x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \cos x = -\sin x (B + A) + \cos x (B - A) + C$$

Comparing both sides we have:

$$C = 0$$

$$B - A = 1 \Rightarrow A = B - 1$$

$$B + A = 0 \Rightarrow 2B - 1 = 0 \Rightarrow B = \frac{1}{2}$$

$$\therefore A = B - 1 = -\frac{1}{2}$$

Thus I can be expressed as:

$$I = \int \frac{-\frac{1}{2}(-\sin x - \cos x) + \frac{1}{2}(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x)}{(\cos x - \sin x)} dx + \int \frac{\frac{1}{2}(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$\therefore \text{Let } I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx \text{ and } I_2 = \frac{1}{2} \int \frac{(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx$$

$$\text{Let, } u = \cos x - \sin x \Rightarrow du = -(\cos x + \sin x) dx$$

So, I_1 reduces to:

$$I_1 = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \log|u| + C_1$$

$$\therefore I_1 = -\frac{1}{2} \log|\cos x - \sin x| + C_1 \dots \text{Equation 2}$$

$$\text{As, } I_2 = \frac{1}{2} \int \frac{(\cos x - \sin x)}{(\cos x - \sin x)} dx = \frac{1}{2} \int dx$$

$$\therefore I_2 = \frac{x}{2} + C_2 \dots \text{Equation 3}$$

From equation 1, 2 and 3 we have:

$$I = -\frac{1}{2} \log|\cos x - \sin x| + C_1 + \frac{x}{2} + C_2$$

$$\therefore I = -\frac{1}{2} \log|\cos x - \sin x| + \frac{x}{2} + C$$

$$3. \int \frac{3 + 2 \cos x + 4 \sin x}{2 \sin x + \cos x + 3} dx$$

Solution:

$$\text{Let, } I = \int \frac{3 + 2 \cos x + 4 \sin x}{2 \sin x + \cos x + 3} dx$$

To solve such integrals involving trigonometric terms in numerator and

denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{3+2 \cos x+4 \sin x}{2 \sin x+\cos x+3} dx$$

As I matches with the form described above, so we will take the steps as described.

$$\therefore 3 + 2 \cos x + 4 \sin x = A \frac{d}{dx} (2 \sin x + \cos x + 3) + B(2 \sin x + \cos x + 3) + C$$

$$\Rightarrow 3 + 2 \cos x + 4 \sin x = A(2 \cos x - \sin x) + B(2 \sin x + \cos x + 3) + C$$

$$\left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow 3 + 2 \cos x + 4 \sin x = \sin x (2B - A) + \cos x (B + 2A) + 3B + C$$

Comparing both sides we have:

$$3B + C = 3$$

$$B + 2A = 2$$

$$2B - A = 4$$

On solving for A, B and C we have:

$$A = 0, B = 2 \text{ and } C = -3$$

Thus I can be expressed as:

$$I = \int \frac{2(2 \sin x + \cos x + 3) - 3}{2 \sin x + \cos x + 3} dx$$

$$I = \int \frac{2(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx + \int \frac{-3}{2 \sin x + \cos x + 3} dx$$

$$\therefore \text{Let } I_1 = 2 \int \frac{(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx \text{ and } I_2 = -3 \int \frac{1}{2 \sin x + \cos x + 3} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = 2 \int \frac{(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx$$

So, I_1 reduces to:

$$I_1 = 2 \int dx = 2x + C_1 \dots \text{Equation 2}$$

$$\text{As, } I_2 = -3 \int \frac{1}{2 \sin x + \cos x + 3} dx$$

To solve the integrals of the form $\int \frac{1}{a \sin x + b \cos x + c} dx$

To apply substitution method we take following procedure.

We substitute:

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\therefore I_2 = -3 \int \frac{1}{2 \sin x + \cos x + 3} dx$$

$$\Rightarrow I_2 = -3 \int \frac{1}{2 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 3 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 3} dx$$

$$\Rightarrow I_2 = -3 \int \frac{1 + \tan^2 \frac{x}{2}}{4 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2} + 3(1 + \tan^2 \frac{x}{2})} dx$$

$$\Rightarrow I_2 = -3 \int \frac{\sec^2 \frac{x}{2}}{2(2 \tan \frac{x}{2} + 2 + 1 \tan^2 \frac{x}{2})} dx$$

$$\text{Let, } t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$$

$$\therefore I_2 = -3 \int \frac{1}{(2t + 2 + t^2)} dt$$

As, the denominator is polynomial without any square root term. So one of the special integral will be used to solve I_2 .

$$I_2 = -3 \int \frac{1}{(2t + 2 + t^2)} dt$$

$$\Rightarrow I_2 = -3 \int \frac{1}{(t^2 + 2(1)t + 1) + 1} dt$$

$$\therefore I_2 = -3 \int \frac{1}{(t+1)^2 + 1} dt \quad \{\because a^2 + 2ab + b^2 = (a + b)^2\}$$

As, I_2 matches with the special integral form

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$I_2 = -3 \tan^{-1}(t + 1)$$

Putting value of t we have:

$$\therefore I_2 = -3 \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3:

$$I = 2x + C_1 - 3 \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + C_2$$

$$\therefore I = 2x - 3 \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + C$$

4. $\int \frac{1}{p + q \tan x} dx$

Solution:

$$\text{Let, } I = \int \frac{1}{p + q \tan x} dx$$

To solve such integrals involving trigonometric terms in numerator and

denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A , B and C are constants

$$\text{We have, } I = \int \frac{1}{p+q \tan x} dx = \int \frac{1}{p+q \frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{p \cos x + q \sin x} dx$$

As I matches with the form described above, so we will take the steps as described.

$$\therefore \cos x = A \frac{d}{dx}(p \cos x + q \sin x) + B(p \cos x + q \sin x) + C$$

$$\Rightarrow \cos x = A(-p \sin x + q \cos x) + B(p \cos x + q \sin x) + C \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \cos x = \sin x (Bq - Ap) + \cos x (Bp + Aq) + C$$

Comparing both sides we have:

$$C = 0$$

$$Bp + Aq = 1$$

$$Bq - Ap = 0$$

On solving above equations, we have:

$$A = \frac{q}{p^2+q^2} \quad B = \frac{p}{p^2+q^2} \quad \text{and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{\frac{q}{p^2+q^2}(-p \sin x + q \cos x) + \frac{p}{p^2+q^2}(p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx$$

$$I = \int \frac{\frac{q}{p^2+q^2}(-p \sin x + q \cos x)}{(p \cos x + q \sin x)} dx + \int \frac{\frac{p}{p^2+q^2}(p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx$$

$$\therefore \text{Let } I_1 = \frac{q}{p^2+q^2} \int \frac{(-p \sin x + q \cos x)}{(p \cos x + q \sin x)} dx \quad \text{and } I_2 = \frac{p}{p^2+q^2} \int \frac{(p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{q}{p^2+q^2} \int \frac{(-p \sin x + q \cos x)}{(p \cos x + q \sin x)} dx$$

$$\text{Let, } u = p \cos x + q \sin x \Rightarrow du = (-p \sin x + q \cos x) dx$$

So, I_1 reduces to:

$$I_1 = \frac{q}{p^2+q^2} \int \frac{du}{u} = \frac{q}{p^2+q^2} \log|u| + C_1$$

$$\therefore I_1 = \frac{q}{p^2+q^2} \log|(p \cos x + q \sin x)| + C_1 \quad \dots \text{Equation 2}$$

$$\text{As, } I_2 = \frac{p}{p^2+q^2} \int \frac{(p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx = \frac{p}{p^2+q^2} \int dx$$

$$\therefore I_2 = \frac{px}{p^2+q^2} + C_2 \quad \dots \text{Equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{q}{p^2+q^2} \log|(p \cos x + q \sin x)| + C_1 + \frac{px}{p^2+q^2} + C_2$$

$$\therefore I = \frac{q}{p^2+q^2} \log|(p \cos x + q \sin x)| + \frac{px}{p^2+q^2} + C$$

5. $\int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx$

Solution:

$$\text{Let, } I = \int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx$$

To solve such integrals involving trigonometric terms in numerator and

denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx$$

As I matches with the form described above, so we will take the steps as described.

$$\therefore 5 \cos x + 6 = A \frac{d}{dx} (2 \cos x + \sin x + 3) + B(2 \cos x + \sin x + 3) + C$$

$$\Rightarrow 5 \cos x + 6 = A(-2 \sin x + \cos x) + B(2 \cos x + \sin x + 3) + C$$

$$\because \frac{d}{dx} \cos x = -\sin x \}$$

$$\Rightarrow 5 \cos x + 6 = \sin x (B - 2A) + \cos x (2B + A) + 3B + C$$

Comparing both sides we have:

$$3B + C = 6$$

$$2B + A = 5$$

$$B - 2A = 0$$

On solving for A, B and C we have:

$$A = 1, B = 2 \text{ and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{(-2 \sin x + \cos x) + 2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$I = \int \frac{(-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx + \int \frac{2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$\therefore \text{Let } I_1 = \int \frac{(-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx \text{ and } I_2 = \int \frac{2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \int \frac{(-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx$$

$$\text{Let, } 2 \cos x + \sin x + 3 = u$$

$$\Rightarrow (-2 \sin x + \cos x) dx = du$$

So, I_1 reduces to:

$$I_1 = \int \frac{du}{u} = \log |u| + C_1$$

$$\therefore I_1 = \log |2 \cos x + \sin x + 3| + C_1 \dots \text{Equation 2}$$

$$\text{As, } I_2 = \int \frac{2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$\Rightarrow I_2 = 2 \int dx = 2x + C_2 \dots \text{Equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \log|2 \cos x + \sin x + 3| + C_1 + 2x + C_2$$

$$\therefore I = \log|2 \cos x + \sin x + 3| + 2x + C$$

