EXERCISE 18.2

Find the 11^{th} term from the beginning and the 11^{th} term from the end in the expansion of $(2x – 1/x^2)^{25}$.

Solution:

Given:

$(2x – 1/x^2)^{25}$

The given expression contains 26 terms.

So, the 11^{th} term from the end is the $(26 − 11 + 1)^{th}$ term from the beginning.

In other words, the 11^{th} term from the end is the 16^{th} term from the beginning.

Then,

$T_{16} = T_{15+1} = {}^{25}C_{15} (2x)^{25-15} (-1/x^2)^{15}$

$= {}^{25}C_{15} (2^{10}) (x)^{10} (-1/x^{30})$

$= - {}^{25}C_{15} (2^{10} / x^{20})$

Now we shall find the 11^{th} term from the beginning.

$T_{11} = T_{10+1} = {}^{25}C_{10} (2x)^{25-10} (-1/x^2)^{10}$

$= {}^{25}C_{10} (2^{15}) (x)^{15} (1/x^{20})$

$= {}^{25}C_{10} (2^{15} / x^5)$

1. Find the 7^{th} term in the expansion of $(3x^2 – 1/x^3)^{10}$.

Solution:

Given:

$(3x^2 – 1/x^3)^{10}$

Let us consider the 7^{th} term as T_7

So,

$T_7 = T_{6+1}$

$= {}^{10}C_6 (3x^2)^{10-6} (-1/x^3)^6$

$= {}^{10}C_6 (3)^4 (x)^8 (1/x^{18})$

$= [10×9×8×7×81] / [4×3×2×x^{10}]$

$= 17010 / x^{10}$

∴ The 7^{th} term of the expression $(3x^2 – 1/x^3)^{10}$ is $17010 / x^{10}$.

2. Find the 5^{th} term in the expansion of $(3x – 1/x^2)^{10}$.

Solution:

Given:

$(3x – 1/x^2)^{10}$

The 5^{th} term from the end is the $(11 – 5 + 1)$th, is., 7^{th} term from the beginning.

So,

$T_7 = T_{6+1}$

$= {}^{10}C_6\,(3x)^{10-6}\,(-1/x^2)^6$

$= {}^{10}C_6\,(3)^4\,(x)^4\,(1/x^{12})$

$= [10\times9\times8\times7\times81] / [4\times3\times2\times x^8]$

$= 17010 / x^8$

∴ The 5^{th} term of the expression $(3x – 1/x^2)^{10}$ is $17010 / x^8$.

3. Find the 8^{th} term in the expansion of $(x^{3/2}\,y^{1/2} – x^{1/2}\,y^{3/2})^{10}$.

Solution:

Given:

$(x^{3/2}\,y^{1/2} – x^{1/2}\,y^{3/2})^{10}$

Let us consider the 8^{th} term as T_8

So,

$T_8 = T_{7+1}$

$= {}^{10}C_7\,(x^{3/2}\,y^{1/2})^{10-7}\,(-x^{1/2}\,y^{3/2})^7$

$= -[10\times9\times8]/[3\times2]\,x^{9/2}\,y^{3/2}\,(x^{7/2}\,y^{21/2})$

$= -120\,x^8y^{12}$

∴ The 8^{th} term of the expression $(x^{3/2}\,y^{1/2} – x^{1/2}\,y^{3/2})^{10}$ is $-120\,x^8y^{12}$.

4. Find the 7^{th} term in the expansion of $(4x/5 + 5/2x)^8$.

Solution:

Given:

$(4x/5 + 5/2x)^8$

Let us consider the 7^{th} term as T_7

So,

$T_7 = T_{6+1}$

$$= {}^8C_6\left(\frac{4x}{5}\right)^{8-6}\left(\frac{5}{2x}\right)^6$$

$$= \frac{8\times7\times4\times4\times125\times125}{2\times1\times25\times64}\,x^2\,\left(\frac{1}{x^6}\right)$$

$$= \frac{4375}{x^4}$$

∴ The 7^{th} term of the expression $(4x/5 + 5/2x)^8$ is $4375/x^4$.

5. Find the 4^{th} term from the beginning and 4^{th} term from the end in the expansion of $(x + 2/x)^9$.

Solution:

Given:

$(x + 2/x)^9$

Let T_{r+1} be the 4th term from the end.

Then, T_{r+1} is (10 − 4 + 1)th, i.e., 7th, term from the beginning.

$T_7 = T_{6+1}$

$$= {}^9C_6\ (x^{9-6})\left(\frac{2}{x}\right)^6$$

$$= \frac{9\times8\times7}{3\times2}\,(x^3)\left(\frac{64}{x^6}\right)$$

$$= \frac{5376}{x^3}$$

4th term from the beginning = $T_4 = T_{3+1}$

$$T_4 = {}^9C_3\ (x^{9-3})\ \left(\frac{2}{x}\right)^3$$

$$= \frac{9\times8\times7}{3\times2}\,(x^6)\left(\frac{8}{x^3}\right)$$

$= 672\ x^3$

6. Find the 4th term from the end in the expansion of $(4x/5 - 5/2x)^9$.

Solution:

Given:

$(4x/5 - 5/2x)^9$

Let T_{r+1} be the 4th term from the end of the given expression.

Then, T_{r+1} is (10 − 4 + 1)th term, i.e., 7th term, from the beginning.

$T_7 = T_{6+1}$

$$= {}^9C_6\left(\frac{4x}{5}\right)^{9-6}\left(\frac{5}{2x}\right)^6$$

$$= \frac{9\times8\times7}{3\times2}\left(\frac{64}{125}x^3\right)\left(\frac{125\times125}{64x^6}\right)$$

$$= \frac{10500}{x^3}$$

∴ The 4th term from the end is $10500/x^3$.

7. Find the 7th term from the end in the expansion of $(2x^2 - 3/2x)^8$.

Solution:

Given:

$(2x^2 - 3/2x)^8$

Let T_{r+1} be the 4th term from the end of the given expression.

Then, T_{r+1} is (9 − 7 + 1)th term, i.e., 3rd term, from the beginning.

$T_3 = T_{2+1}$

$$= {}^8C_2\left(2x^2\right)^{8-2}\left(-\frac{3}{2x}\right)^2$$

$$= \frac{8\times7}{2\times1}\left(64x^{12}\right)\frac{9}{4x^2}$$

$$= 4032\ x^{10}$$

∴ The 7^{th} term from the end is $4032\ x^{10}$.

8. Find the coefficient of:
(i) x^{10} in the expansion of $(2x^2 – 1/x)^{20}$
(ii) x^7 in the expansion of $(x – 1/x^2)^{40}$
(iii) x^{-15} in the expansion of $(3x^2 – a/3x^3)^{10}$
(iv) x^9 in the expansion of $(x^2 – 1/3x)^9$
(v) x^m in the expansion of $(x + 1/x)^n$
(vi) x in the expansion of $(1 – 2x^3 + 3x^5) (1 + 1/x)^8$
(vii) a^5b^7 in the expansion of $(a – 2b)^{12}$
(viii) x in the expansion of $(1 – 3x + 7x^2) (1 - x)^{16}$
Solution:
(i) x^{10} in the expansion of $(2x^2 – 1/x)^{20}$
Given:
$(2x^2 – 1/x)^{20}$
If x^{10} occurs in the (r + 1)th term in the given expression.
Then, we have:
$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$T_{r+1} = {}^{20}C_r \left(2x^2\right)^{20-r} \left(\frac{-1}{x}\right)^r$$
$$= \left(-1\right)^r\ {}^{20}C_r\ (2^{20-r})\ (x^{40-2r-r})$$

For this term to contain x^{10}, we must have:
40 − 3r =10
3r = 30
r = 10

$$\therefore\ \text{Coefficient of } x^{10} = \left(-1\right)^{10}\ {}^{20}C_{10}\ (2^{20-10}) = {}^{20}C_{10}\ (2^{10})$$

(ii) x^7 in the expansion of $(x – 1/x^2)^{40}$
Given:
$(x – 1/x^2)^{40}$
If x^7 occurs at the (r + 1) th term in the given expression.
Then, we have:
$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$T_{r+1} = {}^{40}C_r\ x^{40-r} \left(\frac{-1}{x^2}\right)^r$$
$$= \left(-1\right)^r\ {}^{40}C_r\ x^{40-r-2r}$$

For this term to contain x^7, we must have:

$40 - 3r = 7$
$3r = 40 - 7$
$3r = 33$
$r = 33/3$
$= 11$

$$\therefore \text{Coefficient of } x^7 = (-1)^{11}\ {}^{40}C_{11} = -{}^{40}C_{11}$$

(iii) x^{-15} in the expansion of $(3x^2 - a/3x^3)^{10}$
Given:
$(3x^2 - a/3x^3)^{10}$
If x^{-15} occurs at the (r + 1)th term in the given expression.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{10}C_r\, (3x^2)^{10-r} \left(\frac{-a}{3x^3}\right)^r$$
$$= (-1)^r\, {}^{10}C_r\ (3^{10-r-r})\,(x^{20-2r-3r})\,(a^r)$$

For this term to contain x^{-15}, we must have:
$20 - 5r = -15$
$5r = 20 + 15$
$5r = 35$
$r = 35/5$
$= 7$

$$\therefore \text{Coefficient of } x^{-15} = (-1)^7\ {}^{10}C_7\, 3^{10-14}\ (a^7) = -\frac{10\times9\times8}{3\times2\times9\times9}a^7 = -\frac{40}{27}a^7$$

(iv) x^9 in the expansion of $(x^2 - 1/3x)^9$
Given:
$(x^2 - 1/3x)^9$
If x^9 occurs at the (r + 1)th term in the above expression.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^9C_r\, (x^2)^{9-r} \left(\frac{-1}{3x}\right)^r$$
$$= (-1)^r\, {}^9C_r\ (x^{18-2r-r})\ \left(\frac{1}{3^r}\right)$$

For this term to contain x^9, we must have:
$18 - 3r = 9$
$3r = 18 - 9$

3r = 9
r = 9/3
= 3

$$\therefore \text{Coefficient of } x^9 = (-1)^3\, {}^9C_3 \frac{1}{3^3} = -\frac{9\times8\times7}{2\times9\times9} = \frac{-28}{9}$$

(v) x^m in the expansion of $(x + 1/x)^n$
Given:
$(x + 1/x)^n$
If x^m occurs at the (r + 1)th term in the given expression.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^nC_r\, x^{n-r}\, \frac{1}{x^r}$$
$$= {}^nC_r\, x^{n-2r}$$

For this term to contain x^m, we must have:
n – 2r = m
2r = n – m
r = (n - m)/2

$$\therefore \text{Coefficient of } x^m = {}^nC_{(n-m)/2} = \frac{n!}{\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}$$

(vi) x in the expansion of $(1 – 2x^3 + 3x^5)(1 + 1/x)^8$
Given:
$(1 – 2x^3 + 3x^5)(1 + 1/x)^8$
If x occurs at the (r + 1)th term in the given expression.
Then, we have:
$(1 – 2x^3 + 3x^5)(1 + 1/x)^8 = (1 – 2x^3 + 3x^5)({}^8C_0 + {}^8C_1 (1/x) + {}^8C_2 (1/x)^2 + {}^8C_3 (1/x)^3 + {}^8C_4 (1/x)^4 + {}^8C_5 (1/x)^5 + {}^8C_6 (1/x)^6 + {}^8C_7 (1/x)^7 + {}^8C_8 (1/x)^8)$
So, ‘x’ occurs in the above expression at $-2x^3.{}^8C_2 (1/x^2) + 3x^5.{}^8C_4 (1/x^4)$
∴ Coefficient of x = -2 (8!/(2!6!)) + 3 (8!/(4! 4!))
= -56 + 210
= 154

(vii) a^5b^7 in the expansion of $(a – 2b)^{12}$
Given:
$(a – 2b)^{12}$
If a^5b^7 occurs at the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^{n}C_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{12}C_r\, a^{12-r}\, (-2b)^r$$
$$= (-1)^r\, {}^{12}C_r\, (a^{12-r})\, (b^r)\, (2^r)$$

For this term to contain a^5b^7, we must have:

$12 - r = 5$

$r = 12 - 5$

$= 7$

$$\therefore \text{ Required coefficient } = (-1)^7\, {}^{12}C_7\, (2^7)$$
$$= -\frac{12\times 11\times 10\times 9\times 8\times 128}{5\times 4\times 3\times 2}$$
$$= -101376$$

(viii) x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$

Given:

$(1 - 3x + 7x^2)(1 - x)^{16}$

If x occurs at the (r + 1)th term in the given expression.

Then, we have:

$(1 - 3x + 7x^2)(1 - x)^{16} = (1 - 3x + 7x^2)({}^{16}C_0 + {}^{16}C_1(-x) + {}^{16}C_2(-x)^2 + {}^{16}C_3(-x)^3 + {}^{16}C_4(-x)^4 + {}^{16}C_5(-x)^5 + {}^{16}C_6(-x)^6 + {}^{16}C_7(-x)^7 + {}^{16}C_8(-x)^8 + {}^{16}C_9(-x)^9 + {}^{16}C_{10}(-x)^{10} + {}^{16}C_{11}(-x)^{11} + {}^{16}C_{12}(-x)^{12} + {}^{16}C_{13}(-x)^{13} + {}^{16}C_{14}(-x)^{14} + {}^{16}C_{15}(-x)^{15} + {}^{16}C_{16}(-x)^{16})$

So, 'x' occurs in the above expression at ${}^{16}C_1(-x) - 3x\,{}^{16}C_0$

$\therefore$ Coefficient of x $= -(16!/(1!\,15!)) - 3(16!/(0!\,16!))$

$= -16 - 3$

$= -19$

9. Which term in the expansion of $\left\{\left(\frac{x}{\sqrt{y}}\right)^{1/3} + \left(\frac{y}{x^{1/3}}\right)^{1/2}\right\}^{21}$ contains x and y to one and the same power.

Solution:

Let us consider T_{r+1} th term in the given expansion contains x and y to one and the same power.

Then we have,

$T_{r+1} = {}^{n}C_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{21}C_r\left[\left(\frac{x}{\sqrt{y}}\right)^{1/3}\right]^{21-r}\left[\left(\frac{y}{x^{1/3}}\right)^{1/2}\right]^r$$
$$= {}^{21}C_r\left(\frac{x^{(21-r)/3}}{x^{r/6}}\right)\left(\frac{y^{r/2}}{y^{(21-r)/6}}\right)$$

$$= {}^{21}C_r\ (x)^{7-r/2}(y)^{2r/3-7/2}$$

If x and y have the same power, then

$7 - r/2 = 2r/3 - 7/2$

$2r/3 + r/2 = 7 + 7/2$

$(4r + 3r)/6 = (14+7)/2$

$7r/6 = 21/2$

$r = (21 \times 6)/(2 \times 7)$

$= 3\times3$

$= 9$

Hence, the required term is the 10^{th} term.

10. Does the expansion of $(2x^2 - 1/x)$ contain any term involving x^9?

Solution:

Given:

$(2x^2 - 1/x)$

If x^9 occurs at the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$T_{r+1} = {}^{20}C_r\left(2x^2\right)^{20-r}\left(\frac{-1}{x}\right)^r$$

$$= \left(-1\right)^r\ {}^{20}C_r(2)^{20-r}\ (x)^{40-2r-r}$$

For this term to contain x^9, we must have

$40 - 3r = 9$

$3r = 40 - 9$

$3r = 31$

$r = 31/3$

It is not possible, since r is not an integer.

Hence, there is no term with x^9 in the given expansion.

11. Show that the expansion of $(x^2 + 1/x)^{12}$ does not contain any term involving x^{-1}.

Solution:

Given:

$(x^2 + 1/x)^{12}$

If x^{-1} occurs at the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$T_{r+1} = {}^{12}C_r \left(x^2\right)^{12-r} \left(\frac{1}{x}\right)^r$

$= {}^{12}C_r \, x^{24-2r-r}$

For this term to contain x^{-1}, we must have

24 – 3r = -1

3r = 24 + 1

3r = 25

r = 25/3

It is not possible, since r is not an integer.

Hence, there is no term with x^{-1} in the given expansion.

12. Find the middle term in the expansion of:

(i) $(2/3x – 3/2x)^{20}$

(ii) $(a/x + bx)^{12}$

(iii) $(x^2 – 2/x)^{10}$

(iv) $(x/a – a/x)^{10}$

Solution:

(i) $(2/3x – 3/2x)^{20}$

We have,

$(2/3x – 3/2x)^{20}$ where, n = 20 (even number)

So the middle term is (n/2 + 1) = (20/2 + 1) = (10 + 1) = 11. ie., 11th term

Now,

$T_{11} = T_{10+1}$

$= {}^{20}C_{10} (2/3x)^{20-10} (3/2x)^{10}$

$= {}^{20}C_{10} \, 2^{10}/3^{10} \times 3^{10}/2^{10} \, x^{10-10}$

$= {}^{20}C_{10}$

Hence, the middle term is ${}^{20}C_{10}$.

(ii) $(a/x + bx)^{12}$

We have,

$(a/x + bx)^{12}$ where, n = 12 (even number)

So the middle term is (n/2 + 1) = (12/2 + 1) = (6 + 1) = 7. ie., 7th term

Now,

$T_7 = T_{6+1}$

$= {}^{12}C_6 \left(\frac{a}{x}\right)^{12-6} \left(bx\right)^6$

$= {}^{12}C_6 \, a^6 \, b^6$

$= \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2} a^6 \, b^6$

$= 924\ a^6b^6$

Hence, the middle term is $924\ a^6b^6$.

(iii) $(x^2 – 2/x)^{10}$

We have,

$(x^2 – 2/x)^{10}$ where, n = 10 (even number)

So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6th term

Now,

$T_6 = T_{5+1}$

$$= {}^{10}C_5 \left(x^2\right)^{10-5} \left(\frac{-2}{x}\right)^5$$

$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2} \times 32x^5$$

$= -8064\ x^5$

Hence, the middle term is $-8064x^5$.

(iv) $(x/a – a/x)^{10}$

We have,

$(x/a – a/x)^{10}$ where, n = 10 (even number)

So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6th term

Now,

$T_6 = T_{5+1}$

$$= {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5$$

$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$$

$= -252$

Hence, the middle term is -252.

13. Find the middle terms in the expansion of:

(i) $(3x – x^3/6)^9$

(ii) $(2x^2 – 1/x)^7$

(iii) $(3x – 2/x^2)^{15}$

(iv) $(x^4 – 1/x^3)^{11}$

Solution:

(i) $(3x – x^3/6)^9$

We have,

$(3x – x^3/6)^9$ where, n = 9 (odd number)

So the middle terms are ((n+1)/2) = ((9+1)/2) = 10/2 = 5 and

((n+1)/2 + 1) = ((9+1)/2 + 1) = (10/2 + 1) = (5 + 1) = 6

The terms are 5th and 6th.

Now,

$T_5 = T_{4+1}$

$$= {}^{9}C_{4} \left(3x\right)^{9-4} \left(\frac{-x^3}{6}\right)^{4}$$

$$= \frac{9\times8\times7\times6}{4\times3\times2} \times 27 \times 9 \times \frac{1}{36\times36} x^{17}$$

$$= \frac{189}{8} x^{17}$$

And,

$T_6 = T_{5+1}$

$$= {}^{9}C_{5} \left(3x\right)^{9-5} \left(\frac{-x^3}{6}\right)^{5}$$

$$= -\frac{9\times8\times7\times6}{4\times3\times2} \times 81 \times \frac{1}{216\times36} x^{19}$$

$$= -\frac{21}{16} x^{19}$$

Hence, the middle term are $189/8\ x^{17}$ and $-21/16\ x^{19}$.

(ii) $(2x^2 – 1/x)^7$

We have,

$(2x^2 – 1/x)^7$ where, n = 7 (odd number)

So the middle terms are ((n+1)/2) = ((7+1)/2) = 8/2 = 4 and

((n+1)/2 + 1) = ((7+1)/2 + 1) = (8/2 + 1) = (4 + 1) = 5

The terms are 4th and 5th.

Now,

$T_4 = T_{3+1}$

$$= {}^{7}C_{3} \left(2x^2\right)^{7-3} \left(\frac{-1}{x}\right)^{3}$$

$$= -\frac{7\times6\times5}{3\times2} \times 16\, x^8 \times \frac{1}{x^3}$$

$$= -560\, x^5$$

And,

$T_5 = T_{4+1}$

$$= {}^{7}C_{4} \left(2x^2\right)^{7-4} \left(\frac{-1}{x}\right)^{4}$$

$$= 35 \times 8 \times x^6 \times \frac{1}{x^4}$$

$$= 280\, x^2$$

Hence, the middle term are $-560x^5$ and $280x^2$.

(iii) $(3x - 2/x^2)^{15}$
We have,
$(3x - 2/x^2)^{15}$ where, n = 15 (odd number)
So the middle terms are ((n+1)/2) = ((15+1)/2) = 16/2 = 8 and
((n+1)/2 + 1) = ((15+1)/2 + 1) = (16/2 + 1) = (8 + 1) = 9
The terms are 8th and 9th.
Now,
$T_8 = T_{7+1}$

$$= {}^{15}C_7 \left(3x\right)^{15-7} \left(\frac{-2}{x^2}\right)^7$$
$$= -\frac{15\times14\times13\times12\times11\times10\times9}{7\times6\times5\times4\times3\times2} \times 3^8 \times 2^7\ x^{8-14}$$
$$= \frac{-6435\times3^8\times2^7}{x^6}$$

And,
$T_9 = T_{8+1}$

$$= {}^{15}C_8 \left(3x\right)^{15-8} \left(\frac{-2}{x^2}\right)^8$$
$$= \frac{15\times14\times13\times12\times11\times10\times9}{7\times6\times5\times4\times3\times2} \times 3^7 \times 2^8 \times x^{7-16}$$
$$= \frac{6435\times3^7\times2^8}{x^9}$$

Hence, the middle term are $(-6435\times3^8\times2^7)/x^6$ and $(6435\times3^7\times2^8)/x^9$.

(iv) $(x^4 - 1/x^3)^{11}$
We have,
$(x^4 - 1/x^3)^{11}$
where, n = 11 (odd number)
So the middle terms are ((n+1)/2) = ((11+1)/2) = 12/2 = 6 and
((n+1)/2 + 1) = ((11+1)/2 + 1) = (12/2 + 1) = (6 + 1) = 7
The terms are 6th and 7th.
Now,
$T_6 = T_{5+1}$

$$= {}^{11}C_5 \left(x^4\right)^{11-5} \left(\frac{-1}{x^3}\right)^5$$
$$= -\frac{11\times10\times9\times8\times7}{5\times4\times3\times2} \times (x)^{24-15}$$
$$= -462\ x^9$$

And,
$T_7 = T_{6+1}$

$$= {}^{11}C_6 \left(x^4\right)^{11-6} \left(\frac{-1}{x^3}\right)^6$$
$$= \frac{11\times10\times9\times8\times7}{5\times4\times3\times2}(x)^{20-18}$$
$$= 462\,x^2$$

Hence, the middle term are $-462x^9$ and $462x^2$.

14. Find the middle terms in the expansion of:
(i) $(x - 1/x)^{10}$
(ii) $(1 - 2x + x^2)^n$
(iii) $(1 + 3x + 3x^2 + x^3)^{2n}$
(iv) $(2x - x^2/4)^9$
(v) $(x - 1/x)^{2n+1}$
(vi) $(x/3 + 9y)^{10}$
(vii) $(3 - x^3/6)^7$
(viii) $(2ax - b/x^2)^{12}$
(ix) $(p/x + x/p)^9$
(x) $(x/a - a/x)^{10}$
Solution:
(i) $(x - 1/x)^{10}$
We have,
$(x - 1/x)^{10}$ where, n = 10 (even number)
So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6th term
Now,
$T_6 = T_{5+1}$

$$= {}^{10}C_5\, x^{10-5} \left(\frac{-1}{x}\right)^5$$
$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$$
$$= -252$$

Hence, the middle term is -252.

(ii) $(1 - 2x + x^2)^n$
We have,
$(1 - 2x + x^2)^n = (1 - x)^{2n}$ where, n is an even number.
So the middle term is (2n/2 + 1) = (n + 1)th term.
Now,
$T_n = T_{n+1}$
$= {}^{2n}C_n (-1)^n (x)^n$
$= (2n)!/(n!)^2 (-1)^n x^n$

Hence, the middle term is $(2n)!/(n!)^2 (-1)^n x^n$.

(iii) $(1 + 3x + 3x^2 + x^3)^{2n}$
We have,
$(1 + 3x + 3x^2 + x^3)^{2n} = (1 + x)^{6n}$ where, n is an even number.
So the middle term is (n/2 + 1) = (6n/2 + 1) = (3n + 1)th term.
Now,
$T_{2n} = T_{3n+1}$
$= {}^{6n}C_{3n}\, x^{3n}$
$= (6n)!/(3n!)^2\, x^{3n}$
Hence, the middle term is $(6n)!/(3n!)^2 x^{3n}$.

(iv) $(2x – x^2/4)^9$
We have,
$(2x – x^2/4)^9$ where, n = 9 (odd number)
So the middle terms are ((n+1)/2) = ((9+1)/2) = 10/2 = 5 and
((n+1)/2 + 1) = ((9+1)/2 + 1) = (10/2 + 1) = (5 + 1) = 6
The terms are 5th and 6th.
Now,
$T_5 = T_{4+1}$

$$= {}^9C_4 \left(2x\right)^{9-4} \left(\frac{-x^2}{4}\right)^4$$
$$= \frac{9\times8\times7\times6}{4\times3\times2} \times 2^5\ \frac{1}{4^4} x^{5+8}$$
$$= \frac{63}{4} x^{13}$$

And,
$T_6 = T_{5+1}$

$$= {}^9C_5 \left(2x\right)^{9-5} \left(\frac{-x^2}{4}\right)^5$$
$$= -\frac{9\times8\times7\times6}{4\times3\times2} \times 2^4\ \frac{1}{4^5} x^{4+10}$$
$$= -\frac{63}{32} x^{14}$$

Hence, the middle term is $63/4\, x^{13}$ and $-63/32\, x^{14}$.

(v) $(x – 1/x)^{2n+1}$
We have,
$(x – 1/x)^{2n+1}$ where, n = (2n + 1) is an (odd number)
So the middle terms are ((n+1)/2) = ((2n+1+1)/2) = (2n+2)/2 = (n + 1) and

((n+1)/2 + 1) = ((2n+1+1)/2 + 1) = ((2n+2)/2 + 1) = (n + 1 + 1) = (n + 2)
The terms are $(n + 1)^{th}$ and $(n + 2)^{th}$.
Now,
$T_n = T_{n+1}$

$$= {}^{2n+1}C_n\, x^{2n+1-n} \times \frac{(-1)^n}{x^n}$$

$$= (-1)^n\, {}^{2n+1}C_n\, x$$

And,
$T_{n+2} = T_{n+1+1}$

$$= {}^{2n+1}C_n\, x^{2n+1-n-1}\, \frac{(-1)^{n+1}}{x^{n+1}}$$

$$= (-1)^{n+1}\, {}^{2n+1}C_n \times \frac{1}{x}$$

Hence, the middle term is $(-1)^n.{}^{2n+1}C_n$ x and $(-1)^{n+1}.{}^{2n+1}C_n$ (1/x).

(vi) $(x/3 + 9y)^{10}$
We have,
$(x/3 + 9y)^{10}$ where, n = 10 is an even number.
So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. i.e., 6th term.
Now,
$T_6 = T_{5+1}$

$$= {}^{10}C_5 \left(\frac{x}{3}\right)^{10-5} (9y)^5$$

$$= \frac{10\times9\times8\times7\times6}{5\times4\times3\times2} \times \frac{1}{3^5} \times 9^5 \times x^5\, y^5$$

$$= 61236\, x^5\, y^5$$

Hence, the middle term is $61236x^5y^5$.

(vii) $(3 – x^3/6)^7$
We have,
$(3 – x^3/6)^7$ where, n = 7 (odd number).
So the middle terms are ((n+1)/2) = ((7+1)/2) = 8/2 = 4 and
((n+1)/2 + 1) = ((7+1)/2 + 1) = (8/2 + 1) = (4 + 1) = 5
The terms are 4^{th} and 5^{th}.
Now,
$T_4 = T_{3+1}$
$= {}^7C_3\, (3)^{7-3}\, (-x^3/6)^3$
$= -105/8\, x^9$
And,
$T_5 = T_{4+1}$

$= {}^9C_4 (3)^{9-4} (-x^3/6)^4$

$$= \frac{7\times6\times5}{3\times2} \times 3^5 \times \frac{1}{6^4}\, x^{12}$$

$$= \frac{35}{48} x^{12}$$

Hence, the middle terms are -105/8 x^9 and 35/48 x^{12}.

(viii) $(2ax - b/x^2)^{12}$

We have,

$(2ax - b/x^2)^{12}$ where, n = 12 is an even number.

So the middle term is (n/2 + 1) = (12/2 + 1) = (6 + 1) = 7. i.e., 7th term.

Now,

$T_7 = T_{6+1}$

$$= {}^{12}C_6\, (2ax)^{12-6} \left(\frac{-b}{x^2}\right)^6$$

$$= \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2\times1} \times \left(\frac{2ab}{x}\right)^6$$

$$= \frac{59136\, a^6 b^6}{x^6}$$

Hence, the middle term is $(59136a^6b^6)/x^6$.

(ix) $(p/x + x/p)^9$

We have,

$(p/x + x/p)^9$ where, n = 9 (odd number).

So the middle terms are ((n+1)/2) = ((9+1)/2) = 10/2 = 5 and

((n+1)/2 + 1) = ((9+1)/2 + 1) = (10/2 + 1) = (5 + 1) = 6

The terms are 5^{th} and 6^{th}.

Now,

$T_5 = T_{4+1}$

$$= {}^9C_4 \left(\frac{p}{x}\right)^{9-4} \left(\frac{x}{p}\right)^4$$

$$= \frac{9\times8\times7\times6}{4\times3\times2\times1} \times \left(\frac{p}{x}\right)$$

$$= \frac{126\, p}{x}$$

And,

$T_6 = T_{5+1}$

$= {}^9C_5 (p/x)^{9-5} (x/p)^5$

$$= \frac{9\times8\times7\times6}{4\times3\times2\times1} \times \left(\frac{x}{p}\right)$$

$$= \frac{126\, x}{p}$$

Hence, the middle terms are 126p/x and 126x/p.

(x) $(x/a - a/x)^{10}$
We have,
$(x/a - a/x)^{10}$ where, n = 10 (even number)
So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6^{th} term
Now,
$T_6 = T_{5+1}$
$$= {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5$$
$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$$
= - 252
Hence, the middle term is -252.

15. Find the term independent of x in the expansion of the following expressions:
(i) $(3/2\ x^2 - 1/3x)^9$
(ii) $(2x + 1/3x^2)^9$
(iii) $(2x^2 - 3/x^3)^{25}$
(iv) $(3x - 2/x^2)^{15}$
(v) $((\sqrt{x}/3) + \sqrt{3}/2x^2)^{10}$
(vi) $(x - 1/x^2)^{3n}$
(vii) $(1/2\ x^{1/3} + x^{-1/5})^8$
(viii) $(1 + x + 2x^3)(3/2x^2 - 3/3x)^9$
(ix) $(\sqrt[3]{x} + 1/2\sqrt[3]{x})^{18}$, x > 0
(x) $(3/2x^2 - 1/3x)^6$
Solution:
(i) $(3/2\ x^2 - 1/3x)^9$
Given:
$(3/2\ x^2 - 1/3x)^9$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$
$$= {}^9C_r \left(\frac{3}{2}x^2\right)^{9-r} \left(\frac{-1}{3x}\right)^r$$
$$= \left(-1\right)^r {}^9C_r \cdot \frac{3^{9-2r}}{2^{9-r}} \times x^{18-2r-r}$$
For this term to be independent of x, we must have
18 – 3r = 0
3r = 18
r = 18/3
= 6

So, the required term is 7th term.
We have,
$T_7 = T_{6+1}$
$= {}^9C_6 \times (3^{9-12})/(2^{9-6})$
$= (9\times8\times7)/(3\times2) \times 3^{-3} \times 2^{-3}$
$= 7/18$
Hence, the term independent of x is 7/18.

(ii) $(2x + 1/3x^2)^9$
Given:
$(2x + 1/3x^2)^9$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$
$$= {}^9C_r \left(2x\right)^{9-r} \left(\frac{1}{3x^2}\right)^r$$
$$= {}^9C_r . \frac{2^{9-r}}{3^r}\, x^{9-r-2r}$$
For this term to be independent of x, we must have
$9 – 3r = 0$
$3r = 9$
$r = 9/3$
$= 3$
So, the required term is 4th term.
We have,
$T_4 = T_{3+1}$
$= {}^9C_3 \times (2^6)/(3^3)$
$= {}^9C_3 \times 64/27$
Hence, the term independent of x is ${}^9C_3 \times 64/27$.

(iii) $(2x^2 – 3/x^3)^{25}$
Given:
$(2x^2 – 3/x^3)^{25}$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$
$= {}^{25}C_r\, (2x^2)^{25-r}\, (-3/x^3)^r$
$= (-1)^r\, {}^{25}C_r \times 2^{25-r} \times 3^r\, x^{50-2r-3r}$
For this term to be independent of x, we must have

$50 - 5r = 0$
$5r = 50$
$r = 50/5$
$= 10$
So, the required term is 11^{th} term.
We have,
$T_{11} = T_{10+1}$
$= (-1)^{10}\ {}^{25}C_{10} \times 2^{25-10} \times 3^{10}$
$= {}^{25}C_{10}\ (2^{15} \times 3^{10})$
Hence, the term independent of x is ${}^{25}C_{10}\ (2^{15} \times 3^{10})$.

(iv) $(3x - 2/x^2)^{15}$
Given:
$(3x - 2/x^2)^{15}$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$
$= {}^{15}C_r\ (3x)^{15-r}\ (-2/x^2)^r$
$= (-1)^r\ {}^{15}C_r \times 3^{15-r} \times 2^r\ x^{15-r-2r}$
For this term to be independent of x, we must have
$15 - 3r = 0$
$3r = 15$
$r = 15/3$
$= 5$
So, the required term is 6^{th} term.
We have,
$T_6 = T_{5+1}$
$= (-1)^5\ {}^{15}C_5 \times 3^{15-5} \times 2^5$
$= -3003 \times 3^{10} \times 2^5$
Hence, the term independent of x is $-3003 \times 3^{10} \times 2^5$.

(v) $((\sqrt{x}/3) + \sqrt{3}/2x^2)^{10}$
Given:
$((\sqrt{x}/3) + \sqrt{3}/2x^2)^{10}$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$= {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\frac{3}{2x^2}\right)^r$$

$$= {}^{10}C_r \cdot \frac{3^{r-\frac{10-r}{2}}}{2^r}\, x^{\frac{10-r}{2}-2r}$$

For this term to be independent of x, we must have

(10-r)/2 – 2r = 0

10 – 5r = 0

5r = 10

r = 10/5

= 2

So, the required term is 3rd term.

We have,

$T_3 = T_{2+1}$

$$= {}^{10}C_2 \times \frac{3^{2-\frac{10-2}{2}}}{2^2}$$

$$= \frac{10\times9}{2\times4\times9}$$

= 90/72

= 15/12

= 5/4

Hence, the term independent of x is 5/4.

(vi) $(x – 1/x^2)^{3n}$

Given:

$(x – 1/x^2)^{3n}$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$= {}^{3n}C_r\, x^{3n-r}\, (-1/x^2)^r$

$= (-1)^r\, {}^{3n}C_r\, x^{3n-r-2r}$

For this term to be independent of x, we must have

3n – 3r = 0

r = n

So, the required term is (n+1)th term.

We have,

$(-1)^n\, {}^{3n}C_n$

Hence, the term independent of x is $(-1)^n\, {}^{3n}C_n$

(vii) $(1/2\, x^{1/3} + x^{-1/5})^8$

Given:
$(1/2\ x^{1/3} + x^{-1/5})^8$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$= {}^8C_r \left(\frac{1}{2}x^{1/3}\right)^{8-r} \left(x^{-1/5}\right)^r$$

$$= {}^8C_r \cdot \frac{1}{2^{8-r}}\ x^{\frac{8-r}{3} - \frac{r}{5}}$$

For this term to be independent of x, we must have
(8-r)/3 – r/5 = 0
(40 – 5r – 3r)/15 = 0
40 – 5r – 3r = 0
40 – 8r = 0
8r = 40
r = 40/8
= 5
So, the required term is 6th term.
We have,
$T_6 = T_{5+1}$
$= {}^8C_5 \times 1/(2^{8-5})$
= (8×7×6)/(3×2×8)
= 7
Hence, the term independent of x is 7.

(viii) $(1 + x + 2x^3)\ (3/2x^2 – 3/3x)^9$
Given:
$(1 + x + 2x^3)\ (3/2x^2 – 3/3x)^9$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$(1 + x + 2x^3)\ (3/2x^2 – 3/3x)^9 =$

$$= (1 + x + 2x^3)\left[\left(\frac{3}{2}x^2\right)^9 - {}^9C_1\left(\frac{3}{2}x^2\right)^8\frac{1}{3x}\ldots + {}^9C_6\left(\frac{3}{2}x^2\right)^3\left(\frac{1}{3x}\right)^6 - {}^9C_7\left(\frac{3}{2}x^2\right)^2\left(\frac{1}{3x}\right)^7\right.$$

By computing we get,
The term independent of x

$$= 1\left[{}^9C_6\frac{3^3}{2^3} \times \frac{1}{3^6}\right] - 2x^3\left[{}^9C_7\frac{3^3}{2^3} \times \frac{1}{3^7} \times \frac{1}{x^3}\right]$$

$$= \left[\frac{9 \times 8 \times 7}{1 \times 2 \times 3} \times \frac{1}{8 \times 27}\right] - 2\left[\frac{9 \times 8}{1 \times 2} - \frac{1}{4 \times 243}\right]$$

= 7/18 – 2/27
= (189 - 36)/486
= 153/486 (divide by 9)
= 17/54
Hence, the term independent of x is 17/54.

(ix) $(\sqrt[3]{x} + 1/2\sqrt[3]{x})^{18}$, x > 0
Given:
$(\sqrt[3]{x} + 1/2\sqrt[3]{x})^{18}$, x > 0
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^{18}C_r \left(x^{1/3}\right)^{18-r} \left(\frac{1}{2\,x^{1/3}}\right)^r$$

$$= {}^{18}C_r \times \frac{1}{2^r}\, x^{\frac{18-r}{3}-\frac{r}{3}}$$

For this term to be independent of r, we must have
(18-r)/3 – r/3 = 0
(18 – r – r)/3 = 0
18 – 2r = 0
2r = 18
r = 18/2
= 9
So, the required term is 10th term.
We have,
$T_{10} = T_{9+1}$
$= {}^{18}C_9 \times 1/2^9$
Hence, the term independent of x is ${}^{18}C_9 \times 1/2^9$.

(x) $(3/2x^2 – 1/3x)^6$
Given:
$(3/2x^2 – 1/3x)^6$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^6C_r \left(\frac{3}{2}x^2\right)^{6-r} \left(\frac{-1}{3x}\right)^r$$

$$= (-1)^r\, {}^6C_r \times \frac{3^{6-r-r}}{2^{6-r}}\, x^{12-2r-r}$$

For this term to be independent of r, we must have

12 – 3r = 0
3r = 12
r = 12/3
= 4
So, the required term is 5th term.
We have,
$T_5 = T_{4+1}$
$= {}^6C_4 \times \frac{3^{6-4-4}}{2^{6-4}}$
$= \frac{6\times5}{2\times1\times4\times9}$
$= \frac{5}{12}$
Hence, the term independent of x is 5/12.

16. If the coefficients of (2r + 4)th and (r - 2)th terms in the expansion of $(1 + x)^{18}$ are equal, find r.
Solution:
Given:
$(1 + x)^{18}$
We know, the coefficient of the r term in the expansion of $(1 + x)^n$ is ${}^nC_{r-1}$
So, the coefficients of the (2r + 4) and (r - 2) terms in the given expansion are ${}^{18}C_{2r+4-1}$ and ${}^{18}C_{r-2-1}$
For these coefficients to be equal, we must have
${}^{18}C_{2r+4-1} = {}^{18}C_{r-2-1}$
${}^{18}C_{2r+3} = {}^{18}C_{r-3}$
2r + 3 = r – 3 (or) 2r + 3 + r – 3 = 18 [Since, ${}^nC_r = {}^nC_s$ => r = s (or) r + s = n]
2r – r = -3 – 3 (or) 3r = 18 – 3 + 3
r = -6 (or) 3r = 18
r = -6 (or) r = 18/3
r = -6 (or) r = 6
∴ r = 6 [since, r should be a positive integer.]

17. If the coefficients of (2r + 1)th term and (r + 2)th term in the expansion of $(1 + x)^{43}$ are equal, find r.
Solution:
Given:
$(1 + x)^{43}$
We know, the coefficient of the r term in the expansion of $(1 + x)^n$ is ${}^nC_{r-1}$
So, the coefficients of the (2r + 1) and (r + 2) terms in the given expansion are ${}^{43}C_{2r+1-1}$ and ${}^{43}C_{r+2-1}$

For these coefficients to be equal, we must have

$^{43}C_{2r+1-1} = {}^{43}C_{r+2-1}$

$^{43}C_{2r} = {}^{43}C_{r+1}$

$2r = r + 1$ (or) $2r + r + 1 = 43$ [Since, $^{n}C_{r} = {}^{n}C_{s} \Rightarrow r = s$ (or) $r + s = n$]

$2r - r = 1$ (or) $3r + 1 = 43$

$r = 1$ (or) $3r = 43 – 1$

$r = 1$ (or) $3r = 42$

$r = 1$ (or) $r = 42/3$

$r = 1$ (or) $r = 14$

$\therefore r = 14$ [since, value '1' gives the same term]

18. Prove that the coefficient of (r + 1)th term in the expansion of $(1 + x)^{n+1}$ is equal to the sum of the coefficients of rth and (r + 1)th terms in the expansion of $(1 + x)^{n}$.

Solution:

We know, the coefficients of (r + 1)th term in $(1 + x)^{n+1}$ is $^{n+1}C_{r}$

So, sum of the coefficients of the rth and (r + 1)th terms in $(1 + x)^{n}$ is

$(1 + x)^{n} = {}^{n}C_{r-1} + {}^{n}C_{r}$

$= {}^{n+1}C_{r}$ [since, $^{n}C_{r+1} + {}^{n}C_{r} = {}^{n+1}C_{r+1}$]

Hence proved.