28. The Plane

Exercise 28A

1. Question

Find the equation of the plane passing through each group of points:

(i) A(2, 2, -1), B(3, 4, 2) and C(7, 0, 6)

(ii) A(0, -1, -1), B(4, 5, 1) and C(3, 9, 4)

(iii) A(-2, 6, -6), B(-3, 10, 9) and

Answer

(i) A(2, 2, -1), B(3, 4, 2) and C(7, 0, 6)

Given Points :

A = (2, 2, -1)

B = (3, 4, 2)

C = (7, 0, 6)

To Find : Equation of plane passing through points A, B & C

Formulae :

1) Position vectors :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

2) Vector :

If A and B be two points with position vectors $\bar{a}$ & $\bar{b}$ respectively, where

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then,

$\overline{AB} = \bar{b} - \bar{a}$

$= (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$

3) Cross Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

4) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

5) Equation of Plane :

If A = (a_1, a_2, a_3), B = (b_1, b_2, b_3), C = (c_1, c_2, c_3) are three non-collinear points,

Then, the vector equation of the plane passing through these points is

$\bar{r}.(\overline{AB} \times \overline{AC}) = \bar{a}.(\overline{AB} \times \overline{AC})$

Where,

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

For given points,

A = (2, 2, -1)

B = (3, 4, 2)

C = (7, 0, 6)

Position vectors are given by,

$\bar{a} = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$

$\bar{b} = 3\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$

$\bar{c} = 7\hat{\imath} + 6\hat{k}$

Now, vectors $\overline{AB}$ & $\overline{AC}$ are

$\overline{AB} = \bar{b} - \bar{a}$

$= (3-2)\hat{\imath} + (4-2)\hat{\jmath} + (2+1)\hat{k}$

$\therefore \overline{AB} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\overline{AC} = \bar{c} - \bar{a}$

$= (7-2)\hat{\imath} + (0-2)\hat{\jmath} + (6+1)\hat{k}$

$\therefore \overline{AC} = 5\hat{\imath} - 2\hat{\jmath} + 7\hat{k}$

Therefore,

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix}$$

$= \hat{\imath}(2 \times 7 - (-2) \times 3) - \hat{\jmath}(1 \times 7 - 5 \times 3) + \hat{k}(1 \times (-2) - 5 \times 2)$

$= 20\hat{\imath} + 8\hat{\jmath} - 12\hat{k}$

Now,

$\bar{a}.(\overline{AB} \times \overline{AC}) = (2 \times 20) + (2 \times 8) + ((-1) \times (-12))$

= 40 + 16 + 12

= 68

$\therefore \bar{a}.(\overline{AB} \times \overline{AC}) = 68$ ....eq(1)

And

$\bar{r}.(\overline{AB} \times \overline{AC}) = (x \times 20) + (y \times 8) + (z \times (-12))$

= 20x + 8y - 12z

$\therefore \bar{r}.(\overline{AB} \times \overline{AC}) = 20x + 8y - 12z$ ....eq(2)

Vector equation of the plane passing through points A, B & C is

$\bar{r}.(\overline{AB} \times \overline{AC}) = \bar{a}.(\overline{AB} \times \overline{AC})$

From eq(1) and eq(2)

20x + 8y - 12z = 68

This is 5x + 2y - 3z = 17 vector equation of required plane.

(ii) Given Points :

A = (0, -1, -1)

B = (4, 5, 1)

C = (3, 9, 4)

To Find : Equation of plane passing through points A, B & C

Formulae :

1) Position vectors :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

2) Vector :

If A and B be two points with position vectors $\bar{a}$ & $\bar{b}$ respectively, where

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\overline{AB} = \bar{b} - \bar{a}$

$= (b_1 - a_1)\hat{\imath} + (b_2 - a_2)\hat{\jmath} + (b_3 - a_3)\hat{k}$

3) Cross Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

4) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

5) Equation of Plane :

If A = (a_1, a_2, a_3), B = (b_1, b_2, b_3), C = (c_1, c_2, c_3) are three non-collinear points,

Then, vector equation of the plane passing through these points is

$\bar{r}.(\overline{AB} \times \overline{AC}) = \bar{a}.(\overline{AB} \times \overline{AC})$

Where,

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given points,

A = (0, -1, -1)

B = (4, 5, 1)

C = (3, 9, 4)

Position vectors are given by,

$\bar{a} = -\hat{\jmath} - \hat{k}$

$\bar{b} = 4\hat{\imath} + 5\hat{\jmath} + \hat{k}$

$\bar{c} = 3\hat{\imath} + 9\hat{\jmath} + 4\hat{k}$

Now, vectors $\overline{AB}$ & $\overline{AC}$ are

$\overline{AB} = \bar{b} - \bar{a}$

$= (4 - 0)\hat{\imath} + (5 + 1)\hat{\jmath} + (1 + 1)\hat{k}$

$\therefore \overline{AB} = 4\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$

$\overline{AC} = \bar{c} - \bar{a}$

$= (3 - 0)\hat{\imath} + (9 + 1)\hat{\jmath} + (4 + 1)\hat{k}$

$\therefore \overline{AC} = 3\hat{\imath} + 10\hat{\jmath} + 5\hat{k}$

Therefore,

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 6 & 2 \\ 3 & 10 & 5 \end{vmatrix}$$

$= \hat{\imath}(6 \times 5 - 10 \times 2) - \hat{\jmath}(4 \times 5 - 2 \times 3) + \hat{k}(4 \times 10 - 3 \times 6)$

$= 10\hat{\imath} - 14\hat{\jmath} + 22\hat{k}$

Now,

$\bar{a}.(\overline{AB} \times \overline{AC}) = (0 \times 10) + ((-1) \times (-14)) + ((-1) \times 22)$

= 0 + 14 - 22

= - 8

$\therefore \bar{a}.(\overline{AB} \times \overline{AC}) = -8$ ....eq(1)

And

$\bar{r}.(\overline{AB} \times \overline{AC}) = (x \times 10) + (y \times (-14)) + (z \times 22)$

= 10x - 14y + 22z

$\therefore \bar{r}.(\overline{AB} \times \overline{AC}) = 10x - 14y + 22z$ ....eq(2)

Vector equation of plane passing through points A, B & C is

$\bar{r}.(\overline{AB} \times \overline{AC}) = \bar{a}.(\overline{AB} \times \overline{AC})$

From eq(1) and eq(2)

10x - 14y + 22z = - 8

This is 5x - 7y + 11z = - 4 vector equation of required plane

(iii) Given Points :

A = (-2, 6, -6)

B = (-3, 10, 9)

C = (-5, 0, -6)

To Find : Equation of plane passing through points A, B & C

Formulae :

1) Position vectors :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

2) Vector :

If A and B be two points with position vectors $\bar{a}$ & $\bar{b}$ respectively, where

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then,

$\overline{AB} = \bar{b} - \bar{a}$

$= (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$

3) Cross Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

4) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

5) Equation of Plane :

If A = (a_1, a_2, a_3), B = (b_1, b_2, b_3), C = (c_1, c_2, c_3) are three non-collinear points,

Then, vector equation of the plane passing through these points is

$\bar{r}.(\overline{AB} \times \overline{AC}) = \bar{a}.(\overline{AB} \times \overline{AC})$

Where,

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given points,

A = (-2, 6, -6)

B = (-3, 10, 9)

C = (-5, 0, -6)

Position vectors are given by,

$\bar{a} = -2\hat{\imath} + 6\hat{\jmath} - 6\hat{k}$

$\bar{b} = -3\hat{\imath} + 10\hat{\jmath} + 9\hat{k}$

$\bar{c} = -5\hat{\imath} - 6\hat{k}$

Now, vectors $\overline{AB}$ & $\overline{AC}$ are

$\overline{AB} = \bar{b} - \bar{a}$

$= (-3+2)\hat{\imath} + (10-6)\hat{\jmath} + (9+6)\hat{k}$

$\therefore \overline{AB} = -\hat{\imath} + 4\hat{\jmath} + 15\hat{k}$

$\overline{AC} = \bar{c} - \bar{a}$

$= (-5+2)\hat{\imath} + (0-6)\hat{\jmath} + (-6+6)\hat{k}$

$\therefore \overline{AC} = -3\hat{\imath} - 6\hat{\jmath} + 0\hat{k}$

Therefore,

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 4 & 15 \\ -3 & -6 & 0 \end{vmatrix}$$

$$= \hat{\imath}(4 \times 0 - (-6) \times 15) - \hat{\jmath}((-1) \times 0 - (-3) \times 15) + \hat{k}((-1) \times (-6) - (-3) \times 4)$$

$= 90\hat{\imath} - 45\hat{\jmath} + 18\hat{k}$

Now,

$\bar{a}.(\overline{AB} \times \overline{AC}) = ((-2) \times 90) + (6 \times (-45)) + ((-6) \times 18)$

= - 180 - 270 - 108

= - 558

$\therefore \bar{a}.(\overline{AB} \times \overline{AC}) = -558$ ....eq(1)

And

$\bar{r}.(\overline{AB} \times \overline{AC}) = (x \times 90) + (y \times (-45)) + (z \times 18)$

= 90x - 45y + 18z

$\therefore \bar{r}.(\overline{AB} \times \overline{AC}) = 90x - 45y + 18z$ ....eq(2)

Vector equation of plane passing through points A, B & C is

$\bar{r}.(\overline{AB} \times \overline{AC}) = \bar{a}.(\overline{AB} \times \overline{AC})$

From eq(1) and eq(2)

90x - 45y + 18z = - 558

This is 10x - 5y + 2z = - 62 vector equation of required plane

2. Question

Show that the four points A(3, 2, -5),B(-1, 4, -3), C(-3, 8, -5) and D(-3, 2, 1) are coplanar. Find the equation of the plane containing them.

Answer

Given Points :

A = (3, 2, -5)

B = (-1, 4, -3)

C = (-3, 8, -5)

D = (-3, 2, 1)

To Prove : Points A, B, C & D are coplanar.

To Find : Equation of plane passing through points A, B, C & D.

Formulae :

1) Position vectors :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

2) Equation of line

If A and B are two points having position vectors $\bar{a}$ & $\bar{b}$ then equation of line passing through two points is given by,

$\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

3) Cross Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

4) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

5) Coplanarity of two lines :

If two lines $\overline{r_1} = \bar{a} + \lambda\bar{b}$ & $\overline{r_2} = \bar{c} + \mu\bar{d}$ are coplanar then

$\bar{a}.(\bar{b} \times \bar{d}) = \bar{c}.(\bar{b} \times \bar{d})$

6) Equation of plane :

If two lines $\overline{r_1} = \overline{a_1} + \lambda\overline{b_1}$ & $\overline{r_2} = \overline{a_2} + \lambda\overline{b_2}$ are coplanar then equation of the plane containing them is

$\bar{r}.(\overline{b_1} \times \overline{b_2}) = \overline{a_1}.(\overline{b_1} \times \overline{b_2})$

Where,

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given points,

A = (3, 2, -5)

B = (-1, 4, -3)

C = (-3, 8, -5)

D = (-3, 2, 1)

Position vectors are given by,

$\bar{a} = 3\hat{\imath} + 2\hat{\jmath} - 5\hat{k}$

$\bar{b} = -1\hat{\imath} + 4\hat{\jmath} - 3\hat{k}$

$\bar{c} = -3\hat{\imath} + 8\hat{\jmath} - 5\hat{k}$

$\bar{d} = -3\hat{\imath} + 2\hat{\jmath} + \hat{k}$

Equation of line passing through points A & B is

$\overline{r_1} = \bar{a} + \lambda(\bar{b} - \bar{a})$

$\bar{b} - \bar{a} = (-1-3)\hat{\imath} + (4-2)\hat{\jmath} + (-3+5)\hat{k}$

$= -4\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

$\therefore \overline{r_1} = (3\hat{\imath} + 2\hat{\jmath} - 5\hat{k}) + \lambda(-4\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$

Let, $\overline{r_1} = \overline{a_1} + \lambda b_1$

Where,

$\overline{a_1} = 3\hat{\imath} + 2\hat{\jmath} - 5\hat{k}$ & $b_1 = -4\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

And the equation of the line passing through points C & D is

$\overline{r_2} = \bar{c} + \mu(\bar{d} - \bar{c})$

$\bar{d} - \bar{c} = (-3+3)\hat{\imath} + (2-8)\hat{\jmath} + (1+5)\hat{k}$

$= -6\hat{\jmath} + 6\hat{k}$

$\therefore \overline{r_1} = (-3\hat{\imath} + 8\hat{\jmath} - 5\hat{k}) + \lambda(-6\hat{\jmath} + 6\hat{k})$

Let, $\overline{r_2} = \overline{a_2} + \lambda b_2$

Where,

$\overline{a_2} = -3\hat{\imath} + 8\hat{\jmath} - 5\hat{k}$ & $b_2 = -6\hat{\jmath} + 6\hat{k}$

Now,

$$\overline{b_1} \times \overline{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -4 & 2 & 2 \\ 0 & -6 & 6 \end{vmatrix}$$

$= \hat{\imath}(12+12) - \hat{\jmath}(-24-0) + \hat{k}(24+0)$

$\therefore (\overline{b_1} \times \overline{b_2}) = 24\hat{\imath} + 24\hat{\jmath} + 24\hat{k}$

Therefore,

$\overline{a_1}.(\overline{b_1} \times \overline{b_2}) = (3 \times 24) + (2 \times 24) + ((-5) \times 24)$

= 72 + 48 - 120

= 0

$\therefore \overline{a_1}.(\overline{b_1} \times \overline{b_2}) = 0$ ……… eq(1)

And

$\overline{a_2}.(\overline{b_1} \times \overline{b_2}) = ((-3) \times 24) + (8 \times 24) + ((-5) \times 24)$

= - 72 + 192 - 120

= 0

$\therefore \overline{a_2}.(\overline{b_1} \times \overline{b_2}) = 0$ ……… eq(2)

From eq(1) and eq(2)

$\overline{a_1}.(\overline{b_1} \times \overline{b_2}) = \overline{a_2}.(\overline{b_1} \times \overline{b_2})$

Hence lines $\overline{r_1}$ & $\overline{r_2}$ are coplanar

Therefore, points A, B, C & D are also coplanar.

As lines $\overline{r_1}$ & $\overline{r_2}$ are coplanar therefore equation of the plane passing through two lines containing four given points is

$\bar{r}.(\overline{b_1} \times \overline{b_2}) = \overline{a_1}.(\overline{b_1} \times \overline{b_2})$

Now,

$\bar{r}.(\overline{b_1} \times \overline{b_2}) = (x \times 24) + (y \times 24) + (z \times 24)$

= 24x + 24y + 24z

From eq(1)

$\overline{a_1}.(\overline{b_1} \times \overline{b_2}) = 0$

Therefore, equation of required plane is

24x + 24y + 24z = 0

x + y + z = 0

3. Question

Show that the four points A(0, -1, 0),B(2, 1, -1), C(1, 1, 1) and D(3, 3, 0) are coplanar. Find the equation of the plane containing them.

Answer

Given Points :

A = (0, -1, 0)

B = (2, 1, -1)

C = (1, 1, 1)

D = (3, 3, 0)

To Prove : Points A, B, C & D are coplanar.

To Find : Equation of plane passing through points A, B, C & D.

Formulae :

1) Position vectors :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

2) Equation of line

If A and B are two points having position vectors $\bar{a}$ & $\bar{b}$ then equation of line passing through two points is given by,

$\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

3) Cross Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

4) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

5) Coplanarity of two lines :

If two lines $\overline{r_1} = \bar{a} + \lambda\bar{b}$ & $\overline{r_2} = \bar{c} + \mu\bar{d}$ are coplanar then

$\bar{a}.(\bar{b} \times \bar{d}) = \bar{c}.(\bar{b} \times \bar{d})$

6) Equation of plane :

If two lines $\overline{r_1} = \overline{a_1} + \lambda\overline{b_1}$ & $\overline{r_2} = \overline{a_2} + \lambda\overline{b_2}$ are coplanar then equation of the plane containing them is

$\bar{r}.(\overline{b_1} \times \overline{b_2}) = \overline{a_1}.(\overline{b_1} \times \overline{b_2})$

Where,

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given points,

A = (0, -1, 0)

B = (2, 1, -1)

C = (1, 1, 1)

D = (3, 3, 0)

Position vectors are given by,

$\bar{a} = -\hat{\jmath}$

$\bar{b} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$

$\bar{c} = \hat{\imath} + \hat{\jmath} + \hat{k}$

$\bar{d} = 3\hat{\imath} + 3\hat{\jmath}$

Equation of line passing through points A & D is

$\overline{r_1} = \bar{a} + \lambda(\bar{d} - \bar{a})$

$\bar{d} - \bar{a} = (3-0)\hat{\imath} + (3+1)\hat{\jmath} + (0-0)\hat{k}$

$= 3\hat{\imath} + 4\hat{\jmath}$

$\therefore \overline{r_1} = (-\hat{\jmath}) + \lambda(3\hat{\imath} + 4\hat{\jmath})$

Let, $\overline{r_1} = \overline{a_1} + \lambda b_1$

Where,

$\overline{a_1} = -\hat{\jmath}$ & $b_1 = 3\hat{\imath} + 4\hat{\jmath}$

And equation of line passing through points B & C is

$\overline{r_2} = \bar{b} + \mu(\bar{c} - \bar{b})$

$\bar{c} - \bar{b} = (1-2)\hat{\imath} + (1-1)\hat{\jmath} + (1+1)\hat{k}$

$= -\hat{\imath} + 0\hat{\jmath} + 2\hat{k}$

$\therefore \overline{r_1} = (2\hat{\imath} + \hat{\jmath} - \hat{k}) + \lambda(-\hat{\imath} + 2\hat{k})$

Let, $\overline{r_2} = \overline{a_2} + \lambda b_2$

Where,

$\overline{a_2} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ & $b_2 = -\hat{\imath} + 2\hat{k}$

Now,

$\overline{b_1} \times \overline{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 0 \\ -1 & 0 & 2 \end{vmatrix}$

$= \hat{\imath}(8-0) - \hat{\jmath}(6-0) + \hat{k}(0+4)$

$\therefore (\overline{b_1} \times \overline{b_2}) = 8\hat{\imath} - 6\hat{\jmath} + 4\hat{k}$

Therefore,

$\overline{a_1}.(\overline{b_1} \times \overline{b_2}) = (0 \times 8) + ((-1) \times (-6)) + (0 \times 4)$

$= 0 + 6 + 0$

$= 6$

$\therefore \overline{a_1}.(\overline{b_1} \times \overline{b_2}) = 6$ ……… eq(1)

And

$\overline{a_2}.(\overline{b_1} \times \overline{b_2}) = (2 \times 8) + (1 \times (-6)) + ((-1) \times 4)$

$= 16 - 6 - 4$

$= 6$

$\therefore \overline{a_2}.(\bar{b} \times \bar{d}) = 6$ ……… eq(2)

From eq(1) and eq(2)

$\overline{a_1}.(\overline{b_1} \times \overline{b_2}) = \overline{a_2}.(\overline{b_1} \times \overline{b_2})$

Hence lines $\overline{r_1}$ & $\overline{r_2}$ are coplanar

Therefore, points A, B, C & D are also coplanar.

As lines $\overline{r_1}$ & $\overline{r_2}$ are coplanar therefore equation of the plane passing through two lines containing four given points is

$\bar{r}.(\overline{b_1} \times \overline{b_2}) = \overline{a_1}.(\overline{b_1} \times \overline{b_2})$

Now,

$\bar{r}.(\overline{b_1} \times \overline{b_2}) = (x \times 8) + (y \times (-6)) + (z \times 4)$

$= 8x - 6y + 4z$

From eq(1)

$\overline{a_1}.(\overline{b_1} \times \overline{b_2}) = 6$

Therefore, equation of required plane is

$8x - 6y + 4z = 6$

$4x - 3y + 2z = 3$

4. Question

Write the equation of the plane whose intercepts on the coordinate axes are 2, -4 and 5 respectively.

Answer

Given :

X - intercept, a = 2

Y - intercept, b = - 4

Z - intercept, c = 5

To Find : Equation of plane

Formula :

If a, b & c are the intercepts made by plane on X, Y & Z axes respectively, then equation of the plane is given by,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\therefore \frac{x}{2}+\frac{y}{-4}+\frac{z}{5}=1$$

Multiplying above equation throughout by 40,

$$\therefore \frac{40x}{2}+\frac{40y}{-4}+\frac{40z}{5}=40$$

20x - 10y + 8z = 40

10x - 5y + 4z = 20

This the equation of the required plane.

5. Question

Reduce the equation of the plane 4x - 3y + 2z = 12 to the intercept form, and hence find the intercepts made by the plane with the coordinate axes.

Answer

Given :

Equation of plane : 4x - 3y + 2z = 12

To Find :

1) Equation of plane in intercept form

2) Intercepts made by the plane with the co-ordinate axes.

Formula :

If $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

is the equation of a plane in intercept form then intercept made by it with co-ordinate axes are

X-intercept = a

Y-intercept = b

Z-intercept = c

Given the equation of plane:

4x - 3y + 2z = 12

Dividing the above equation throughout by 12

$$\therefore \frac{4x}{12}+\frac{-3y}{12}+\frac{2z}{12}=1$$

$$\therefore \frac{x}{3}+\frac{y}{-4}+\frac{z}{6}=1$$

This is the equation of a plane in intercept form.

Comparing the above equation with

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

We get,

a = 3

b = -4

c = 6

Therefore, intercepts made by plane with co-ordinate axes are

X-intercept = 3

Y-intercept = -4

Z-intercept = 6

6. Question

Find the equation of the plane which passes through the point (2, -3,7) and makes equal intercepts on the coordinate axes.

Answer

Equation of the plane making a, b & c intercepts with X, Y & Z axes respectively is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

But, the plane makes equal intercepts on the co-ordinate axes

Therefore, a = b = c

Therefore the equation of the plane is

$$\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$$

x + y + z = a

As plane passes through the point (2, -3, 7),

Substituting x = 2, y = -3 & z = 7

2 - 3 + 7 = a

Therefore, a = 6

Hence, required equation of plane is

x + y + z = 6

7. Question

A plane meets the coordinate axes at A, B and C respectively such that the centroid of ΔABC is (1, -2, 3). Find the equation of the plane.

Answer

Given :

X-intercept = A

Y-intercept = B

Z-intercept = C

Centroid of ΔABC = (1, -2, 3)

To Find : Equation of a plane

Formulae :

1) Centroid Formula :

For ΔABC if co-ordinates of A, B & C are

A = (x_1, x_2, x_3)

B = (y_1, y_2, y_3)

C = (z_1, z_2, z_3)

Then co-ordinates of the centroid of ΔABC are

$$G=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$$

2) Equation of plane :

Equation of the plane making a, b & c intercepts with X, Y & Z axes respectively is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

As the plane makes intercepts at points A, B & C on X, Y & Z axes respectively, let co-ordinates of A, B, C be

A = (a, 0, 0)

B = (0, b, 0)

C = (0, 0, c)

By centroid formula,

The centroid of ΔABC is given by

$$G=\left(\frac{a+0+0}{3},\frac{0+b+0}{3},\frac{0+0+c}{3}\right)$$

$G = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

But, Centroid of ΔABC = (1, -2, 3) ...... given

$\therefore \frac{a}{3} = 1, \frac{b}{3} = -2, \frac{c}{3} = 3$

Therefore, a = 3, b = - 6, c = 9

Therefore,

X-intercept = a = 3

Y-intercept = b = - 6

Z-intercept = c = 9

Therefore, equation of required plane is

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\therefore \frac{x}{3} + \frac{y}{-6} + \frac{z}{9} = 1$

8. Question

Find the Cartesian and vector equations of a plane passing through the point (1, 2, 3) and perpendicular to a line with direction ratios 2, 3, -4.

Answer

Given :

A = (1, 2, 3)

Direction ratios of perpendicular vector = (2, 3, -4)

To Find : Equation of a plane

Formulae :

1) Position vectors :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

3) Equation of plane :

If a plane is passing through point A, then the equation of a plane is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

Where, $\bar{a} = position\ vector\ of\ A$

$\bar{n} = vector\ perpendicular\ to\ the\ plane$

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For point A = (1, 2, 3), position vector is

$\bar{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Vector perpendicular to the plane with direction ratios (2, 3, -4) is

$\bar{n} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$

Now, $\bar{a}.\bar{n} = (1 \times 2) + (2 \times 3) + (3 \times (-4))$

= 2 + 6 - 12

= - 4

Equation of the plane passing through point A and perpendicular to vector $\bar{n}$ is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

$\therefore \bar{r}.(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}) = -4$

As $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\therefore \bar{r}.(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k})$

= 2x + 3y - 4z

Therefore, equation of the plane is

2x + 3y - 4z = -4

Or

2x + 3y - 4z + 4 = 0

9. Question

If O is the origin and P(1, 2, -3) be a given point, then find the equation of the plane passing through P and perpendicular to OP.

Answer

Given :

P = (1, 2, -3)

O = (0, 0, 0)

$\bar{n} = \overline{OP}$

To Find : Equation of a plane

Formulae :

1) Position vectors :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

2) Vector :

If A and B be two points with position vectors $\bar{a}$ & $\bar{b}$ respectively, where

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\overline{AB} = \bar{b} - \bar{a}$

$= (b_1 - a_1)\hat{\imath} + (b_2 - a_2)\hat{\jmath} + (b_3 - a_3)\hat{k}$

3) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

4) Equation of plane :

If a plane is passing through point A, then the equation of a plane is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

Where, $\bar{a} = position\ vector\ of\ A$

$\bar{n} = vector\ perpendicular\ to\ the\ plane$

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For points,

P = (1, 2, -3)

O = (0, 0, 0)

Position vectors are

$\bar{p} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\bar{o} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

Vector

$\overline{OP} = \bar{p} - \bar{o}$

$= (1-0)\hat{\imath} + (2-0)\hat{\jmath} + (3-0)\hat{k}$

$\therefore \overline{OP} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Now,

$\bar{p}.\overline{OP} = (1 \times 1) + (2 \times 2) + (3 \times 3)$

$= 1 + 4 + 9$

$= 14$

And

$\bar{r}.\overline{OP} = (x \times 1) + (y \times 2) + (z \times 3)$

$= x + 2y + 3z$

Equation of the plane passing through point A and perpendicular to the vector $\bar{n}$ is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

But, $\bar{n} = \overline{OP}$

Therefore, the equation of the plane is

$\bar{r}.\overline{OP} = \bar{p}.\overline{OP}$

$x + 2y + 3z = 14$

$x + 2y + 3z - 14 = 0$

Exercise 28B

1. Question

Find the vector and Cartesian equations of a plane which is at a distance of 5 units from the origin and which has $\hat{k}$ as the unit vector normal to it.

Answer

Given :

$d = 5$

$\hat{n} = \hat{k}$

To Find : Equation of a plane

Formulae :

1) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

2) Equation of plane :

Equation of plane which is at a distance of 5 units from the origin and having $\hat{n}$ as a unit vector normal to it is

$\bar{r}.\hat{n} = d$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given d = 5 and $\hat{n} = \hat{k}$.

Equation of plane is

$\bar{r}.\hat{n} = d$

$\therefore \bar{r}.\hat{k} = 5$

This is a vector equation of the plane

Now,

$\bar{r}.\hat{k} = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).\hat{k}$

$= (x \times 0) + (y \times 0) + (z \times 1)$

$= z$

$\therefore \bar{r}.\hat{k} = z$

Therefore, the equation of the plane is

This is - the Cartesian z = 5 equation of the plane.

2. Question

Find the vector and Cartesian equations of a plane which is at a distance of 7 units from the origin and whose normal vector from the origin is $(3\hat{i} - 5\hat{j} - 6\hat{k})$.

Answer

Given :

d = 7

$\bar{n} = 3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}$

To Find : Equation of plane

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

3) Equation of plane :

Equation of plane which is at a distance of 5 units from the origin and having $\hat{n}$ as a unit vector normal to it is

$\bar{r}.\hat{n} = d$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given normal vector

$\bar{n} = 3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}$

Unit vector normal to the plane is

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$\therefore \hat{n} = \frac{3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}}{\sqrt{3^2 + 5^2 + (-6)^2}}$$

$$\therefore \hat{n} = \frac{3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}}{\sqrt{9 + 25 + 36}}$$

$$\therefore \hat{n} = \frac{3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}}{\sqrt{70}}$$

Equation of the plane is

$\bar{r}.\hat{n} = d$

$$\therefore \bar{r}.\left(\frac{3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}}{\sqrt{70}}\right) = 7$$

$\therefore \bar{r}.\left(3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}\right) = 7\sqrt{70}$

This is a vector equation of the plane.

Now,

$\bar{r}.\left(3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}\right) = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}\right)$

$= (x \times 3) + (y \times 5) + (z \times (-6))$

$= 3x + 5y - 6z$

Therefore equation of the plane is

$3x + 5y - 6z = 7\sqrt{70}$

This is the Cartesian equation of the plane.

3. Question

Find the vector and Cartesian equations of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and whose normal vector from the origin is $(2\hat{i} - 3\hat{j} - 4\hat{k})$.

Answer

Given :

$$d = \frac{6}{\sqrt{29}}$$

$$\bar{n} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$$

To Find : Equation of a plane

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then the unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

then,

$$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$$

3) Equation of plane :

Equation of plane which is at a distance of 5 units from the origin and having $\hat{n}$ as a unit vector normal to it is

$$\bar{r}.\hat{n} = d$$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given normal vector

$$\bar{n} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$$

Unit vector normal to the plane is

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}}{\sqrt{2^2 + (-3)^2 + 4^2}}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}}{\sqrt{4 + 9 + 16}}$$

$$\therefore \hat{n} = \frac{2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}}{\sqrt{29}}$$

Equation of the plane is

$$\bar{r}.\hat{n} = d$$

$$\therefore \bar{r}.\left(\frac{2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}}{\sqrt{29}}\right) = \frac{6}{\sqrt{29}}$$

$$\therefore \bar{r}.\left(2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}\right) = 6$$

This is a vector equation of the plane.

Now,

$\bar{r}.(2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(2\hat{\imath} - 3\hat{\jmath} + 4\hat{k})$

$= (x \times 2) + (y \times (-3)) + (z \times 4)$

$= 2x - 3y + 4z$

Therefore equation of the plane is

$2x - 3y + 4z = 6$

This is the Cartesian equation of the plane.

4. Question

Find the vector and Cartesian equations of a plane which is at a distance of 6 units from the origin and which has a normal with direction ratios 2, -1, -2.

Answer

Given :

$d = 6$

direction ratios of $\bar{n}$ are (2, -1, -2)

$\therefore \bar{n} = 2\hat{\imath} - \hat{\jmath} - 2\hat{k}$

To Find : Equation of plane

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then the unit vector of $\bar{a}$ is

$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

3) Equation of plane :

Equation of plane which is at a distance of 5 units from the origin and having $\hat{n}$ as a unit vector normal to it is

$\bar{r}.\hat{n} = d$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For given normal vector

$\bar{n} = 2\hat{\imath} - \hat{\jmath} - 2\hat{k}$

Unit vector normal to the plane is

$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$

$\therefore \hat{n} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$

$\therefore \hat{n} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{\sqrt{4 + 1 + 4}}$

$\therefore \hat{n} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{\sqrt{9}}$

$\therefore \hat{n} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}$

Equation of the plane is

$\bar{r}.\hat{n} = d$

$\therefore \bar{r}.\left(\frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}\right) = 6$

$\therefore \bar{r}.(2\hat{\imath} - \hat{\jmath} - 2\hat{k}) = 18$

This is vector equation of the plane.

Now,

$\bar{r}.(2\hat{\imath} - \hat{\jmath} - 2\hat{k}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(2\hat{\imath} - \hat{\jmath} - 2\hat{k})$

$= (x \times 2) + (y \times (-1)) + (z \times (-2))$

$= 2x - y - 2z$

Therefore equation of the plane is

$2x - y - 2z = 18$

This is Cartesian equation of the plane.

5. Question

Find the vector, and Cartesian equations of a plane which passes through the point (1, 4, 6) and the normal vector to the plane is $(\hat{\imath} - 2\hat{\jmath} + \hat{k})$.

Answer

Given :

A = (1, 4, 6)

$\bar{n} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

To Find : Equation of plane.

Formulae :

1) Position Vector :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

3) Equation of plane :

Equation of plane passing through point A and having $\bar{n}$ as a unit vector normal to it is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

Where, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Position vector of point A = (1, 4, 6) is

$\bar{a} = \hat{\imath} + 4\hat{\jmath} + 6\hat{k}$

Now,

$\bar{a}.\bar{n} = (\hat{\imath} + 4\hat{\jmath} + 6\hat{k}).(\hat{\imath} - 2\hat{\jmath} + \hat{k})$

$= (1\times1) + (4\times(-2)) + (6\times1)$

$= 1 - 8 + 6$

$= -1$

Equation of plane is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

$\therefore \bar{r}.(\hat{\imath} - 2\hat{\jmath} + \hat{k}) = -1$

This is vector equation of the plane.

As $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Therefore

$\bar{r}.(\hat{\imath} - 2\hat{\jmath} + \hat{k}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(\hat{\imath} - 2\hat{\jmath} + \hat{k})$

= (x × 1) + (y × (-2)) + (z × 1)

= x - 2y + z

Therefore equation of the plane is

$x - 2y + z = -1$

This is Cartesian equation of the plane.

6. Question

Find the length of the perpendicular from the origin to the plane $\bar{r}\cdot(3\hat{\imath} - 12\hat{\jmath} - 4\hat{k}) + 39 = 0$. Also write the unit normal vector from the origin to the plane.

Answer

Given :

Equation of plane : $\bar{r}.(3\hat{\imath} - 12\hat{\jmath} - 4\hat{k}) + 39 = 0$

To Find :

i) Length of perpendicular = d

ii) Unit normal vector = $\hat{n}$

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Length of perpendicular :

The length of the perpendicular from the origin to the plane

$\bar{r}.\bar{n} = p$ is given by,

$d = \frac{p}{|\bar{n}|}$

Given the equation of the plane is

$\bar{r}.(3\hat{\imath} - 12\hat{\jmath} - 4\hat{k}) + 39 = 0$

$\therefore \bar{r}.(3\hat{\imath} - 12\hat{\jmath} - 4\hat{k}) = -39$

$\therefore \bar{r}.(-3\hat{\imath} + 12\hat{\jmath} + 4\hat{k}) = 39$

Comparing the above equation with

$\bar{r}.\bar{n} = p$

We get,

$\bar{n} = -3\hat{\imath} + 12\hat{\jmath} + 4\hat{k}$ & p = 39

Therefore,

$|\bar{n}| = \sqrt{(-3)^2 + 12^2 + 4^2}$

$= \sqrt{9 + 144 + 16}$

$= \sqrt{169}$

= 13

The length of the perpendicular from the origin to the given plane is

$d = \frac{p}{|\bar{n}|}$

$\therefore d = \frac{39}{13}$

$\therefore d = 3$

Vector normal to the plane is

$\bar{n} = -3\hat{\imath} + 12\hat{\jmath} + 4\hat{k}$

Therefore, the unit vector normal to the plane is

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$\therefore \hat{n} = \frac{-3\hat{\imath} + 12\hat{\jmath} + 4\hat{k}}{13}$$

$$\therefore \hat{n} = \frac{-3\hat{\imath}}{13} + \frac{12\hat{\jmath}}{13} + \frac{4\hat{k}}{13}$$

7. Question

Find the Cartesian equation of the plane whose vector equation is $\bar{r} \cdot (3\hat{i} + 5\hat{j} - 9\hat{k}) = 8$.

Answer

Given :

Vector equation of the plane is

$\bar{r}.(3\hat{\imath} + 5\hat{\jmath} - 9\hat{k}) = 8$

To Find : Cartesian equation of the given plane.

Formulae :

1) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

Given the equation of the plane is

$\bar{r}.(3\hat{\imath} + 5\hat{\jmath} - 9\hat{k}) = 8$

Here,

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\therefore \bar{r}.(3\hat{\imath} + 5\hat{\jmath} - 9\hat{k}) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(3\hat{\imath} + 5\hat{\jmath} - 9\hat{k})$

$= (x \times 3) + (y \times 5) + (z \times (-9))$

$= 3x + 5y - 9z$

Therefore equation of the plane is

$3x + 5y - 9z = 8$

This is the Cartesian equation of the given plane.

8. Question

Find the vector equation of a plane whose Cartesian equation is $5x - 7y + 2z + 4 = 0$.

Answer

Given :

Cartesian equation of the plane is

$5x - 7y + 2z + 4 = 0$

To Find : Vector equation of the given plane.

Formulae :

1) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

Given the equation of the plane is

5x - 7y + 2z + 4 = 0

⇒ 5x - 7y + 2z = - 4

The term (5x - 7y + 2z) can be written as

$(5x - 7y + 2z) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(5\hat{\imath} - 7\hat{\jmath} + 2\hat{k})$

But $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\therefore (5x - 7y + 2z) = \bar{r}.(5\hat{\imath} - 7\hat{\jmath} + 2\hat{k})$

Therefore the equation of the plane is

$\bar{r}.(5\hat{\imath} - 7\hat{\jmath} + 2\hat{k}) = -4$

or

$\bar{r}.(-5\hat{\imath} + 7\hat{\jmath} - 2\hat{k}) = 4$

This is Vector equation of the given plane.

9. Question

Find a unit vector normal to the planex - 2y + 2z = 6.

Answer

Given :

Equation of plane : x - 2y + 2z = 6

To Find : unit normal vector $= \hat{n}$

Formula :

Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then the unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

From the given equation of a plane

x - 2y + 2z = 6

direction ratios of vector normal to the plane are (1, -2, 2).

Therefore, the equation of normal vector is

$\bar{n} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

Therefore unit normal vector is given by

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

$$\therefore \hat{n} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{\sqrt{1^2 + (-2)^2 + 2^2}}$$

$$\therefore \hat{n} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{\sqrt{1 + 4 + 4}}$$

$$\therefore \hat{n} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{\sqrt{9}}$$

$$\therefore \hat{n} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{3}$$

$$\therefore \hat{n} = \frac{\hat{\imath}}{3} - \frac{2\hat{\jmath}}{3} + \frac{2\hat{k}}{3}$$

10. Question

Find the direction cosines of the normal to the plane 3x - 6y + 2z = 7.

Answer

Given :

Equation of plane : 3x - 6y + 2z = 7

To Find : Direction cosines of the normal, i.e. $l, m \& n$

Formula :

1) Direction cosines :

If a, b & c are direction ratios of the vector, then its direction cosines are given by

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

For the given equation of a plane

3x - 6y + 2z = 7

Direction ratios of normal vector are (3, -6, 2)

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{3^2 + (-6)^2 + 2^2}$$

$$= \sqrt{9 + 36 + 4}$$

$$= \sqrt{49}$$

$$= \pm 7$$

Therefore, direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{3}{7}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \mp\frac{6}{7}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{2}{7}$$

$$(l, m, n) = \pm\left(\frac{3}{7}, \frac{-6}{7}, \frac{2}{7}\right)$$

11. Question

For each of the following planes, find the direction cosines of the normal to the plane and the distance of the plane from the origin:

(i) 2x + 3y - z = 5

(ii) z = 3

(iii) 3y + 5 = 0

Answer

(i) 2x + 3y - z = 5

Given :

Equation of plane : 2x + 3y - z = 5

To Find :

Direction cosines of the normal i.e. $l, m \& n$

Distance of the plane from the origin = d

Formulae :

1) Direction cosines :

If a, b & c are direction ratios of the vector then its direction cosines are given by

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

2) The distance of the plane from the origin :

Distance of the plane from the origin is given by,

$$d = \frac{p}{|\bar{n}|}$$

For the given equation of plane

2x + 3y - z = 5

Direction ratios of normal vector are (2, 3, -1)

Therefore, equation of normal vector is

$\bar{n} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\sqrt{a^2 + b^2 + c^2} = \sqrt{2^2 + 3^2 + (-1)^2}$

$= \sqrt{4 + 9 + 1}$

$= \sqrt{14}$

Therefore, direction cosines are

$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{2}{\sqrt{14}}$

$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{3}{\sqrt{14}}$

$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{-1}{\sqrt{14}}$

$(l, m, n) = \left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right)$

Now, the distance of the plane from the origin is

$d = \frac{p}{|\bar{n}|}$

$\therefore d = \frac{5}{\sqrt{14}}$

(ii) Given :

Equation of plane : z = 3

To Find :

Direction cosines of the normal, i.e. l, m & n

The distance of the plane from the origin = d

Formulae :

3) Direction cosines :

If a, b & c are direction ratios of the vector, then its direction cosines are given by

$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$

$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$

$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

4) The distance of the plane from the origin :

Distance of the plane from the origin is given by,

$d = \frac{p}{|\bar{n}|}$

For the given equation of a plane

z = 3

Direction ratios of normal vector are (0, 0, 1)

Therefore, equation of normal vector is

$\bar{n} = \hat{k}$

$\sqrt{a^2 + b^2 + c^2} = \sqrt{0^2 + 0^2 + 1^2}$

$= \sqrt{1}$

$= 1$

Therefore, direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{0}{1} = 0$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{0}{1} = 0$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{1} = 1$$

$(l, m, n) = (0,0,1)$

Now, the distance of the plane from the origin is

$$d = \frac{p}{|\bar{n}|}$$

$$\therefore d = \frac{3}{1}$$

$$\therefore d = 3$$

(iii) Given :

Equation of plane : 3y + 5 = 0

To Find :

Direction cosines of the normal, i.e. l, m & n

The distance of the plane from the origin = d

Formulae :

1) Direction cosines :

If a, b & c are direction ratios of the vector, then its direction cosines are given by

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

2) Distance of the plane from the origin :

Distance of the plane from the origin is given by,

$$d = \frac{p}{|\bar{n}|}$$

For the given equation of a plane

3y + 5 = 0

⇒-3y = 5

Direction ratios of normal vector are (0, -3, 0)

Therefore, equation of normal vector is

$$\bar{n} = -3\hat{j}$$

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{0^2 + (-3)^2 + 0^2}$$

$$= \sqrt{9}$$

$$= 3$$

Therefore, direction cosines are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{0}{3} = 0$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{-3}{3} = -1$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{0}{3} = 0$$

$(l, m, n) = (0,-1,0)$

Now, distance of the plane from the origin is

$$d = \frac{p}{|\bar{n}|}$$

$\therefore d = \frac{5}{3}$

12. Question

Find the vector and Cartesian equations of the plane passing through the point (2, -1, 1) and perpendicular to the line having direction ratios 4, 2, -3.

Answer

Given :

A = (2, -1, 1)

Direction ratios of perpendicular vector = (4, 2, -3)

To Find : Equation of a plane

Formulae :

1) Position vectors :

If A is a point having co-ordinates (a_1, a_2, a_3), then its position vector is given by,

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

2) Dot Product :

If $\bar{a}$ & $\bar{b}$ are two vectors

$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

then,

$\bar{a}.\bar{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

3) Equation of plane :

If a plane is passing through point A, then the equation of a plane is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

Where, $\bar{a} = position\ vector\ of\ A$

$\bar{n} = vector\ perpendicular\ to\ the\ plane$

$\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

For point A = (2, -1, 1), position vector is

$\bar{a} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

Vector perpendicular to the plane with direction ratios (4, 2, -3) is

$\bar{n} = 4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$

Now, $\bar{a}.\bar{n} = (2 \times 4) + ((-1) \times 2) + (1 \times (-3))$

= 8 - 2 - 3

= 3

Equation of the plane passing through point A and perpendicular to vector $\bar{n}$ is

$\bar{r}.\bar{n} = \bar{a}.\bar{n}$

$\therefore \bar{r}.\left(4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) = 3$

As $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\therefore \bar{r}.\left(4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right)$

= 4x + 2y - 3z

Therefore, the equation of the plane is

4x + 2y - 3z = 3

Or

4x + 2y - 3z - 3 = 0

13. Question

Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

(i) 2x + 3y + 4z -12 = 0

(ii) 5y + 8 = 0

Answer

(i) 2x + 3y + 4z - 12 = 0

Given :

Equation of plane : 2x + 3y + 4z + 12 = 0

To Find :

coordinates of the foot of the perpendicular

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

From the given equation of the plane

2x + 3y + 4z – 12 = 0

⇒ 2x + 3y + 4z = 12

Direction ratios of the vector normal to the plane are (2, 3, 4)

Let, P = (x, y, z) be the foot of perpendicular perpendicular drawn from origin to the plane.

Therefore perpendicular drawn is $\overline{OP}$.

$$\therefore \overline{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Let direction ratios of $\overline{OP}$ are (x, y, z)

As normal vector and $\overline{OP}$ are parallel

$$\therefore \frac{x}{2} = \frac{y}{3} = \frac{z}{4} = k(say)$$

⇒x = 2k, y = 3k, z = 4k

As point P lies on the plane, we can write

2(2k) + 3(3k) + 4(4k) = 12

⇒ 4k + 9k + 16k = 12

⇒ 29k = 12

$$\therefore k = \frac{12}{29}$$

$$\therefore x = 2k = \frac{24}{29},$$

$$y = 3k = \frac{36}{29}$$

$$z = 4k = \frac{48}{29}$$

Therefore co-ordinates of the foot of perpendicular are

$$P(x, y, z) = \left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$$

$$P = \left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$$

(ii) Given :

Equation of plane : 5y + 8 = 0

To Find :

coordinates of the foot of the perpendicular

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

From the given equation of the plane

5y + 8 = 0

⇒ 5y = - 8

Direction ratios of the vector normal to the plane are (0, 5, 0)

Let, P = (x, y, z) be the foot of perpendicular perpendicular drawn from origin to the plane.

Therefore perpendicular drawn is $\overline{OP}$.

$\therefore \overline{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

Let direction ratios of $\overline{OP}$ are (x, y, z)

As normal vector and $\overline{OP}$ are parallel

$\therefore \frac{0}{x} = \frac{5}{y} = \frac{0}{z} = \frac{1}{k}\,(say)$

⇒x = 0, y = 5k, z = 0

As point P lies on the plane, we can write

5(5k) = -8

⇒ 25k = -8

$\therefore k = \frac{-8}{25}$

$\therefore x = 0,$

$y = 5k = 5 \times \frac{-8}{25} = \frac{-8}{5}$

$z = 0$

Therefore co-ordinates of the foot of perpendicular are

$P(x, y, z) = \left(0, \frac{-8}{5}, 0\right)$

$P = \left(0, \frac{-8}{5}, 0\right)$

14. Question

Find the length and the foot of perpendicular drawn from the point (2, 3, 7) to the plane 3x - y - z = 7.

Answer

Given :

Equation of plane : 3x - y - z = 7

A = (2, 3, 7)

To Find :

i) Length of perpendicular = d

ii) coordinates of the foot of the perpendicular

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$

Where, $|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

2) Length of perpendicular :

The length of the perpendicular from point A with position vector $\bar{a}$ to the plane is given by,

$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Given equation of the plane is

3x - y - z = 7 ....eq(1)

Therefore direction ratios of normal vector of the plane are

(3, -1, -1)

Therefore normal vector of the plane is

$\bar{n} = 3\hat{\imath} - \hat{\jmath} - \hat{k}$

$\therefore |\bar{n}| = \sqrt{3^2 + (-1)^2 + (-1)^2}$

$= \sqrt{9 + 1 + 1}$

$= \sqrt{11}$

From eq(1), p = 7

Given point A = (2, 3, 7)

Position vector of A is

$\bar{a} = 2\hat{\imath} + 3\hat{\jmath} + 7\hat{k}$

Now,

$\bar{a}.\bar{n} = (2\hat{\imath} + 3\hat{\jmath} + 7\hat{k}).(3\hat{\imath} - \hat{\jmath} - \hat{k})$

= (2×3) + (3×(-1)) + (7×(-1))

= 6 - 3 - 7

= -4

Length of the perpendicular from point A to the plane is

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

$$\therefore d = \frac{|-4 - 7|}{\sqrt{11}}$$

$$\therefore d = \frac{11}{\sqrt{11}}$$

$\therefore d = \sqrt{11}$

Let P be the foot of perpendicular drawn from point A to the given plane,

Let P = (x, y, z)

$\overline{AP} = (x-2)\hat{\imath} + (y-3)\hat{\jmath} + (z-7)\hat{k}$

As normal vector and $\overline{AP}$ are parallel

$$\therefore \frac{x-2}{3} = \frac{y-3}{-1} = \frac{z-7}{-1} = k(say)$$

⇒x = 3k+2, y = - k+3, z = -k+7

As point P lies on the plane, we can write

3(3k+2) - (- k+3) - (-k+7) = 7

⇒ 9k + 6 + k - 3 + k - 7 = 7

⇒ 11k = 11

$\therefore k = 1$

$\therefore x = 3k + 2 = 5,$

$y = -k + 3 = 2$

$z = -k + 7 = 6$

Therefore co-ordinates of the foot of perpendicular are

P(x, y, z) = $(5, 2, 6)$

P = (5, 2, 6)

15. Question

Find the length and the foot of the perpendicular drawn from the point (1, 1, 2) to the plane $\bar{r}\cdot(2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}) + 5 = 0$.

Answer

Given :

Equation of plane : $\bar{r}.(2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}) + 5 = 0$

A = (1, 1, 2)

To Find :

i) Length of perpendicular = d

ii) coordinates of the foot of the perpendicular

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$

2) Length of perpendicular :

The length of the perpendicular from point A with position vector $\bar{a}$ to the plane is given by,

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Given equation of the plane is

$\bar{r}.(2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}) + 5 = 0$ ....eq(1)

$\therefore \bar{r}.(2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}) = -5$

As $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Therefore equation of plane is

$2x - 2y + 4z = -5$ .... eq(2)

From eq(1) normal vector of the plane is

$\bar{n} = 2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$

$\therefore |\bar{n}| = \sqrt{2^2 + (-2)^2 + 4^2}$

$= \sqrt{4 + 4 + 16}$

$= \sqrt{24}$

From eq(1), p = -5

Given point A = (1, 1, 2)

Position vector of A is

$\bar{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$

Now,

$\bar{a}.\bar{n} = (\hat{\imath} + \hat{\jmath} + 2\hat{k}).(2\hat{\imath} - 2\hat{\jmath} + 4\hat{k})$

$= (1\times2) + (1\times(-2)) + (2\times4)$

$= 2 - 2 + 8$

$= 8$

Length of the perpendicular from point A to the plane is

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

$$\therefore d = \frac{|8 + 5|}{\sqrt{24}}$$

$$\therefore d = \frac{13}{\sqrt{24}}$$

$$\therefore d = \frac{13\sqrt{6}}{\sqrt{24}.\sqrt{6}}$$

$$\therefore d = \frac{13\sqrt{6}}{\sqrt{144}}$$

$\therefore d = \frac{13\sqrt{6}}{12}$

Let P be the foot of perpendicular drawn from point A to the given plane,

Let P = (x, y, z)

$\overline{AP} = (x-1)\hat{\imath} + (y-1)\hat{\jmath} + (z-2)\hat{k}$

As normal vector and $\overline{AP}$ are parallel

$\therefore \frac{x-1}{2} = \frac{y-1}{-2} = \frac{z-2}{4} = k(say)$

⇒x = 2k+1, y = -2k+1, z = 4k+2

As point P lies on the plane, we can write

2(2k+1) - 2(-2k+1) + 4(4k+2) = -5

⇒ 4k + 2 + 4k - 2 + 16k + 8 = -5

⇒ 24k = -13

$\therefore k = \frac{-13}{24}$

$\therefore x = 2\left(\frac{-13}{24}\right) + 1 = \frac{-1}{12}$,

$y = -2\left(\frac{-13}{24}\right) + 1 = \frac{25}{12}$

$z = 4\left(\frac{-13}{24}\right) + 2 = \frac{-1}{6}$

Therefore co-ordinates of the foot of perpendicular are

P(x, y, z) = $\left(\frac{-1}{12}, \frac{25}{12}, \frac{-1}{6}\right)$

P ≡ $\left(\frac{-1}{12}, \frac{25}{12}, \frac{-1}{6}\right)$

16. Question

From the point P(1, 2, 4), a perpendicular is drawn on the plane 2x + y - 2z + 3 = 0. Find the equation, the length and the coordinates of the foot of the perpendicular.

Answer

Given :

Equation of plane : 2x + y - 2z + 3 = 0

P = (1, 2, 4)

To Find :

i) Equation of perpendicular

ii) Length of perpendicular = d

iii) coordinates of the foot of the perpendicular

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$

Where, $|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

2) Length of perpendicular :

The length of the perpendicular from point A with position vector $\bar{a}$ to the plane is given by,

$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Given equation of the plane is

2x + y - 2z + 3 = 0

⇒2x + y - 2z = -3 ....eq(1)

From eq(1) direction ratios of normal vector of the plane are

(2, 1, -2)

Therefore, equation of normal vector is

$\bar{n} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$

$\therefore |\bar{n}| = \sqrt{2^2 + 1^2 + (-2)^2}$

$= \sqrt{4 + 1 + 4}$

$= \sqrt{9}$

= 3

From eq(1), p = -3

Given point P = (1, 2, 4)

Position vector of A is

$\bar{p} = \hat{\imath} + 2\hat{\jmath} + 4\hat{k}$

Here, $\bar{a} = \bar{p}$

Now,

$\therefore \bar{a}.\bar{n} = (\hat{\imath} + 2\hat{\jmath} + 4\hat{k}).(2\hat{\imath} + \hat{\jmath} - 2\hat{k})$

= (1×2) + (2×1) + (4×(-2))

= 2 + 2 - 8

= -4

Length of the perpendicular from point A to the plane is

$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$

$\therefore d = \frac{|-4 + 3|}{3}$

$\therefore d = \frac{1}{3}$

Let Q be the foot of perpendicular drawn from point P to the given plane,

Let Q = (x, y, z)

$\overline{PQ} = (x - 1)\hat{\imath} + (y - 2)\hat{\jmath} + (z - 4)\hat{k}$

As normal vector and $\overline{PQ}$ are parallel, we can write,

$\therefore \frac{x - 1}{2} = \frac{y - 2}{1} = \frac{z - 4}{-2}$

This is the equation of perpendicular.

$\therefore \frac{x - 1}{2} = \frac{y - 2}{1} = \frac{z - 4}{-2} = k(say)$

⇒x = 2k+1, y = k+2, z = -2k+4

As point Q lies on the plane, we can write

2(2k+1) + (k+2) - 2(-2k+4) = -3

⇒ 4k + 2 + k + 2 + 4k - 8 = -3

⇒ 9k = 1

$\therefore k = \frac{1}{9}$

$\therefore x = 2\left(\frac{1}{9}\right) + 1 = \frac{11}{9},$

$y = \frac{1}{9} + 2 = \frac{19}{9}$

$z = -2\left(\frac{1}{9}\right) + 4 = \frac{34}{9}$

Therefore co-ordinates of the foot of perpendicular are

$Q(x, y, z) = \left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$

$Q \equiv \left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$

17. Question

Find the coordinates of the foot of the perpendicular and the perpendicular distance from the point P(3, 2, 1) to the plane 2x – y + z + 1 = 0.

Find also the image of the point P in the plane.

Answer

Given :

Equation of plane : 2x - y + z + 1 = 0

P = (3, 2, 1)

To Find :

i) Length of perpendicular = d

ii) coordinates of the foot of the perpendicular

iii) Image of point P in the plane.

Formulae :

1) Unit Vector :

Let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be any vector

Then unit vector of $\bar{a}$ is

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Where, $|\bar{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

2) Length of perpendicular :

The length of the perpendicular from point A with position vector $\bar{a}$ to the plane is given by,

$$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$$

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Given equation of the plane is

2x - y + z + 1 = 0

⇒2x - y + z = -1 ....eq(1)

From eq(1) direction ratios of normal vector of the plane are

(2, -1, 1)

Therefore, equation of normal vector is

$\bar{n} = 2\hat{i} - \hat{j} + \hat{k}$

$\therefore |\bar{n}| = \sqrt{2^2 + (-1)^2 + 1^2}$

$= \sqrt{4 + 1 + 1}$

$= \sqrt{6}$

From eq(1), p = -1

Given point P = (3, 2, 1)

Position vector of A is

$\bar{p} = 3\hat{i} + 2\hat{j} + \hat{k}$

Here, $\bar{a} = \bar{p}$

Now,

$\therefore \bar{a}.\bar{n} = (3\hat{i} + 2\hat{j} + \hat{k}).(2\hat{i} - \hat{j} + \hat{k})$

= (3×2) + (2×(-1)) + (1×1)

= 6 - 2 + 1

= 5

Length of the perpendicular from point A to the plane is

$d = \frac{|\bar{a}.\bar{n} - p|}{|\bar{n}|}$

$\therefore d = \frac{|5+1|}{\sqrt{6}}$

$\therefore d = \frac{6}{\sqrt{6}}$

$\therefore d = \sqrt{6}$

Let Q be the foot of perpendicular drawn from point P to the given plane,

Let Q = (x, y, z)

$\overline{PQ} = (x-3)\hat{\imath} + (y-2)\hat{\jmath} + (z-1)\hat{k}$

As normal vector and $\overline{PA}$ are parallel, we can write,

$\therefore \frac{x-3}{2} = \frac{y-2}{-1} = \frac{z-1}{1} = k(say)$

⇒x = 2k+3, y = -k+2, z = k+1

As point A lies on the plane, we can write

2(2k+3) - (-k+2) + (k+1) = -1

⇒ 4k + 6 + k - 2 + k + 1 = -1

⇒ 6k = -6

$\therefore k = -1$

$\therefore x = 2(-1) + 3 = 1,$

$y = -(-1) + 2 = 3$

$z = (-1) + 1 = 0$

Therefore, co-ordinates of the foot of perpendicular are

Q(x, y, z) = (1,3,0)

Q ≡ (1,3,0)

Let, R(a, b, c) be image of point P in the given plane.

Therefore, the power of points P and R in the given plane will be equal and opposite.

2a - b + c + 1 = - (2(3) - 2 + 1 + 1)

⇒2a - b + c + 1 = - 6

⇒2a - b + c = - 7 ....eq(2)

Now, $\overline{PR} = (a-3)\hat{\imath} + (b-2)\hat{\jmath} + (c-1)\hat{k}$

As $\overline{PR}$ & $\bar{n}$ are parallel

$\therefore \frac{a-3}{2} = \frac{b-2}{-1} = \frac{c-1}{1} = k(say)$

⇒a = 2k+3, b = -k+2, c = k+1

substituting a, b, c in eq(2)

2(2k+3) - (-k+2) + (k+1) = -7

⇒ 4k + 6 + k - 2 + k + 1 = -7

⇒ 6k = -12

$\therefore k = -2$

$\therefore a = 2(-2) + 3 = -1,$

$b = -(-2) + 2 = 4$

$c = (-2) + 1 = -1$

Therefore, co-ordinates of the image of P are

R(a, b, c) = $(-1,4,-1)$

R ≡ $(-1,4,-1)$

18. Question

Find the coordinates of the image of the point P(1, 3, 4) in the plane 2x - y + z + 3 = 0.

Answer

Given :

Equation of plane : 2x - y + z + 3 = 0

P = (1, 3, 4)

To Find : Image of point P in the plane.

Note :

If two vectors with direction ratios (a_1, a_2, a_3) & (b_1, b_2, b_3) are parallel then

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Given equation of the plane is

2x - y + z + 3 = 0

⇒2x - y + z = -3 ....eq(1)

From eq(1) direction ratios of normal vector of the plane are

(2, -1, 1)

Therefore, equation of normal vector is

$\bar{n} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

Given point is P = (1, 3, 4)

Let, R(a, b, c) be image of point P in the given plane.

Therefore, the power of points P and R in the given plane will be equal and opposite.

⇒2a – b + c + 3 = - (2(1) - 3 + 4 + 3)

⇒2a – b + c + 3 = - 6

⇒2a – b + c = - 9 ....eq(2)

Now, $\overline{PR} = (a-1)\hat{\imath} + (b-3)\hat{\jmath} + (c-4)\hat{k}$

As $\overline{PR}$ & $\bar{n}$ are parallel

$\therefore \frac{a-1}{2} = \frac{b-3}{-1} = \frac{c-4}{1} = k(say)$

⇒a = 2k+1, b = -k+3, c = k+4

substituting a, b, c in eq(2)

2(2k+1) - (-k+3) + (k+4) = -9

⇒ 4k + 2 + k – 3 + k + 4 = -9

⇒ 6k = -12

$\therefore k = -2$

$\therefore a = 2(-2) + 1 = -3,$

$b = -(-2) + 3 = 5$

$c = (-2) + 4 = 2$

Therefore, co-ordinates of the image of P are

R(a, b, c) = $(-3, 5, 2)$

19. Question

Find the point where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{4}$ meets the plane 2x + 4y - z = 1.

Answer

Given :

Equation of plane : 2x + 4y - z = 1

Equation of line :

$\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z+3}{4}$

To Find : Point of intersection of line and plane.

Let P(a, b, c) be point of intersection of plane and line.

As point P lies on the line, we can write,

$\frac{a-1}{2}=\frac{b-2}{-3}=\frac{c+3}{4}=k(say)$

⇒a = 2k+1, b = -3k+2, c = 4k-3 ....(1)

Also point P lies on the plane

2a + 4b - c = 1

⇒2(2k+1) + 4(-3k+2) - (4k-3) = 1 .....from (1)

⇒4k + 2 - 12k + 8 - 4k + 3 = 1

⇒-12k = -12

⇒k = 1

$\therefore a=2(1)+1=3,$

$b=-3(1)+2=-1$

$c=4(1)-3=1$

Therefore, co-ordinates of point of intersection of given line and plane are

P ≡ $(3,-1,1)$

20. Question

Find the coordinates of the point where the line through (3, -4, -5) and (2, -3, 1) crosses the plane 2x + y + z = 7.

Answer

Given :

Equation of plane : 2x + y + z = 7

Points :

A = (3, -4, -5)

B = (2, -3, 1)

To Find : Point of intersection of line and plane.

Formula :

Equation of line passing through A = (x_1, y_1, z_1) &

B = (x_2, y_2, z_2) is

$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$

Equation of line passing through A = (3, -4, -5) & B = (2, -3, 1) is

$\frac{x-3}{3-2}=\frac{y+4}{-4+3}=\frac{z+5}{-5-1}$

$\therefore \frac{x-3}{1}=\frac{y+4}{-1}=\frac{z+5}{-6}$

Let P(a, b, c) be point of intersection of plane and line.

As point P lies on the line, we can write,

$\frac{a-3}{1}=\frac{b+4}{-1}=\frac{c+5}{-6}=k(say)$

⇒a = k+3, b = -k - 4, c = -6k-5 ....(1)

Also point P lies on the plane

2a + b + c = 7

⇒2(k+3) + (-k-4) + (-6k-5) = 7 .....from (1)

⇒2k + 6 - k - 4 - 6k - 5 = 7

⇒-5k = 10

⇒k = -2