EXERCISE 29.9

Evaluate the following limits:

1. $\lim\limits_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$

Solution:

Given: $\lim\limits_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$

The limit $\lim\limits_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$

When x = π, the expression $\lim\limits_{x\to\pi} \frac{1+\cos x}{\tan^2 x}$ assumes the form (0/0).

So, let us multiply the expression by $\cos^2 x$

$$\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi}\left[\frac{(1+\cos x)}{\sin^2 x}\times\cos^2 x\right]$$

$$= \lim_{x\to\pi}\left[\frac{(1+\cos x)}{1-\cos^2 x}\times\cos^2 x\right]$$

Upon expansion, we get

$$= \lim_{x\to\pi}\left[\frac{(1+\cos x)}{(1-\cos x)(1+\cos x)}\times\cos^2 x\right]$$

$$= \lim_{x\to\pi}\left[\frac{\cos^2 x}{(1-\cos x)}\right]$$

Now, substitute the value of x, we get

$$= \frac{\cos^2 \pi}{1-\cos\pi}$$

$$= \frac{(-1)^2}{1-(-1)}$$

$$= \frac{1}{2}$$

∴ The value of $\lim\limits_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \frac{1}{2}$

2. $\lim_{x\to\frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$

Solution:

Given: $\lim_{x\to\frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$

The limit $\lim_{x\to\frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$

When x = π/4, the expression $\lim_{x\to\frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$ assumes the form (0/0).

So,

$$\lim_{x\to\frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1} = \lim_{x\to\frac{\pi}{4}} \left[\frac{1+\cot^2 x - 2}{\cot x - 1}\right] \text{ [Since, } \operatorname{cosec}^2 x = 1 + \cot^2 x\text{]}$$

$$= \lim_{x\to\frac{\pi}{4}} \left[\frac{\cot^2 x - 1}{\cot x - 1}\right]$$

Upon expansion, we get

$$= \lim_{x\to\frac{\pi}{4}} \left[\frac{(\cot x - 1)(\cot x + 1)}{(\cot x - 1)}\right]$$

Now, substitute the value of x, we get

$$= \cot\frac{\pi}{4} + 1$$

$$= 2$$

∴ The value of $\lim_{x\to\frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1} = 2$

3. $\lim_{x\to\frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

Solution:

Given: $\lim_{x\to\frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

The limit

When x = π/6, the expression $\lim_{x\to\frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$ assumes the form (0/0).

$$\lim_{x\to\frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = \lim_{x\to\frac{\pi}{6}} \left[\frac{\operatorname{cosec}^2 x - 1 - 3}{\operatorname{cosec} x - 2}\right] \text{ [Since, } \cot^2 x = \operatorname{cosec}^2 x - 1\text{]}$$

$$= \lim_{x\to\frac{\pi}{6}} \left[\frac{cosec^2 x - 4}{cosec x - 2}\right]$$

Upon expansion, we get

$$= \lim_{x\to\frac{\pi}{6}} \left[\frac{(cosec x - 2)(cosec x + 2)}{(cosec x - 2)}\right]$$

Now, substitute the value of x, we get

$$= cosec\frac{\pi}{6} + 2$$

$= 2 + 2$

$= 4$

∴ The value of $\lim_{x\to\frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = 4$

4. $\lim_{x\to\frac{\pi}{4}} \frac{2 - \operatorname{cosec}^2 x}{1 - \cot x}$

Solution:

Given: $\lim_{x\to\frac{\pi}{4}} \frac{2 - \operatorname{cosec}^2 x}{1 - \cot x}$

The limit

When x = π/4, the expression $\lim_{x\to\frac{\pi}{4}} \frac{2 - \operatorname{cosec}^2 x}{1 - \cot x}$ assumes the form (0/0).

So,

$$\lim_{x\to\frac{\pi}{4}} \frac{2 - \operatorname{cosec}^2 x}{1 - \cot x} = \lim_{x\to\frac{\pi}{4}} \left[\frac{2 - (1 + \cot^2 x)}{1 - \cot x}\right]$$ [Since, $\operatorname{cosec}^2 x = 1 + \cot^2 x$]

$$= \lim_{x\to\frac{\pi}{4}} \left[\frac{1 - \cot^2 x}{1 - \cot x}\right]$$

Upon expansion, we get

$$= \lim_{x\to\frac{\pi}{4}} \left[\frac{(1 - \cot x)(1 + \cot x)}{(1 - \cot x)}\right]$$

Now, substitute the value of x, we get

$$= 1 + \cot\left(\frac{\pi}{4}\right)$$

$= 1 + 1$

$= 2$

$\therefore$ The value of $\lim_{x\to\frac{\pi}{4}} \frac{2-\operatorname{cosec}^2 x}{1-\cot x} = 2$

5. $\lim_{x\to\pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$

Solution:

Given: $\lim_{x\to\pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$

The limit

When $x=\pi$, the expression $\lim_{x\to\pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$ assumes the form (0/0).

So, let us rationalize the numerator, we get

$$\lim_{x\to\pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2} = \lim_{x\to\pi}\left[\frac{\left(\sqrt{2+\cos x}-1\right)\times\left(\sqrt{2+\cos x}+1\right)}{(\pi-x)^2\left(\sqrt{2+\cos x}+1\right)}\right]$$

Let us simplify the above expression, we get

$$= \lim_{x\to\pi}\left[\frac{2+\cos x-1}{(\pi-x)^2\left(\sqrt{2+\cos x}+1\right)}\right]$$

$$= \lim_{x\to\pi}\left[\frac{1+\cos x}{(\pi-x)^2\left[\sqrt{2+\cos x}+1\right]}\right]$$

Now, let $x=\pi-h$

When $x=\pi$, then $h=0$

So,

$$= \lim_{h\to 0}\left[\frac{1+\cos(\pi-h)}{[\pi-(\pi-h)]^2\left[\sqrt{2+\cos(\pi-h)}+1\right]}\right]$$

$$= \lim_{h\to 0}\left[\frac{1-\cos h}{h^2\left[\sqrt{2-\cos h}+1\right]}\right] \quad \{\because \cos(\pi-\theta)=-\cos\theta\}$$

Let us simplify further,

$$= \lim_{h\to 0}\left[\frac{2\sin^2\left(\frac{h}{2}\right)}{4\times\frac{h^2}{4}\left[\sqrt{2-\cos h}+1\right]}\right]$$

$$= \frac{1}{2} \lim_{h \to 0} \left[\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \times \frac{1}{\left[\sqrt{2 - \cos h} + 1\right]} \right]$$

Now, substitute the value of h, we get

$$= \frac{1}{2} \times 1 \times \frac{1}{\left(\sqrt{2 - \cos 0} + 1\right)}$$

$$= \frac{1}{2} \times \frac{1}{\left(\sqrt{1} + 1\right)}$$

$$= \frac{1}{2 \times 2}$$

$$= \frac{1}{4}$$

∴ The value of $\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} = \frac{1}{4}$