

EXERCISE 19.31

Evaluate the following integrals:

$$1. \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx$$

Solution:

The given equation can be written as,

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx$$

$$\text{Let } x - \frac{1}{x} = t$$

$$\text{Then, } \left(1 + \frac{1}{x^2}\right) dx = dt$$

$$= \int \frac{1}{t^2 + 3} dt$$

Using standard identity we get

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) + c$$

Substituting t as $x - \frac{1}{x}$, we get

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{\left(x - \frac{1}{x}\right)}{\sqrt{3}} \right) + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) + c$$

$$2. \int \sqrt{\cot \theta} d\theta$$

Solution:

$$\text{Let } \cot \theta = x^2$$

$$-\operatorname{cosec}^2 \theta d\theta = 2x dx$$

$$d\theta = -\frac{2x}{1 + \cot^2 \theta} dx$$

$$d\theta = -\frac{2x}{1 + x^4} dx$$

$$\int -\frac{2x^2}{1 + x^4} dx$$

Re-writing the given equation as

$$-\int \frac{1 + \frac{1}{x^2} + 1 - \frac{1}{x^2}}{\frac{1}{x^2} + x^2} dx$$

$$-\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2} dx - \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2} dx$$

$$\text{Let } x - \frac{1}{x} = t \text{ and } x + \frac{1}{x} = z$$

$$\text{So } \left(1 + \frac{1}{x^2}\right) dx = dt \text{ and } \left(1 - \frac{1}{x^2}\right) dx = dz$$

$$-\int \frac{dt}{(t)^2 + 2} - \int \frac{dz}{(z)^2 - 2}$$

$$\text{Using identity } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \text{ and } \int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$$

$$-\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}} \log \left| \frac{z - \sqrt{2}}{z + \sqrt{2}} \right| + c$$

Now, substituting t as $x - 1/x$ and z as $x + 1/x$ we have

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{x^2 + 1 - \sqrt{2}x}{x^2 + 1 + \sqrt{2}x} \right| + c$$

Lastly, substituting x^2 as $\cot \theta$ we get

$$I = -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\cot \theta - 1}{\sqrt{2} \cot \theta} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{\cot \theta + 1 - \sqrt{2} \cot \theta}{\cot \theta + 1 + \sqrt{2} \cot \theta} \right| + c$$

3. $\int \frac{x^2 + 9}{x^4 + 81} dx$

Solution:

$$\int \frac{1 + \frac{9}{x^2}}{x^2 + \frac{81}{x^2}} dx$$

$$= \int \frac{1 + \frac{9}{x^2}}{\left(x - \frac{9}{x}\right)^2 + 18} dx \quad (\text{By completing the square})$$

Let $x - \frac{9}{x} = t$

$$\left(1 + \frac{9}{x^2}\right) dx = dt$$

$$= \int \frac{dt}{t^2 + 18}$$

Using identity $\int \frac{1}{x^2 + 1} dx = \tan^{-1}(x)$

$$= \frac{1}{3\sqrt{2}} \tan^{-1} \left(\frac{t}{3\sqrt{2}} \right) + c$$

Substituting t as $x - \frac{9}{x}$

$$= \frac{1}{3\sqrt{2}} \tan^{-1} \left(\frac{x - \frac{9}{x}}{3\sqrt{2}} \right) + c$$



$$4. \int \frac{1}{x^4 + x^2 + 1} dx$$

Solution:

$$\int \frac{\frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \frac{1}{2} \int \frac{1 + \frac{1}{x^2} + \frac{1}{x^2} - 1}{x^2 + 1 + \frac{1}{x^2}} dx \quad \text{(Manipulating the numerator by multiplying and dividing by 2)}$$

$$= \frac{1}{2} \left[\int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx + \int \frac{-1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \right]$$

$$= \frac{1}{2} \left[\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx + \int \frac{-1 + \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 1} dx \right]$$

Let $x - \frac{1}{x} = t$ and $x + \frac{1}{x} = z$

Then, $\left(1 + \frac{1}{x^2}\right) dx = dt$ and $\left(1 - \frac{1}{x^2}\right) dx = dz$

$$= \frac{1}{2} \left[\int \frac{dt}{(t)^2 + 3} - \int \frac{dz}{(z)^2 - 1} \right]$$

Using identity $\int \frac{1}{x^2+1} dx = \tan^{-1}(x)$ and $\int \frac{dz}{(z)^2-1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) - \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| \right]$$

Substituting t as $x - \frac{1}{x}$ and z as $x + \frac{1}{x}$

We get,

$$= \frac{1}{2} \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{3}} \right) - \frac{1}{2} \log \left| \frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} \right| \right] + c$$

$$I = \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) - \frac{1}{4} \log \left| \frac{x^2 + 1 - x}{x^2 + 1 + x} \right| + c$$

5. $\int \frac{x^2 - 3x + 1}{x^4 + x^2 + 1} dx$

Solution:

The given equation can be written as

$$\int \frac{1 - \frac{3}{x} + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx - \int \frac{3x}{x^4 + x^2 + 1} dx$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$\left(1 + \frac{1}{x^2}\right) dx = dt \quad \text{And } 2x dx = dz$$

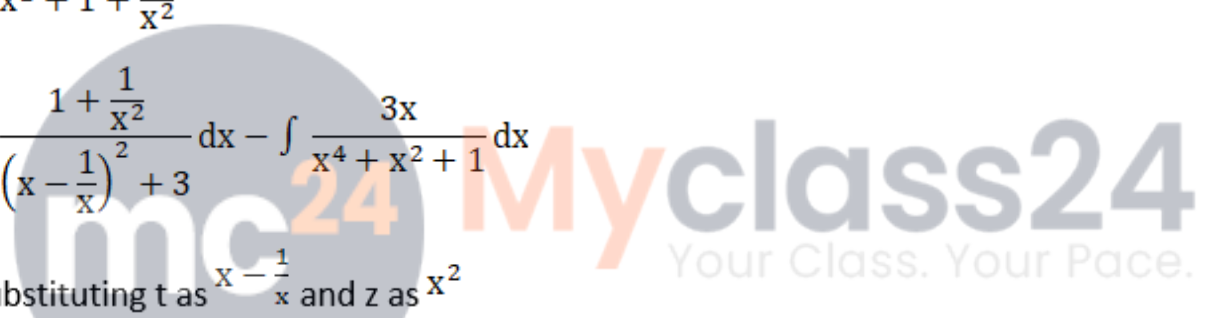
$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2} \int \frac{dz}{z^2 + z + 1}$$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2} \int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Using identity $\int \frac{1}{x^2 + 1} dx = \tan^{-1}(x)$

$$\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) - \sqrt{3} \tan^{-1} \left(\frac{2z + 1}{\sqrt{3}} \right) + c$$

Substituting t as $x - \frac{1}{x}$ and z as x^2



$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{3}} \right) - \sqrt{3} \tan^{-1} \left(\frac{2x^2 + 1}{\sqrt{3}} \right) + c$$

Hence,

$$I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) - \sqrt{3} \tan^{-1} \left(\frac{2x^2 + 1}{\sqrt{3}} \right) + c$$



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