$$\frac{du}{dx} = \frac{1-x^2}{1-x^2+x^2} \cdot \frac{\sqrt{1-x^2} - \frac{-x^2}{\sqrt{1-x^2}}}{1-x^2} = (1-x^2) \cdot \frac{\frac{1-x^2+x^2}{\sqrt{1-x^2}}}{(1-x^2)^2} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Differentiating v with respect to x

$$\frac{dv}{dx} = \frac{d\,[\cos^{-1}(2x^2-1)]}{dx} = \frac{-1}{\sqrt{1-(2x^2-1)^2}} \cdot \frac{d(2x^2-1)}{dx} = \frac{-1}{\sqrt{1-4x^4-1+4x^2}} \cdot 4x$$

$$\frac{dv}{dx} = \frac{-4x}{\sqrt{4x^2-4x^4}} = \frac{-4x}{2x\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

$$\frac{dv}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

$$\frac{du}{dv} = \frac{du}{dx} \Big/ \frac{dv}{dx}$$

$$\frac{du}{dv} = \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{-2}{\sqrt{1-x^2}}} = \frac{-1}{2}$$

$$\frac{du}{dv} = \frac{-1}{2}$$

Ans. $\frac{-1}{2}$

7. Question

Differentiate $\sin^3 x$ with respect to $\cos^3 x$.

Answer

Given : Let u = $\sin^3 x$ and v = $\cos^3 x$

To differentiate : $\sin^3 x$ with respect to $\cos^3 x$

Formula used : $\frac{d\,(x^n)}{dx} = n.x^{n-1}$

$\frac{d(\sin x)}{dx}$ = cos x

$\frac{d(\cos x)}{dx}$ = - sin x

The CHAIN RULE states that the derivative of f(g(x)) is f′(g(x)).g′(x)

Let u = $\sin^3 x$ and v = $\cos^3 x$

Differentiating u with respect to x

$\frac{du}{dx} = 3\sin^2 x.\frac{d(\sin x)}{dx} = 3\sin^2 x \cos x$

$\frac{du}{dx} = 3\sin^2 x \cos x$

Differentiating v with respect to x

$\frac{dv}{dx} = 3\cos^2 x.\frac{d(\cos x)}{dx} = -3\cos^2 x \ \sin x$

$\frac{dv}{dx} = -3\cos^2 x \ \sin x$

$\frac{du}{dv} = \frac{du}{dx} \Big/ \frac{dv}{dx}$

$\frac{du}{dv} = \frac{3\sin^2 x\cos x}{-3\cos^2 x \sin x} = \frac{\sin x}{-\cos x} = -\tan x$

$\frac{du}{dv} = -\tan x$

Ans. $-\tan x$

8. Question

Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Answer

Given : Let u = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and v = $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

To differentiate : $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Formula used : $\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta} = \tan 3\theta$

$\frac{d(x^n)}{dx} = n.x^{n-1}$

$\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$

$\frac{d(\cos^{-1}x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$

The CHAIN RULE states that the derivative of f(g(x)) is f′(g(x)).g′(x)

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{vdu - udv}{v^2}$$

Let u = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and v = $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Differentiating u with respect to x

$$\frac{du}{dx} = \frac{d\cos^{-1}\frac{1-x^2}{1+x^2}}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{d\left(\frac{1-x^2}{1+x^2}\right)}{dx} = \frac{-(1+x^2)}{\sqrt{(1+x^2)^2-(1-x^2)^2}} \cdot \frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}$$

$$\frac{du}{dx} = \frac{-(1+x^2)}{\sqrt{1+x^4+2x^2-1-x^4+2x^2}} \cdot \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2} = \frac{-(1+x^2)}{\sqrt{4x^2}} \cdot \frac{-4x}{(1+x^2)^2} = \frac{+2}{1+x^2}$$

$$\frac{du}{dx} = \frac{+2}{1+x^2}$$

For v = $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Let x = tan θ

$$\tan^{-1}\frac{3x-x^3}{1-3x^2} = \tan^{-1}\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta} = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x$$

$$\tan^{-1}\frac{3x-x^3}{1-3x^2} = 3\tan^{-1}x$$

Differentiating v with respect to x ,

$$\frac{dv}{dx} = \frac{d(3\tan^{-1}x)}{dx} = \frac{3}{1+x^2}$$

$$\frac{dv}{dx} = \frac{3}{1+x^2}$$

$$\frac{du}{dv} = \frac{du}{dx}\bigg/\frac{dv}{dx}$$

$$\frac{du}{dv} = \frac{\frac{+2}{1+x^2}}{\frac{+3}{1+x^2}} = \frac{2}{3}$$

$$\frac{du}{dv} = \frac{2}{3}$$

Ans. $\frac{2}{3}$

9. Question

Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Answer

Given : Let u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and v = $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

To differentiate : $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Formula used : $\frac{d(x^n)}{dx} = n.x^{n-1}$

$$\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

The CHAIN RULE states that the derivative of f(g(x)) is f'(g(x)).g'(x)

Let u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and v = $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Put x = cot θ or θ = $\cot^{-1} x$ in u

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\frac{\sqrt{1+\cot^2\theta}-1}{\cot\theta} = \tan^{-1}\frac{\operatorname{cosec}\theta-1}{\cot\theta}$$

$$\tan^{-1}\frac{\operatorname{cosec}\theta-1}{\cot\theta} = \tan^{-1}\frac{\frac{1}{\sin\theta}-1}{\cot\theta} = \tan^{-1}\frac{\frac{1-\sin\theta}{\sin\theta}}{\cot\theta} = \tan^{-1}\frac{\frac{1-\sin\theta}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}}$$

$$\tan^{-1}\frac{\frac{1-\sin\theta}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}} = \tan^{-1}\frac{1-\sin\theta}{\cos\theta}$$

We know that 1 - $\sin\theta = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}$ - $2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ and $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

1 - $\sin\theta = (\cos\frac{\theta}{2} - \sin\frac{\theta}{2})^2$

Substituting the above values in $\tan^{-1}\frac{1-\sin\theta}{\cos\theta}$,we get

$$\tan^{-1}\frac{1-\sin\theta}{\cos\theta} = \tan^{-1}\frac{(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})^2}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}} = \tan^{-1}\frac{(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})}$$

$$\tan^{-1}\frac{1-\sin\theta}{\cos\theta} = \tan^{-1}\frac{(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})}{(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})}$$

Dividing by $\cos\frac{\theta}{2}$ on numerator and denominator,we get

$$\tan^{-1}\frac{(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})}{(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})} = \tan^{-1}\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}} = \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{\pi}{4}-\frac{\theta}{2}$$

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \frac{\pi}{4}-\frac{\theta}{2} = \frac{\pi}{4}-\frac{\cot^{-1}x}{2}$$

Differentiating u with respect to x

$$\frac{d(\tan^{-1}\frac{\sqrt{1+x^2}-1}{x})}{dx} = \frac{d(\frac{\pi}{4}-\frac{\cot^{-1}x}{2})}{dx} = \frac{1}{2(1+x^2)}$$

$$\frac{du}{dx} = \frac{1}{2(1+x^2)}$$

$$v = \sin^{-1}\frac{2x}{1+x^2}$$

Put $x = \tan\theta$

$$V = \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}\frac{2\frac{\sin\theta}{\cos\theta}}{\sec^2\theta} = \sin^{-1}\frac{2\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}} = \sin^{-1}(2\sin\theta\cos\theta)$$

$$V = \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$$

$$V = \sin^{-1}\frac{2x}{1+x^2} = 2\tan^{-1}x$$

Differentiating v with respect to x

$$\frac{dv}{dx} = \frac{2}{1+x^2}$$

$$\frac{du}{dv} = \frac{du}{dx}\Big/\frac{dv}{dx}$$

$$\frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}} = \frac{1}{4}$$

$\frac{du}{dv} = \frac{1}{4}$

Ans. $\frac{1}{4}$

10. Question

Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$ when $x \neq 0.$

Answer

Given : Let u = $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ and v = $\cos^{-1}(2x\sqrt{1-x^2})$

To differentiate : $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$

Formula used : $\frac{d(x^n)}{dx} = n.x^{n-1}$

$$\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d(\cos^{-1}x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

The CHAIN RULE states that the derivative of f(g(x)) is f′(g(x)).g′(x)

Let u = $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ and v = $\cos^{-1}(2x\sqrt{1-x^2})$

Substitute x = cosθ in u

u = $\tan^{-1}(\frac{\sqrt{1-x^2}}{x})$ = $\tan^{-1}(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta})$ = $\tan^{-1}(\frac{\sqrt{\sin^2\theta}}{\cos\theta})$

u = $\tan^{-1}(\frac{\sin\theta}{\cos\theta})$ = $\tan^{-1}(\tan\theta)$ = θ

u = θ = $\cos^{-1}x$

Differentiating u with respect to x

$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$

Substitute x = sinθ in v ,

v = $\cos^{-1}(2x\sqrt{1-x^2}) = \cos^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}) = \cos^{-1}(2\sin\theta\sqrt{\cos^2\theta})$

v = $\cos^{-1}(2\sin\theta\sqrt{\cos^2\theta}) = \cos^{-1}(2\sin\theta \,.\cos\theta) = \cos^{-1}(\sin 2\theta)$

v = $\cos^{-1}(\sin 2\theta) = \cos^{-1}(\cos[\frac{\pi}{2} - 2\theta]) = \frac{\pi}{2} - 2\theta$

v = $\frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\sin^{-1}x$

v = $\frac{\pi}{2} - 2\sin^{-1}x$

Differentiating v with respect to x

$\frac{dv}{dx} = \frac{-2}{\sqrt{1-x^2}}$

$\frac{du}{dv} = \frac{du}{dx}\Big/\frac{dv}{dx}$

$\frac{du}{dv} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{-2}{\sqrt{1-x^2}}} = \frac{1}{2}$

Ans. $-\frac{1}{2}$

Exercise 10I

1. Question

Find $\frac{dy}{dx}$, when

$x = at^2$, $y = 2at$

Answer

Theorem: y and x are given in a different variable that is t. We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$\frac{dy}{dt} = \frac{d(2at)}{dt}$

= 2a. ......(1)

$\frac{dx}{dt} = \frac{d(at^2)}{dt}$

= 2at ......(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{2a}{2at}$

$= \frac{1}{t}$

2. Question

Find $\frac{dy}{dx}$, when

x = a cos θ , y = b sin θ

Answer

y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{dbsin\theta}{d\theta}$ ($\frac{dsin\theta}{d\theta}$=cosθ)

= bcosθ. ......(1)

$\frac{dx}{d\theta} = \frac{d(acos\theta)}{d\theta}$ ($\frac{dcos\theta}{d\theta} = -sin\theta$)

= -asinθ ....(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{bcos\theta}{-asin\theta}$ ($\frac{cos\theta}{sin\theta}$ = cotθ)

$= \frac{-bcot\theta}{a}$.

3. Question

Find $\frac{dy}{dx}$, when

x = a cos²θ , y = b sin²θ

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{dbsin^2\theta}{d\theta}$

= b× 2sinθ × cosθ (using the chain rule $\frac{dsin^2\theta}{d\theta}$ = 2sinθ× $\frac{dsin\theta}{d\theta}$ = 2sinθ × cosθ)

= 2bsinθcosθ . ...(1)

$$\frac{dx}{d\theta} = \frac{dacos^2\theta}{d\theta}$$

= a × (2cosθ)× (-sinθ) (using chain rule $\frac{dcos^2\theta}{d\theta}$ = 2cosθ× $\frac{dcos\theta}{d\theta}$ = 2 cosθ × (-sinθ))

= -2asinθcosθ.

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{2bsin\theta cos\theta}{-2asin\theta cos\theta}$

= $\frac{-b}{a}$.

4. Question

Find $\frac{dy}{dx}$, when

x = a cos^{3}θ, y = a sin^{3}θ

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{dasin^3\theta}{d\theta}$

= a× 3 sin^{2}θ × cosθ (using the chain rule $\frac{dsin^3\theta}{d\theta}$ = 3sin^{2}θ× $\frac{dsin\theta}{d\theta}$ = 2sin^{2}θ × cosθ)

= 3asin^{2}θcosθ . ...(1)

$$\frac{dx}{d\theta} = \frac{dacos^3\theta}{d\theta}$$

= a × (3cos^{2}θ)× (-sinθ) (using chain rule $\frac{dcos^2\theta}{d\theta}$ = 2cosθ× $\frac{dcos\theta}{d\theta}$ = 2 cosθ × (-sinθ))

= -3asinθcos^{2}θ.

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{3\,a\sin^2\theta\cos\theta}{-3a\sin\theta\cos^2\theta}$

$= \frac{-sin\theta}{cos\theta}$.

= -tanθ

5. Question

Find $\frac{dy}{dx}$, when

x = a(1 – cos θ), y = a(θ + sin θ)

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{da(\theta+sin\theta)}{d\theta}$

= a× (1+cosθ) ....(1)

$$\frac{dx}{d\theta} = \frac{da(1-cos\theta)}{d\theta}$$

= asinθ. ....(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{a(1+co\theta)}{asin\theta}$

$= \frac{1+cos\theta}{sin\theta}$.

$= \frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$ (1+cosθ=2cos^{2}θ/2 and sinθ = 2sin(θ/2)cos(θ/2))

= cot(θ/2)

6. Question

Find $\frac{dy}{dx}$, when

x = a log t, y = b sin t

Answer

Theorem: y and x are given in a different variable that is t. We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$\frac{dy}{dt} = \frac{dbsint}{dt}$

= bcost ....(1)

$\frac{dx}{dt} = \frac{d(a\ logt)}{dt}$

= $\frac{a}{t}$ ....(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{b\ cost}{a/t}$

= $\frac{bt\ cost}{a}$.

7. Question

Find $\frac{dy}{dx}$, when

x = (log t + cos t), y = (e^t + sin t)

Answer

Theorem: *y* and *x* are given in a different variable that is *t* . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$\frac{dy}{dt} = \frac{d(e^t + sint)}{dt}$

= e^t + cost ....(1) ($\frac{de^t}{dt} = e^t$)

$\frac{dx}{dt} = \frac{d(logt + \ cost)}{dt}$

= $\frac{1}{t}$ - sint. ....(2) ($\frac{d\ logt}{dt} = \frac{1}{t}$)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{e^t + cost}{\frac{1}{t} - sint}$

= $\frac{t(e^t + cost)}{1 - tsint}$.

8. Question

Find $\frac{dy}{dx}$, when

$x = \cos\theta + \cos 2\theta$, $y = \sin\theta + \sin 2\theta$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{d(\sin\theta+\sin 2\theta)}{d\theta}$

= cosθ + cos2θ×2 ....(1) $(using\ chain\ rule\ \frac{d\sin 2\theta}{d\theta} = \cos 2\theta \times \frac{d\,2\theta}{d\theta})$

$$\frac{dx}{d\theta} = \frac{d(\cos\theta + \cos 2\theta)}{d\theta}$$

= -sinθ -2sin2θ ....(2) $(using\ chain\ rule\ \frac{d\cos 2\theta}{d\theta} = \sin 2\theta \times \frac{d\,2\theta}{d\theta})$

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{\cos\theta+2\cos 2\theta}{-(\sin\theta+2\sin 2\theta)}$

9. Question

Find $\frac{dy}{dx}$, when

$x = \sqrt{\sin 2\theta}$, $y = \sqrt{\cos 2\theta}$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dx}{d\theta} = \frac{d\sqrt{\sin 2\theta}}{d\theta}$

$= \frac{2\cos 2\theta}{2\sqrt{\sin 2\theta}}$ $(using\ chain\ rule\ \frac{d\sqrt{\sin 2\theta}}{d\theta} = \frac{1}{2\sqrt{\sin 2\theta}} \times \frac{d\sin 2\theta}{d\theta})$

$\frac{dx}{d\theta} = \frac{\cos 2\theta}{\sqrt{\sin 2\theta}}$ ....(1)

$$\frac{dy}{d\theta} = \frac{d(\sqrt{\cos 2\theta})}{d\theta}$$

$= \frac{-2\sin 2\theta}{2\sqrt{\cos 2\theta}}$ $(using\ chain\ rule\ \frac{d\sqrt{\sin 2\theta}}{d\theta} = \frac{1}{2\sqrt{\sin 2\theta}} \times \frac{d\sin 2\theta}{d\theta})$

$= \frac{-sin2\theta}{\sqrt{cos2\theta}}$ ....(2)

Dividing (2) and (2), we get

$\frac{dy}{dx} = -\frac{sin2\theta/\sqrt{cos2\theta}}{cos2\theta/\sqrt{sin2\theta}}$

$= -\frac{\sqrt{\sin^3 2\theta}}{\sqrt{\cos^3 2\theta}}$

$= -(\tan2\theta)^{3/2}$

10. Question

Find $\frac{dy}{dx}$, when

$x = e^{\theta}(\sin\theta + \cos\theta)$, $y = e^{\theta}(\sin\theta - \theta\cos\theta)$

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{d\,e^{\theta}(sin\theta-cos\theta)}{d\theta}$

$= e^{\theta}(\cos\theta + \sin\theta) + (\sin\theta - \cos\theta)e^{\theta}$ ....(1) {by using product rule, $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$}

$\frac{dx}{d\theta} = \frac{d\,e^{\theta}(sin\theta+cos\theta)}{d\theta}$

$= e^{\theta}(\cos\theta - \sin\theta) + e^{\theta}(\sin\theta + \cos\theta)$ ....(2) {by using product rule, $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$}

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{e^{\theta}(2sin\theta)}{e^{\theta}(2cos\theta)}$

$=\tan\theta$.

11 `. Question

Find $\frac{dy}{dx}$, when

$x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

Answer

Theorem: *y* and *x* are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{d\, a(sin\theta - \theta cos\theta)}{d\theta}$

= a(cosθ - (-θ sinθ + cosθ)) {by using product rule, $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ while differentiating θcosθ }

= a(θsinθ) ....(1)

$$\frac{dx}{d\theta} = \frac{d\, a(cos\theta + \theta sin\theta)}{d\theta}$$

= a(-sinθ +θ cosθ +sinθ) {by using product rule, $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ while differentiating θcosθ }

= a× θcosθ ....(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{a\times\theta sin\theta}{a\times\theta cos\theta}$

= tanθ ANS

12. Question

Find $\frac{dy}{dx}$, when

$$x = \frac{3at}{(1+t^2)},\ y = \frac{3at^2}{(1+t^2)}$$

Answer

Theorem: *y* and *x* are given in a different variable that is *t* . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$\frac{dy}{dt} = \frac{d\frac{3at^2}{(1+t^2)}}{dt}$

$= \frac{(1+t^2)6at - 3at^2(2t)}{(1+t^2)^2}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ }

$= \frac{6at + 6at^3 - 6at^3}{(1+t^2)^2}$

$= \frac{6at}{(1+t^2)^2}$ ....(1)

$$\frac{dx}{dt} = \frac{d\,(\frac{3at}{1+t^2})}{d\theta}$$

$= \frac{(1+t^2)3a-3at(2t)}{(1+t^2)^2}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ }

$= \frac{3a+3at^2-6at^2}{(1+t^2)^2}$

$= \frac{3a-3at^2}{(1+t^2)^2}$ ....(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{6at/(1+t^2)^2}{3a(1-t^2)/(1+t^2)^2}$

$= \frac{2t}{(1-t^2)}$

13. Question

Find $\frac{dy}{dx}$, when

$x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$

Answer

Theorem: *y* and *x* are given in a different variable that is *t* . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$\frac{dy}{dt} = \frac{d\frac{2t}{(1+t^2)}}{dt}$

$= \frac{(1+t^2)2-2t(2t)}{(1+t^2)^2}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ }

$= \frac{2+2t^2-4t^2}{(1+t^2)^2}$

$= \frac{2-2t^2}{(1+t^2)^2}$ ....(1)

$$\frac{dx}{dt} = \frac{d\,(\frac{1-t^2}{1+t^2})}{d\theta}$$

$= \frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ }

$= \frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}$

$= \frac{-4t}{(1+t^2)^2}$ ....(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{2-2t^2/\left(1+t^2\right)^2}{-4t/(1+t^2)^2}$

$= \frac{t^2-1}{(2t)}$

14. Question

Find $\frac{dy}{dx}$, when

$x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$, $y = \sin^{-1}\frac{t}{\sqrt{1+t^2}}$

Answer

Theorem: *y* and *x* are given in a different variable that is *t* . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

Let us assume u= $\frac{t}{\sqrt{(1+t^2)}}$

$\frac{dy}{dt} = \frac{d\sin^{-1}(u)}{dt}$

$= \frac{1}{\sqrt{(1-u^2)}} \times \frac{du}{dt}$

$= \frac{1}{\sqrt{(1-u^2)}} \times \frac{\sqrt{1+t^2}\times 1 - t(2t/2\sqrt{(1+t^2)})}{\left(\sqrt{1+t^2}\right)^2}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ }

Putting value of u

$= \frac{\sqrt{(1+t^2)}}{1} \times \frac{1}{(1+t^2)^{\left(\frac{3}{2}\right)}}$

$= \frac{1}{1+t^2}$ ....(1)

Let assume v= $\frac{1}{\sqrt{(1+t^2)}}$

$$\frac{dx}{dt} = \frac{d\,(\cos^{-1} v)}{dv} \times \frac{dv}{dt}$$

$= \frac{-1}{\sqrt{(1-v^2)}} \times \left(\frac{-1}{\left(\sqrt{1+t^2}\right)^2}\right) \times \frac{2t}{2\sqrt{(1+t^2)}}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ }

Putting value of v

$= \frac{t\sqrt{(1+t^2)}}{t \times (1+t^2)^{\frac{3}{2}}}$

$= \frac{\sqrt{(1+t^2)}}{(1+t^2)^{\frac{3}{2}}}$

$= \frac{1}{(1+t^2)}$ ....(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{1}{1+t^2} \times \frac{(1+t^2)}{1}$

$= 1$

15. Question

If x = 2 cos t – 2 cos^{3}t, y = sin t – 2 sin^{3}t, show that $\frac{dy}{dx}$ =cot t.

Answer

Theorem: *y* and *x* are given in a different variable that is *t* . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$\frac{dy}{dt} = \frac{d(sint - 2\sin^3 t)}{dt}$

$= cost - 6\, sin^2t \times cost$ ....(1) (using chain rule)

$$\frac{dx}{dt} = \frac{d(2cost - 2\cos^3 t)}{dt}$$

$= -2sint + 6cos^2t \times sint$ ....(2) (using chain rule)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{cost(1-6\sin^2 t)}{2sint\,(3\cos^2 t - 1)}$

$= \frac{t(e^t + cost)}{1 - tsint}$.

16. Question

If $x = \frac{1+\log t}{t^2}$ and $y = \frac{3+2\log t}{t}$ $\frac{dy}{dx}$ =t.

Answer

Theorem: y and x are given in a different variable that is t . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$\frac{dy}{dt} = \frac{d(3+2logt)/t}{dt}$

$= \frac{t\left(\frac{2}{t}\right)-(3+2logt)\times 1}{t^2}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$}

$= -\frac{1+2logt}{t^2}$ ....(1)

$\frac{dx}{dt} = \frac{d(1+logt)/t^2}{dt}$

$= \frac{t^2\left(\frac{1}{t}\right)-(2t+2t\ logt)}{t^4}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ }

$= -\frac{2logt+1}{t^3}$ ....(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{-(1+2\ logt)/t^2}{-(1+2\ logt)/t^3}$

= t.

17. Question

If x = a(θ – sin θ), y = a(1 – cos θ), find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$.

Answer

Theorem: y and x are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{d\ a(1-cos\theta)}{d\theta}$

= asinθ ....(1)

$\frac{dx}{d\theta} = \frac{d\ a(\theta - sin\theta\ )}{d\theta}$

= a(1-cosθ) ……….(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{a sin\theta}{a\times(1-cos\theta)}$

Putting θ= π/2

$= \frac{\sin(\pi/2)}{1-\cos(\pi/2)}$

= 1.

18. Question

If x = 2 cosθ – cos 2θ and y = 2 sin θ – sin 2θ, show that $\frac{dy}{dx}$ =tan $\frac{3\theta}{2}$.

Answer

Theorem: *y* and *x* are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{d\,(2sin\theta - sin2\theta)}{d\theta}$

= 2cosθ – 2cos2θ ……….(1)

$\frac{dx}{d\theta} = \frac{d\,(2cos\theta - cos2\theta)}{d\theta}$

= -2sinθ + 2sin2θ ……….(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{2cos\theta - 2cos2\theta}{2sin2\theta - 2sin\theta}$

$= \frac{cos\theta - cos2\theta}{sin2\theta - sin\theta}$

$= \frac{cos\theta - (2\cos^2\theta - 1)}{2sin\theta cos\theta - sin\theta}$ $\{sin2t = 2sintcost\}$ $\{cos2t = 2\cos^2 t - 1\}$

By factorising numerator, we get

$= \frac{(1-cos\theta)(cos\theta + \frac{1}{2})}{2sin\theta(cos\theta - \frac{1}{2})}$

$= \frac{1-cos\theta}{2sin\theta} \times \frac{cos\theta + \frac{1}{2}}{cos\theta - \frac{1}{2}}$ $\{\frac{1-cos\theta}{sin\theta} = \tan(\frac{\theta}{2})\}$

$= \frac{\tan(\frac{\theta}{2})}{1} \times \frac{(2(1-\tan^2(\frac{\theta}{2}))+(1+\tan^2(\frac{\theta}{2}))}{2(1-\tan^2(\frac{\theta}{2}))-(1+\tan^2(\frac{\theta}{2}))}$

Foe simplicity let's take θ/2 as *x*.

$= \frac{tanx}{2} \times \frac{2-2\tan^2 x+1+\tan^2 x}{2-2\tan^2 x-1-\tan^2 x}$

$= \frac{tanx}{2} \times \frac{3-\tan^2 x}{1-3\tan^2 x}$

$= \frac{3tanx-\tan^3 x}{1-3\tan^2 x}\ \frac{3tanx-\tan^3 x}{1-3\tan^2 x} = \tan 3x$

$= \frac{tan3x}{2}\ x = \frac{\theta}{2}$

$= \frac{\tan(\frac{3\theta}{2})}{2}$.

19. Question

If $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$, find $\frac{dy}{dx}$.

Answer

Theorem: *y* and *x* are given in a different variable that is *t* . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then dividing them to get the required thing.

$\frac{dx}{dt} = \frac{d\left(\frac{(\sin^3 t)}{(\sqrt{cos2t})}\right)}{dt}$

$= \frac{\sqrt{cos2t}(3\sin^2 t\times cost)-\sin^3 t\left(\frac{(-sin2t)}{\sqrt{cos2t}}\right)}{cos2t}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ }

$= \frac{cos2t\times(3\sin^2 t\times cost)+\sin^3 t\times(2sintcost)}{(cos2t)^{\frac{3}{2}}}$ $\{sin2t = 2sintcost\}$

$= \frac{\sin^2 t\ cost(3cos2t+2\sin^2 t)}{(cos2t)^{\frac{3}{2}}}$ $\{cos2t = 1 - 2\sin^2 t\}$

$= \frac{\sin^2 t\ cost(3-4\sin^2 t)}{(cos2t)^{\frac{3}{2}}}$

$= \frac{sint\ cost\ (3sint-4\sin^3 t)}{2(cos2t)^{\frac{3}{2}}}$ $\{sin3t = 3sint - 4\sin^3 t\}$

$= \frac{sin2t\times sin3t}{(cos2t)^{\frac{3}{2}}}$ ....(1)

$$\frac{dy}{dt} = \frac{d\frac{\cos^3 t}{\sqrt{cos2t}}}{dv}$$

$= \frac{\sqrt{cos2t}\left(3\cos^2 t\times(-sint)-\cos^3 t\left(\frac{(-sin2t)}{\sqrt{cos2t}}\right)\right)}{cos2t}$ {by using divide rule, $\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ }

$= \frac{cos2t\times(-3\cos^2 t\times sint)+\cos^3 t\times(2sintcost)}{(cos2t)^{\frac{3}{2}}}$ $\{sin2t = 2sintcost\}$

$= \frac{\cos^2 t\ sint(-3cos2t+2\cos^2 t)}{(cos2t)^{\frac{3}{2}}}$ $\{cos2t = 2\cos^2 t - 1\}$

$= \frac{\cos^2 t\ sint(3-4\cos^2 t)}{(cos2t)^{\frac{3}{2}}}$

$= \frac{sint\ cost\ (3cost-4\cos^3 t)}{(cos2t)^{\frac{3}{2}}}$ $\{cos3t = 4\cos^3 t - 3cost\}$

$= -\frac{sin2t\times cos3t}{2(cos2t)^{\frac{3}{2}}}$ ....(1)

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{-\frac{sin2t\times cos3t}{2(cos2t)^{\frac{3}{2}}}}{\frac{sin2t\times sin3t}{(cos2t)^{\frac{3}{2}}}}$$

$= -cot3t$

20. Question

If x = (2 cos θ – cos 2θ) and = (2sin θ – sin 2θ), find $\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}}$.

Answer

here we have to find the double derivative, so to find double derivative we will just differentiate the first derivative once again with a similar method.

Theorem: *y* and *x* are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{d\ (2sin\theta-sin2\theta)}{d\theta}$

= 2cosθ – 2cos2θ ....(1)

$$\frac{dx}{d\theta} = \frac{d\ (2cos\theta - cos2\theta)}{d\theta}$$

= -2sinθ + 2sin2θ ……….(2)

Dividing (1) and (2), we get

$$\frac{dy}{dx} = \frac{cos\theta - cos2\theta}{sin2\theta - sin\theta}$$

$= \tan(\frac{3\theta}{2})$ {as shown in question no. 18}

Let $\frac{dy}{dx} = f'$

$$\frac{d^2\ y}{dx^2} = f''$$

⇒ To find *f''* we will differentiate *f'* with θ and then divide with equation (2).

$$\frac{d\frac{dy}{dx}}{d\theta} = \frac{d\tan(\frac{3\theta}{2})}{d\theta}$$

$$= \frac{\sec^2(\frac{3\theta}{2})}{1} \times \frac{3}{2}$$

Now divide by equation (2).

$$\frac{d^2y}{dx^2} = \frac{3\sec^2(\frac{3\theta}{2})}{4} \times \frac{1}{(sin2\theta - sin\theta)}$$

Putting θ = π/2

$$\frac{d^2y}{dx^2} = \frac{3}{4} \times (-2)$$

$$= -\frac{3}{2}.$$

21. Question

If x = a (θ – sin θ), y = a(1+cos θ), find $\frac{d^2y}{dx^2}$.

Answer

here we have to find the double derivative, so to find double derivative we will just differentiate the first derivative once again with a similar method.

Theorem: *y* and *x* are given in a different variable that is θ . We can find $\frac{dy}{dx}$ by finding $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ and then dividing them to get the required thing.

$\frac{dy}{d\theta} = \frac{d\,a(1+cos\theta)}{d\theta}$

$= -asin\theta$ ....(1)

$\frac{dx}{d\theta} = \frac{d\ a(\theta - sin\theta\ )}{d\theta}$

= a(1-cosθ) ....(2)

Dividing (1) and (2), we get

$\frac{dy}{dx} = \frac{-asin\theta}{a\times(1-cos\theta)}$

$= \frac{-2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{2\sin^2\frac{\theta}{2}}\ \{sin2t = 2sintcost\}\,\{cos2t = 1 - 2\sin^2 t\}$

= -cot (θ/2)

⇒ To find f′′ we will differentiate f′ with θ and then divide with equation (2).

$\frac{d\frac{dy}{dx}}{d\theta} = \frac{cosec^2(\frac{\theta}{2})}{2} \times \frac{1}{a(1-cos\theta)}$

$= \frac{-1}{2\,a\sin^2(\frac{\theta}{2})\times(2\sin^2(\frac{\theta}{2}))}\ \left\{1 - cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)\right\}\,\{cosec^2\theta = \frac{1}{\sin^2\theta}\}$

$= \frac{1}{4a}cosec^4\left(\frac{\theta}{2}\right).$

Exercise 10J

1. Question

Find the second derivate of :

(i) x^{11} (ii) 5^x

(iii) tan x (iv) $\cos^{-1}x$

Answer

(i) x^{11}

Differentiating with respect to x

$f'(x) = 11x^{11-1}$

$f'(x)=11x^{10}$

Differentiating with respect to x

$f''(x) = 110x^{10-1}$

$f''(x) = 110x^9$

(ii) 5^x

Differentiating with respect to x

$f'(x) = 5^x \log_e 5$ [Formula: $a^x = a^x \log_e a$]

Differentiating with respect to x

$f''(x) = \log_e 5 \,.\, 5^x \log_e 5$

$= 5^x(\log_e 5)^2$

(iii) tan x

Differentiating with respect to x

$f'(x) = \sec^2 x$

Differentiating with respect to x

$f''(x) = 2\sec x \,.\, \sec x \tan x$

$= 2\sec^2 x \tan x$

(iv) $\cos^{-1} x$

Differentiating with respect to x

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

Differentiating with respect to x

$$f''(x) = \frac{-1}{2} \times \frac{-1}{(1-x^2)^{\frac{3}{2}}} \times -2x$$

$$= \frac{-x}{(1-x^2)^{\frac{3}{2}}}$$

2. Question

Find the second derivative of:

(i) x sin x

(ii) $e^{2x} \cos 3x$

(iii) $x^3 \log x$

Answer

Differentiating with respect to x

$f'(x) = \sin x + x\cos x$

Differentiating with respect to x

$f''(x) = \cos x + \cos x - x\sin x$

$= -\sin x + 2\cos x$

(ii) $e^{2x} \cos 3x$

Differentiating with respect to x

$f'(x) = 2e^{2x}\cos 3x + e^{2x}(-\sin 3x).3$

$= 2e^{2x}\cos 3x - 3e^{2x}\sin 3x$

Differentiating with respect to x

$f''(x) = 2.2e^{2x}\cos 3x + 2e^{2x}(-\sin 3x).3 - 3.2e^{2x}\sin 3x - 3e^{2x}\cos 3x.3$

$= 4e^{2x}\cos 3x - 6e^{2x}\sin 3x - 6e^{2x}\sin 3x - 9e^{2x}\cos 3x$

$=-12e^{2x}\sin 3x - 5e^{2x}\cos 3x$

(iii) $x^3 \log x$

Differentiating with respect to x

$f'(x) = 3x^2 \log x + \frac{x^3}{x}$

$f'(x) = 3x^2\log x + x^2$

Differentiating with respect to x

$f''(x) = 6x \log x + \frac{3x^2}{x} + 2x$

$= 6x \log x + 3x + 2x$

$= 6x \log x + 5x$

3. Question

If $y = x + \tan x$, show that $\cos^2 . \frac{d^2y}{dx^2} - 2y + 2x = 0$.

Answer

$y = x + \tan x, \Rightarrow \tan x = y-x$.... (i)

Differentiating with respect to x

$\frac{dy}{dx} = 1 + \sec^2 x$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = 2\sec x \,.\sec x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\sec^2 x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2\tan x}{\cos^2 x}$$

$\Rightarrow \cos^2 x \frac{d^2y}{dx^2} = 2\tan x$ [putting value of tan x from (i)]

$$\Rightarrow \cos^2 x \frac{d^2y}{dx^2} = 2y - 2x$$

$$\Rightarrow \cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$$

4. Question

If y = 2 sin x + 3 cos x, s how that $y + \frac{d^2y}{dx^2} = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = 2\cos x - 3\sin x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = -2\sin x - 3\cos x$$

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

Hence Proved

5. Question

If y = 3 cos (log x) + 4 sin (log x), prove that $x^2 y_2 + x y_1 + y = 0$.

Answer

Differentiating with respect to x

$$y_1 = -3\sin(\log x)\frac{1}{x} + 4\cos(\log x)\frac{1}{x}$$

$\Rightarrow y_1 = \frac{-3\sin(\log x) + 4\cos(\log x)}{x}$ [we can also write this as xy_1 = -3sin(log x) +4cos(log x)

Differentiating with respect to x

$$y_2 = \frac{x\left(-3\cos(\log x)\frac{1}{x} - 4\sin(\log x)\frac{1}{x}\right) - (-3\sin(\log x) + 4\cos(\log x))}{x^2}$$

$\Rightarrow x^2y_2 = \frac{-x}{x}(3\cos(\log x) - 4\sin(\log x)) - (y_1x)$

$\Rightarrow x^2y_2 = -y - xy_1$

$\Rightarrow x^2y_2 + xy_1 + y = 0$

Hence Proved

6. Question

If y = e⁻ cos x , show that $\frac{d^2y}{dx^2}$ = $2e^{-x}$sin x .

Answer

Differentiating with respect to x

$\frac{dy}{dx}\frac{dy}{dx} = -e^{-x}\cos x x + e^{-x}(-\sin x x)$

$\Rightarrow \frac{dy}{dx} = -ee^{-xx}\cos x - e^{-x}\sin x$

$\Rightarrow \frac{dy}{dx} = -e^{-x}(\cos x + \sin x)$

Differentiating with respect to x

$\frac{d^2y}{dx^2} = e^{-x}(\cos x + \sin x) - e^{-x}(-\sin x + \cos x)$

$\Rightarrow \frac{d^2y}{dx^2} = e^{-x}(\cos x + \sin x - (-\sin x) - \cos x)$

$\Rightarrow \frac{d^2y}{dx^2} = e^{-x}(\sin x + \sin x)$

$\Rightarrow \frac{d^2y}{dx^2} = 2e^{-x}\sin x$

Hence proved

7. Question

If y = sec x – tan x. show that (cos x) $\frac{d^2y}{dx^2}$ = y^2.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \sec x \tan x - \sec^2 x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec x\ \tan x \times \tan x + \sec x \times \sec^2 x - 2\sec x \times \sec x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec x \tan^2 x + \sec^3 x - 2\sec^2 x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec x\,(\tan^2 x + \sec^2 x\ - 2\sec x \tan x\,)$$

$$\Rightarrow \frac{1}{\sec x}\frac{d^2y}{dx^2} = (\sec x - \tan x)^2$$

$$\Rightarrow \cos x\frac{d^2y}{dx^2} = y^2$$

Hence Proved

8. Question

If y = (cosec x + cot x), prove that (sin x) $\frac{d^2y}{dx^2}$ - y^2 = 0.

Answer

$$\frac{dy}{dx} = -\text{cosec}\,x \cot x - \text{cosec}^2 x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \text{cosec}\,x \cot^2 x + \text{cosec}^3 x + 2\text{cosec}\,x \times \text{cosec}\,x \cot x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \text{cosec}\,x\,(\cot^2 x + \text{cosec}^2 x + 2\text{cosec}\,x \cot x)$$

$$\Rightarrow \frac{1}{\text{cosec}\,x}\frac{d^2y}{dx^2} = (\cot x + \text{cosec}\,x)^2$$

$$\Rightarrow \sin x\frac{d^2y}{dx^2} = y^2$$

$\Rightarrow \sin x \frac{d^2y}{dx^2} - y^2 = 0$

Hence proved

9. Question

If $y = \tan^{-1}$, show that $(1 + x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

$\Rightarrow (1 + x^2) \frac{dy}{dx} = 1$

Differentiating with respect to x

$$(1 + x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$$

Hence Proved

10. Question

If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + (\tan)\frac{dy}{dx} + y \cos^2 x = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \cos(\sin x)\ \cos x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = -\sin(\sin x) \cos x \cos x - \sin x \cos(\sin x)$$

$\Rightarrow \frac{d^2y}{dx^2} = -y\cos^2 x - \sin x \frac{\frac{dy}{dx}}{\cos x}$

$\Rightarrow \frac{d^2y}{dx^2} = -y\cos^2 x - \tan x \frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} + y\cos^2 x + \tan x \frac{dy}{dx} = 0$

Hence Proved

11. Question

If y = a cos (log x) + b sin (log x), prove that $x^2y_2 + xy_1 + y = 0$.

Answer

Differentiating with respect to x

$y_1 = -\,a\sin(\log x)\frac{1}{x}$ [can also be written as $-xy_1$= a sin (log x)]

Differentiating with respect to x

$$y_2 = \frac{-x\,a\cos(\log x)\frac{1}{x} + a\sin(\log x)}{x^2}$$

$\Rightarrow x^2y_2 = -y - xy_1$

$\Rightarrow x^2y_2 + xy_1 + y = 0$

Hence Proved

12. Question

Find the second derivative of e^{3x} sin 4x .

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = 3e^{3x}\sin 4x + 4e^{3x}\cos 4x$$

Differentiating with respect to x

$\Rightarrow \frac{d^2y}{dx^2} = 9e^{3x}\sin 4x + 12e^{3x}\cos 4x + 12e^{3x}\cos 4x - 16e^{3x}\sin 4x$

$\Rightarrow \frac{d^2y}{dx^2} = 24e^{3x}\cos 4x - 7e^{3x}\sin x$

$\Rightarrow \frac{d^2y}{dx^2} = e^{3x}(24\cos x - 7\sin x)$

13. Question

Find the second derivative of sin 3 x cos 5x.

Answer

$$y = \frac{1}{2}[\sin(5x + 3x) + \sin(5x - 3x)]$$

$$y = \frac{1}{2}\sin 8x + \frac{1}{2}\sin 2x$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{8}{2}\cos 8x + \frac{2}{2}\cos 2x$$

$$\Rightarrow \frac{dy}{dx} = 4\cos 8x + \cos 2x$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = -32\sin 8x - 2\sin 2x$$

Hence Proved

14. Question

If y = $e^{\tan}$, prove that ($\cos^2 x$) $\frac{d^2y}{dx^2}$ - (1+ sin 2x). $\frac{dy}{dx}$ = 0.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \sec^2 x\, e^{\tan x}$$

$$\Rightarrow \frac{1}{\sec^2 x}\frac{dy}{dx} = e^{\tan x}$$

$$\Rightarrow \cos^2 x\,\frac{dy}{dx} = e^{\tan x}$$

Differentiating with respect to x

$$(\cos^2 x)\frac{d^2y}{dx^2} - (2\cos x \sin x)\frac{dy}{dx} = \sec^2 x\, e^{\tan x}$$

$$\Rightarrow (\cos^2 x)\frac{d^2y}{dx^2} - \sin 2x\frac{dy}{dx} = \frac{dy}{dx}$$

$$\Rightarrow (\cos^2 x)\frac{d^2y}{dx^2} - \sin 2x\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\Rightarrow (\cos^2 x)\frac{d^2y}{dx^2} - (\sin 2x + 1)\frac{dy}{dx} = 0$$

Hence Proved

15. Question

If $y = \frac{\log x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{(2\log x - 3)}{x^3}$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{\frac{1}{x} \times x - \log x}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - \log x}{x^2}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{\frac{-1}{x} \times x^2 - 2x(1 - \log x)}{x^4}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x - 2x(1 - \log x)}{x^4}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1 - 2 + 2\log x}{x^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(2\log x - 3)}{x^3}$$

Hence proved

16. Question

If $y = e^{ax} \cos bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2) = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = ae^{ax}\cos bx - be^{ax}\sin bx$$

$$be^{ax}\sin bx = ae^{ax}\cos bx - \frac{dy}{dx}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = a^2e^{ax}\cos bx - abe^{ax}\sin bx - abe^{ax}\sin bx - b^2e^{ax}\cos bx$$

$$\Rightarrow \frac{d^2y}{dx^2} = a^2e^{ax}\cos bx - 2abe^{ax}\sin bx - b^2e^{ax}\cos bx$$

$\Rightarrow \frac{d^2y}{dx^2} = a^2e^{ax}\cos bx - 2a\left(ae^{ax}\cos bx - \frac{dy}{dx}\right) - b^2e^{ax}\cos bx$

$\Rightarrow \frac{d^2y}{dx^2} = a^2e^{ax}\cos bx - 2a^2e^{ax}\cos bx + 2a\frac{dy}{dx} - b^2e^{ax}\cos bx \Rightarrow$
$\frac{d^2y}{dx^2} = -a^2e^{ax}\cos bx - b^2e^{ax}\cos bx + 2a\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = -(a^2 + b^2)(e^{ax}\cos bx) + 2a\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = -(a^2 + b^2)y + 2a\frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$

Hence Proved

17. Question

If $y = e^{a \cos^{-1}x}$, $-1 \le x \le 1$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.

Answer

Taking log on both sides

$\log y = a\cos^{-1}x \log e$

$\log y = a\cos^{-1}x$

Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{-a}{\sqrt{1 - x^2}}$$

$\Rightarrow \frac{dy}{dx} = \frac{-ae^{a\cos^{-1}x}}{\sqrt{1-x^2}}$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{\frac{a^2e^{a\cos^{-1}x}}{\sqrt{1-x^2}} \times \sqrt{1-x^2} - ae^{\cos^{-1}x} \times \frac{2x}{2\sqrt{1-x^2}}}{(1-x^2)}$$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = a^2e^{\cos^{-1}x} - \frac{axe^{\cos^{-1}x}}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = a^2y + x\frac{dy}{dx}$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - a^2y - x\frac{dy}{dx} = 0$

Hence Proved

18. Question

If $x = at^2$ and $y = 2at$, find $\frac{d^2y}{dx^2}$ at $t = 2$.

Answer

Differentiating with t

$$\frac{dx}{dt} = 2at\ \frac{dy}{dt} = 2a$$

$$\frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{-1}{t^2}\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4} \times \frac{1}{2at}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4} \times \frac{1}{4a}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{16a}$$

19. Question

If $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \pi$.

Answer

Differentiating with respect to θ

$$\frac{dx}{d\theta} = a(1 - \cos\theta)\ \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)}$$

$\Rightarrow \frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{\cos\theta(1-\cos\theta) - \sin^2\theta}{(1-\cos\theta)^2} \times \frac{d\theta}{dx}$$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^2} \times \frac{1}{a(1-\cos\theta)}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos\theta - 1}{(1-\cos\theta)^2} \times \frac{1}{a(1-\cos\theta)}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-(1-\cos\theta)}{(1-\cos\theta)^2} \times \frac{1}{a(1-\cos\theta)}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{a(1-\cos\theta)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{a(1-(-1))^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4a}$

20. Question

If y = sin (log x), prove that $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

Answer

Differentiating with respect to

$$\frac{dy}{dx} = \cos(\log x)\,\frac{1}{x}$$

$\Rightarrow x\frac{dy}{dx} = \cos(\log x)$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{-\sin(\log x)\frac{1}{x}x - \cos(\log x)}{x^2}$$

$\Rightarrow x^2\frac{d^2y}{dx^2} = -\sin(\log x) - \cos(\log x)$

$\Rightarrow x^2\frac{d^2y}{dx^2} = -y - x\frac{dy}{dx}$

$\Rightarrow x^2\frac{d^2y}{dx^2}+y+x\frac{dy}{dx}=0$

Hence Proved

21. Question

If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, show that $(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$.

Answer

$\sqrt{1-x^2}\,y=\sin^{-1}x$

Differentiating with respect to x

$\sqrt{1-x^2}\frac{dy}{dx}-\frac{2xy}{2\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\frac{dy}{dx}-xy=1$

Differentiating with respect to x

$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-x\frac{dy}{dx}-y=0$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0$

Hence Proved

22. Question

If $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

Answer

$y=e^x\sin x$

Differentiating with respect to x

$\frac{dy}{dx}=e^x\sin x+e^x\cos x$

$\left[e^x\cos x=\frac{dy}{dx}-e^x\sin x\right]$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2e^x \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 2e^x \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 2y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

23. Question

If x = $a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$ and y = a sin θ, show that the value of $\frac{d^2y}{dx^2}$ at θ = $\frac{\pi}{4}$ is $\frac{2\sqrt{2}}{a}$.

Answer

$$\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{\sec^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}\right) \frac{dy}{d\theta} = a\cos\theta$$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right)$$

$$= a\left(\frac{-\sin^2\theta + 1}{\sin\theta}\right)$$

$$= \frac{a\cos^2\theta}{\sin\theta}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\frac{dy}{dx} = a\cos\theta \times \frac{\sin\theta}{a\cos^2\theta}$$

$$\frac{dy}{dx} = \tan\theta$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec^2\theta \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\sqrt{2}\right)^2 \times \frac{\sin\theta}{a\cos^2\theta}$$

$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \frac{\frac{1}{\sqrt{2}}}{a\left(\frac{1}{\sqrt{2}}\right)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{2\sqrt{2}}{a}$

24. Question

If x = cos t + log tan $\frac{t}{2}$, y = sin t then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at t = $\frac{\pi}{4}$.

Answer

$\frac{dx}{dt} = -\sin t + \frac{\sec^2\frac{t}{2}}{2\tan\frac{t}{2}}\ \frac{dy}{dt} = \cos t$

$= -\sin t + \frac{1}{\sin t}$

$= \frac{-\sin^2 t + 1}{\sin t}$

$= \frac{\cos^2 t}{\sin t}$

$\frac{dy}{dt} = \cos t$

Differentiating with respect to t

$\Rightarrow \frac{d^2y}{dt^2} = -\sin t$ [Putting t = π /4]

$\Rightarrow \frac{d^2y}{dt^2} = -\frac{1}{\sqrt{2}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dx}{dt}$

$\Rightarrow \frac{dy}{dx} = \cos t \times \frac{\sin t}{\cos^2 t}$

$\Rightarrow \frac{dy}{dx} = \tan t$

Differentiating with respect to x

$\frac{d^2y}{dx^2} = \sec^2 t \frac{dt}{dx}$ [Putting t = π /4]

$\Rightarrow \frac{d^2y}{dx^2} = (\sqrt{2}) \times \frac{\sin t}{\cos^2 t}$

$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = 2\sqrt{2}$

25. Question

If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.

Answer

$y = x^x$

Taking log on both sides

log y = x log x

Differentiating with respect to x

$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$ ...(i)

$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$

Differentiating with respect to x

$\frac{d^2y}{dx^2} = \frac{y}{x} + (1 + \log x)\frac{dy}{dx}$ [putting value of (1 + log x) from (i)]

$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2$

$\Rightarrow \frac{d^2y}{dx^2} - \frac{y}{x} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 0$

Hence Proved

26. Question

If $y = (\cot^{-1}\ )^2$, then show that $(x^2 + 1)^2 \frac{d^2y}{dx^2} + 2x(x^2 + 1)\frac{dy}{dx} = 2$.

Answer

$y = (\cot^{-1} x)^2$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{-2\cot^{-1}x}{1+x^2}$$

$$\Rightarrow -2\cot^{-1}x = (1+x^2)\frac{dy}{dx}$$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{2+4x\ \cot^{-1}x}{(1+x^2)^2}$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} - 4x\cot^{-1}x = 2$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} - 2x\left(-(1+x^2)\frac{dy}{dx}\right) = 2$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$$

Hence Proved

27. Question

If $y = \left\{x + \sqrt{x^2+1}\right\}^m$, then show that $(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$.

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = m\left\{x + \sqrt{x^2+1}\right\}^{m-1}\left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = m\left\{x + \sqrt{x^2+1}\right\}^{m-1}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = m\frac{\left\{x+\sqrt{x^2+1}\right\}^m}{\sqrt{x^2+1}}$$

$$\Rightarrow \frac{dy}{dx} = m\frac{y}{\sqrt{x^2+1}}$$

$[\frac{dy}{dx}\sqrt{x^2+1} = my]$

Differentiating with respect to x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{m\frac{dy}{dx}\sqrt{1+x^2} - \frac{2xmy}{2\sqrt{x^2+1}}}{(1+x^2)}$$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} = m^2y - x\frac{dy}{dx}$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$

Hence Proved

28. Question

If y = log [x + $\left[x+\sqrt{x^2+a^2}\right]$, then prove that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Answer

$$\frac{dy}{dx} = \frac{1+\frac{2x}{2\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}$$

$\Rightarrow \frac{dy}{dx} = \frac{2\sqrt{x^2+a^2}+2x}{2\sqrt{x^2+a^2}} \times \frac{1}{x+\sqrt{x^2+a^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2+a^2}}$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \frac{-2x}{2(x^2+a^2)\sqrt{x^2+a^2}}$$

$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2} = \frac{-x}{\sqrt{x^2+a^2}}$

$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2} = -x\frac{dy}{dx}$

$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Hence Proved

29. Question

If x = a(cos θ + θ sin θ) and y = a(sin θ - θ cos θ), show that $\frac{d^2y}{dx^2} = \frac{1}{a}\left(\frac{\sec^3\theta}{\theta}\right)$

Answer

Differentiating with respect to θ

$$\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)\ \frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$$

$\Rightarrow \frac{dx}{d\theta} = a\theta\cos\theta \Rightarrow \frac{dy}{d\theta} = a\theta\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$\Rightarrow \frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow \frac{dy}{dx} = \tan\theta$

Differentiating with respect to x

$$\frac{d^2y}{dx^2} = \sec^2\theta \times \frac{d\theta}{dx}$$

$\Rightarrow \frac{d^2y}{dx^2} = \sec^2\theta \times \frac{1}{a\theta\cos\theta}$

$\Rightarrow \frac{d^2y}{dx^2} = \sec^2\theta \times \frac{\sec\theta}{a\theta}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta}$

Hence Proved

30. Question

If x = a cos θ + b sin θ and y = a sin θ – b cos θ , show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

Answer

$$\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta \quad \frac{dy}{d\theta} = a\cos\theta + b\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$\Rightarrow \frac{dy}{dx} = \frac{a\cos\theta + b\sin\theta}{-a\sin\theta + b\cos\theta}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$

Differentiating with respect to x

$\Rightarrow \frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2}$

$\Rightarrow y^2\frac{d^2y}{dx^2} = y - x\frac{dy}{dx}$

$$\Rightarrow y^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Hence Proved