EXERCISE 6.1

1. Write the minors and cofactors of each element of the first column of the following matrices and hence evaluate the determinant in each case:

$(i) A = \begin{bmatrix} 5 & 20 \\ 0 & -1 \end{bmatrix}$

$(ii) A = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$

$(iii) A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$

$(iv) A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$

$(v) A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{bmatrix}$

$(vi) A = \begin{bmatrix} a & h & g \\ h & b & f \\ f & f & c \end{bmatrix}$

$(vii) A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix}$

Solution:

(i) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column.The minor of the matrix can be obtained for a particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$

Given,

$A = \begin{bmatrix} 5 & 20 \\ 0 & -1 \end{bmatrix}$

From the given matrix we have,

$M_{11} = -1$

$M_{21} = 20$

$C_{11} = (-1)^{1+1} \times M_{11}$
$= 1 \times -1$
$= -1$
$C_{21} = (-1)^{2+1} \times M_{21}$
$= 20 \times -1$
$= -20$
Now expanding along the first column we get
$|A| = a_{11} \times C_{11} + a_{21} \times C_{21}$
$= 5 \times (-1) + 0 \times (-20)$
$= -5$

(ii) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.
Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$
Given
$$A = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$$
From the above matrix we have
$M_{11} = 3$
$M_{21} = 4$
$C_{11} = (-1)^{1+1} \times M_{11}$
$= 1 \times 3$
$= 3$
$C_{21} = (-1)^{2+1} \times 4$
$= -1 \times 4$
$= -4$
Now expanding along the first column we get
$|A| = a_{11} \times C_{11} + a_{21} \times C_{21}$
$= -1 \times 3 + 2 \times (-4)$
$= -11$

(iii) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of the matrix can be obtained for a particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$

Given,

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$$

From given matrix we have,

$$\Rightarrow M_{11} = \begin{bmatrix} -1 & 2 \\ 5 & 2 \end{bmatrix}$$

$M_{11} = -1\times2 - 5\times2$

$M_{11} = -12$

$$\Rightarrow M_{21} = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$$

$M_{21} = -3\times2 - 5\times2$

$M_{21} = -16$

$$\Rightarrow M_{31} = \begin{bmatrix} -3 & 2 \\ -1 & 2 \end{bmatrix}$$

$M_{31} = -3 \times 2 - (-1) \times 2$

$M_{31} = -4$

$C_{11} = (-1)^{1+1} \times M_{11}$

$= 1 \times -12$

$= -12$

$C_{21} = (-1)^{2+1} \times M_{21}$

$= -1 \times -16$

$= 16$

$C_{31} = (-1)^{3+1} \times M_{31}$

$= 1 \times -4$

$= -4$

Now expanding along the first column we get

$|A| = a_{11} \times C_{11} + a_{21}\times C_{21} + a_{31}\times C_{31}$

$= 1\times (-12) + 4 \times 16 + 3\times (-4)$

$= -12 + 64 - 12$

$= 40$

(iv) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of the matrix can be obtained for a particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$

Given,

$$A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$$

$$\Rightarrow M_{11} = \begin{bmatrix} b & ca \\ c & ab \end{bmatrix}$$

$M_{11} = b \times ab - c \times ca$

$M_{11} = ab^2 - ac^2$

$$\Rightarrow M_{21} = \begin{bmatrix} a & bc \\ c & ab \end{bmatrix}$$

$M_{21} = a \times ab - c \times bc$

$M_{21} = a^2b - c^2b$

$$\Rightarrow M_{31} = \begin{bmatrix} a & bc \\ b & ca \end{bmatrix}$$

$M_{31} = a \times c\, a - b \times bc$

$M_{31} = a^2c - b^2c$

$C_{11} = (-1)^{1+1} \times M_{11}$

$= 1 \times (ab^2 - ac^2)$

$= ab^2 - ac^2$

$C_{21} = (-1)^{2+1} \times M_{21}$

$= -1 \times (a^2b - c^2b)$

$= c^2b - a^2b$

$C_{31} = (-1)^{3+1} \times M_{31}$

$= 1 \times (a^2c - b^2c)$

$= a^2c - b^2c$

Now expanding along the first column we get

$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31}$

$= 1 \times (ab^2 - ac^2) + 1 \times (c^2b - a^2b) + 1 \times (a^2c - b^2c)$

$= ab^2 - ac^2 + c^2b - a^2b + a^2c - b^2c$

(v) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.

Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$

Given,

$$A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{bmatrix}$$

From the above matrix we have,

$$\Rightarrow M_{11} = \begin{bmatrix} 5 & 0 \\ 7 & 1 \end{bmatrix}$$

$M_{11} = 5\times1 - 7\times0$

$M_{11} = 5$

$$\Rightarrow M_{21} = \begin{bmatrix} 2 & 6 \\ 7 & 1 \end{bmatrix}$$

$M_{21} = 2\times1 - 7\times6$

$M_{21} = -40$

$$\Rightarrow M_{31} = \begin{bmatrix} 2 & 6 \\ 5 & 0 \end{bmatrix}$$

$M_{31} = 2\times0 - 5\times6$

$M_{31} = -30$

$C_{11} = (-1)^{1+1} \times M_{11}$

$= 1 \times 5$

$= 5$

$C_{21} = (-1)^{2+1} \times M_{21}$

$= -1 \times -40$

$= 40$

$C_{31} = (-1)^{3+1} \times M_{31}$

$= 1 \times -30$

$= -30$

Now expanding along the first column we get

$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31}$
$= 0 \times 5 + 1 \times 40 + 3 \times (-30)$
$= 0 + 40 - 90$
$= 50$

(vi) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.
Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$
Given,

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

From the given matrices we have,

$$\Rightarrow M_{11} = \begin{bmatrix} b & f \\ f & c \end{bmatrix}$$

$M_{11} = b \times c - f \times f$

$M_{11} = bc - f^2$

$$\Rightarrow M_{21} = \begin{bmatrix} h & g \\ f & c \end{bmatrix}$$

$M_{21} = h \times c - f \times g$

$M_{21} = hc - fg$

$$\Rightarrow M_{31} = \begin{bmatrix} h & g \\ b & f \end{bmatrix}$$

$M_{31} = h \times f - b \times g$
$M_{31} = hf - bg$
$C_{11} = (-1)^{1+1} \times M_{11}$
$= 1 \times (bc - f^2)$
$= bc - f^2$
$C_{21} = (-1)^{2+1} \times M_{21}$
$= -1 \times (hc - fg)$
$= fg - hc$

$C_{31} = (-1)^{3+1} \times M_{31}$
$= 1 \times (hf - bg)$
$= hf - bg$
Now expanding along the first column we get
$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31}$
$= a \times (bc - f^2) + h \times (fg - hc) + g \times (hf - bg)$
$= abc - af^2 + hgf - h^2c + ghf - bg^2$

(vii) Let M_{ij} and C_{ij} represents the minor and co–factor of an element, where i and j represent the row and column. The minor of matrix can be obtained for particular element by removing the row and column where the element is present. Then finding the absolute value of the matrix newly formed.
Also, $C_{ij} = (-1)^{i+j} \times M_{ij}$
Given,

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix}$$

From the given matrix we have,

$$\Rightarrow M_{11} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ -1 & 5 & 0 \end{bmatrix}$$

$M_{11} = 0(-1 \times 0 - 5 \times 1) - 1(1 \times 0 - (-1) \times 1) + (-2)(1 \times 5 - (-1) \times (-1))$

$M_{11} = -9$

$$\Rightarrow M_{21} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 5 & 0 \end{bmatrix}$$

$M_{21} = -1(-1 \times 0 - 5 \times 1) - 0(1 \times 0 - (-1) \times 1) + 1(1 \times 5 - (-1) \times (-1))$

$M_{21} = 9$

$$\Rightarrow M_{31} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 5 & 0 \end{bmatrix}$$

$M_{31} = -1(1 \times 0 - 5 \times (-2)) - 0(0 \times 0 - (-1) \times (-2)) + 1(0 \times 5 - (-1) \times 1)$
$M_{31} = -9$

$$\Rightarrow M_{41} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$M_{41} = -1(1\times1 - (-1) \times (-2)) - 0(0 \times 1 - 1 \times (-2)) + 1(0 \times (-1) - 1 \times 1)$

$M_{41} = 0$

$C_{11} = (-1)^{1+1} \times M_{11}$

$= 1 \times (-9)$

$= -9$

$C_{21} = (-1)^{2+1} \times M_{21}$

$= -1 \times 9$

$= -9$

$C_{31} = (-1)^{3+1} \times M_{31}$

$= 1 \times -9$

$= -9$

$C_{41} = (-1)^{4+1} \times M_{41}$

$= -1 \times 0$

$= 0$

Now expanding along the first column we get

$|A| = a_{11} \times C_{11} + a_{21} \times C_{21} + a_{31} \times C_{31} + a_{41} \times C_{41}$

$= 2 \times (-9) + (-3) \times -9 + 1 \times (-9) + 2 \times 0$

$= -18 + 27 - 9$

$= 0$

2. Evaluate the following determinants:

$(i) \begin{vmatrix} x & -7 \\ x & 5x+1 \end{vmatrix}$

$(ii) \begin{vmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{vmatrix}$

$(iii) \begin{vmatrix} \cos 15^0 & \sin 15^0 \\ sin75^0 & \cos 75^0 \end{vmatrix}$

$(iv) \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

Solution:

(i) Given

$$\begin{vmatrix} x & -7 \\ x & 5x+1 \end{vmatrix}$$

⇒ |A| = x (5x + 1) – (–7) x
$|A| = 5x^2 + 8x$

(ii) Given

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{vmatrix}$$

⇒ |A| = cos θ × cos θ – (–sin θ) x sin θ
$|A| = \cos^2\theta + \sin^2\theta$
We know that $\cos^2\theta + \sin^2\theta = 1$
|A| = 1

(iii) Given

$$\begin{vmatrix} \cos 15^0 & \sin 15^0 \\ sin75^0 & \cos 75^0 \end{vmatrix}$$

⇒ |A| = cos15° × cos75° + sin15° x sin75°
We know that cos (A – B) = cos A cos B + Sin A sin B
By substituting this we get, |A| = cos (75 – 15)°
|A| = cos60°
|A| = 0.5

(iv) Given

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$$

⇒ |A| = (a + ib) (a – ib) – (c + id) (–c + id)
= (a + ib) (a – ib) + (c + id) (c – id)
$= a^2 - i^2 b^2 + c^2 - i^2 d^2$
We know that $i^2 = -1$
$= a^2 - (-1) b^2 + c^2 - (-1) d^2$
$= a^2 + b^2 + c^2 + d^2$

3. Evaluate:

$$\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}^2$$

Solution:

Since |AB| = |A||B|

$$|A| = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

$$|A| = 2\begin{vmatrix} 17 & 5 \\ 20 & 12 \end{vmatrix} - 3\begin{vmatrix} 13 & 5 \\ 15 & 12 \end{vmatrix} + 7\begin{vmatrix} 13 & 17 \\ 15 & 20 \end{vmatrix}$$

= 2(17 × 12 – 5 × 20) – 3(13 × 12 – 5 × 15) + 7(13 × 20 – 15 × 17)

= 2 (204 – 100) – 3 (156 – 75) + 7 (260 – 255)

= 2×104 – 3×81 + 7×5

= 208 – 243 +35

= 0

Now $|A|^2 = |A|\times|A|$

$|A|^2 = 0$

4. Show that

$$\begin{vmatrix} \sin 10^0 & -\cos 10^0 \\ \sin 80^0 & \cos 80^0 \end{vmatrix}$$

Solution:

Given

$$\begin{vmatrix} \sin 10^0 & -\cos 10^0 \\ \sin 80^0 & \cos 80^0 \end{vmatrix}$$

Let the given determinant as A

Using sin (A+B) = sin A × cos B + cos A × sin B

⇒ |A| = sin 10° × cos 80° + cos 10° x sin 80°

|A| = sin (10 + 80)°

|A| = sin90°

|A| = 1

Hence Proved

5. Evaluate $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$ **by two methods.**

Solution:

Given,

$$|A| = \begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

Expanding along the first row

$$|A| = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 7 & -2 \\ -3 & 1 \end{vmatrix} - 5\begin{vmatrix} 7 & 1 \\ -3 & 4 \end{vmatrix}$$

= 2(1 × 1 – 4 × (–2)) – 3(7 × 1 – (–2) × (–3)) – 5(7 × 4 – 1 × (–3))
= 2(1 + 8) – 3(7 – 6) – 5(28 + 3)
= 2 × 9 – 3 × 1 – 5 × 31
= 18 – 3 – 155
= –140

Now by expanding along the second column

$$|A| = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 7\begin{vmatrix} 3 & -5 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & -5 \\ 1 & -2 \end{vmatrix}$$

= 2(1 × 1 – 4 × (–2)) – 7(3 × 1 – 4 × (–5)) – 3(3 × (–2) – 1 × (–5))
= 2 (1 + 8) – 7 (3 + 20) – 3 (–6 + 5)
= 2 × 9 – 7 × 23 – 3 × (–1)
= 18 – 161 +3
= –140

6. Evaluate : $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

Solution:

Given

$$\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$$

Expanding along the first row

$$|A| = 0\begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} - \cos\alpha\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

⇒ |A| = 0 (0 – sinβ (–sinβ)) –sinα (–sinα × 0 – sinβ cosα) – cosα ((–sinα) (–sinβ) – 0 × cosα)
|A| = 0 + sinα sinβ cosα – cosα sinα sinβ
|A| = 0