

## EXERCISE 9.2

**Prove that:**

**1.  $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$**

**Solution:**

Let us consider LHS:

$$\sin 5x$$

Now,

$$\sin 5x = \sin (3x + 2x)$$

But we know,

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \dots (i)$$

So,

$$\begin{aligned} \sin 5x &= \sin 3x \cos 2x + \cos 3x \sin 2x \\ &= \sin (2x + x) \cos 2x + \cos (2x + x) \sin 2x \dots (ii) \end{aligned}$$

And

$$\cos (x + y) = \cos x \cos y - \sin x \sin y \dots (iii)$$

Now substituting equation (i) and (iii) in equation (ii), we get

$$\begin{aligned} \sin 5x &= (\sin 2x \cos x + \cos 2x \sin x) \cos 2x + (\cos 2x \cos x - \sin 2x \sin x) \sin 2x \\ &= \sin 2x \cos 2x \cos x + \cos^2 2x \sin x + (\sin 2x \cos 2x \cos x - \sin^2 2x \sin x) \\ &= 2\sin 2x \cos 2x \cos x + \cos^2 2x \sin x - \sin^2 2x \sin x \dots (iv) \end{aligned}$$

$$\text{Now } \sin 2x = 2\sin x \cos x \dots (v)$$

$$\text{And } \cos 2x = \cos^2 x - \sin^2 x \dots (vi)$$

Substituting equation (v) and (vi) in equation (iv), we get

$$\begin{aligned} \sin 5x &= 2(2\sin x \cos x) (\cos^2 x - \sin^2 x) \cos x + (\cos^2 x - \sin^2 x)^2 \sin x - (2\sin x \cos x)^2 \sin x \\ &= 4(\sin x \cos^2 x) ([1 - \sin^2 x] - \sin^2 x) + ([1 - \sin^2 x] - \sin^2 x)^2 \sin x - (4\sin^2 x \cos^2 x) \sin x \end{aligned}$$

$$\text{(as } \cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x)$$

$$\begin{aligned} \sin 5x &= 4(\sin x [1 - \sin^2 x]) (1 - 2\sin^2 x) + (1 - 2\sin^2 x)^2 \sin x - 4\sin^3 x [1 - \sin^2 x] \\ &= 4\sin x (1 - \sin^2 x) (1 - 2\sin^2 x) + (1 - 4\sin^2 x + 4\sin^4 x) \sin x - 4\sin^3 x + 4\sin^5 x \\ &= (4\sin x - 4\sin^3 x) (1 - 2\sin^2 x) + \sin x - 4\sin^3 x + 4\sin^5 x - 4\sin^3 x + 4\sin^5 x \\ &= 4\sin x - 8\sin^3 x - 4\sin^3 x + 8\sin^5 x + \sin x - 8\sin^3 x + 8\sin^5 x \\ &= 5\sin x - 20\sin^3 x + 16\sin^5 x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

**2.  $4 (\cos^3 10^\circ + \sin^3 20^\circ) = 3 (\cos 10^\circ + \sin 20^\circ)$**

**Solution:**

Let us consider LHS:

$$4 (\cos^3 10^\circ + \sin^3 20^\circ)$$

We know that,  $\sin 60^\circ = \sqrt{3}/2 = \cos 30^\circ$

$$\sin 30^\circ = \cos 60^\circ = 1/2$$

So,

$$\sin (3 \times 20^\circ) = \cos (3 \times 10^\circ)$$

$$3 \sin 20^\circ - 4 \sin^3 20^\circ = 4 \cos^3 10^\circ - 3 \cos 10^\circ$$

(we know,  $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$  and  $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$ )

So,

$$4(\cos^3 10^\circ + \sin^3 20^\circ) = 3(\sin 20^\circ + \cos 10^\circ)$$

= RHS

Hence proved.

$$3. \cos^3 x \sin 3x + \sin^3 x \cos 3x = 3/4 \sin 4x$$

**Solution:**

We know that,

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\text{So, } 4 \cos^3 \theta = \cos 3\theta + 3 \cos \theta$$

$$\cos^3 \theta = [\cos 3\theta + 3 \cos \theta]/4 \dots\dots (i)$$

Similarly,

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

$$4 \sin^3 \theta = 3 \sin \theta - \sin 3\theta$$

$$\sin^3 \theta = [3 \sin \theta - \sin 3\theta]/4 \dots\dots\dots (ii)$$

Now,

Let us consider LHS:

$$\cos^3 x \sin 3x + \sin^3 x \cos 3x$$

Substituting the values from equation (i) and (ii), we get

$$\begin{aligned} \cos^3 x \sin 3x + \sin^3 x \cos 3x &= (\cos 3x + 3 \cos x)/4 \sin 3x + (3 \sin x - \sin 3x)/4 \cos 3x \\ &= 1/4 (\sin 3x \cos 3x + 3 \sin 3x \cos x + 3 \sin x \cos 3x - \sin 3x \cos 3x) \\ &= 1/4 (3(\sin 3x \cos x + \sin x \cos 3x) + 0) \\ &= 1/4 (3 \sin (3x + x)) \\ &\text{(We know, } \sin(x + y) = \sin x \cos y + \cos x \sin y) \\ &= 3/4 \sin 4x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$4. \sin 5x = 5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x$$

**Solution:**

Let us consider LHS:

$$\sin 5x$$

Now,

$$\sin 5x = \sin (3x + 2x)$$

But we know,

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \dots (i)$$

So,

$$\begin{aligned} \sin 5x &= \sin 3x \cos 2x + \cos 3x \sin 2x \\ &= \sin (2x + x) \cos 2x + \cos (2x + x) \sin 2x \dots (ii) \end{aligned}$$

And

$$\cos (x + y) = \cos x \cos y - \sin x \sin y \dots (iii)$$

Now substituting equation (i) and (iii) in equation (ii), we get

$$\sin 5x = (\sin 2x \cos x + \cos 2x \sin x) \cos 2x + (\cos 2x \cos x - \sin 2x \sin x) \sin 2x \dots (iv)$$

$$\text{Now } \sin 2x = 2 \sin x \cos x \dots (v)$$

$$\text{And } \cos 2x = \cos^2 x - \sin^2 x \dots (vi)$$

Substituting equation (v) and (vi) in equation (iv), we get

$$\begin{aligned} \sin 5x &= [(2 \sin x \cos x) \cos x + (\cos^2 x - \sin^2 x) \sin x] (\cos^2 x - \sin^2 x) + [(\cos^2 x - \sin^2 x) \cos x - (2 \sin x \cos x) \sin x] (2 \sin x \cos x) \\ &= [2 \sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x] (\cos^2 x - \sin^2 x) + [\cos^3 x - \sin^2 x \cos x - 2 \sin^2 x \cos x] (2 \sin x \cos x) \\ &= \cos^2 x [3 \sin x \cos^2 x - \sin^3 x] - \sin^2 x [3 \sin x \cos^2 x - \sin^3 x] + 2 \sin x \cos^4 x - 2 \sin^3 x \cos^2 x - 4 \sin^3 x \cos^2 x \\ &= 3 \sin x \cos^4 x - \sin^3 x \cos^2 x - 3 \sin^3 x \cos^2 x - \sin^5 x + 2 \sin x \cos^4 x - 2 \sin^3 x \cos^2 x - 4 \sin^3 x \cos^2 x \\ &= 5 \sin x \cos^4 x - 10 \sin^3 x \cos^2 x + \sin^5 x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

**5.  $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$**

**Solution:**

Let us consider LHS:

$$\sin 5x$$

Now,

$$\sin 5x = \sin (3x + 2x)$$

But we know,

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \dots (i)$$

So,

$$\begin{aligned} \sin 5x &= \sin 3x \cos 2x + \cos 3x \sin 2x \\ &= \sin (2x + x) \cos 2x + \cos (2x + x) \sin 2x \dots (ii) \end{aligned}$$

And

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \dots\dots(iii)$$

Now substituting equation (i) and (iii) in equation (ii), we get

$$\begin{aligned} \sin 5x &= (\sin 2x \cos x + \cos 2x \sin x) \cos 2x + (\cos 2x \cos x - \sin 2x \sin x) \sin 2x \\ &= \sin 2x \cos 2x \cos x + \cos^2 2x \sin x + (\sin 2x \cos 2x \cos x - \sin^2 2x \sin x) \\ &= 2\sin 2x \cos 2x \cos x + \cos^2 2x \sin x - \sin^2 2x \sin x \dots\dots(iv) \end{aligned}$$

$$\text{Now } \sin 2x = 2\sin x \cos x \dots\dots(v)$$

$$\text{And } \cos 2x = \cos^2 x - \sin^2 x \dots\dots(vi)$$

Substituting equation (v) and (vi) in equation (iv), we get

$$\begin{aligned} \sin 5x &= 2(2\sin x \cos x) (\cos^2 x - \sin^2 x) \cos x + (\cos^2 x - \sin^2 x)^2 \sin x - (2\sin x \cos x)^2 \sin x \\ &= 4(\sin x \cos^2 x) ([1 - \sin^2 x] - \sin^2 x) + ([1 - \sin^2 x] - \sin^2 x)^2 \sin x - (4\sin^2 x \cos^2 x) \sin x \end{aligned}$$

$$\text{(as } \cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x)$$

$$\begin{aligned} \sin 5x &= 4(\sin x [1 - \sin^2 x]) (1 - 2\sin^2 x) + (1 - 2\sin^2 x)^2 \sin x - 4\sin^3 x [1 - \sin^2 x] \\ &= 4\sin x (1 - \sin^2 x) (1 - 2\sin^2 x) + (1 - 4\sin^2 x + 4\sin^4 x) \sin x - 4\sin^3 x + 4\sin^5 x \\ &= (4\sin x - 4\sin^3 x) (1 - 2\sin^2 x) + \sin x - 4\sin^3 x + 4\sin^5 x - 4\sin^3 x + 4\sin^5 x \\ &= 4\sin x - 8\sin^3 x - 4\sin^3 x + 8\sin^5 x + \sin x - 8\sin^3 x + 8\sin^5 x \\ &= 5\sin x - 20\sin^3 x + 16\sin^5 x \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$7. \tan x + \tan\left(\frac{\pi}{3} + x\right) - \tan\left(\frac{\pi}{3} - x\right) = 3 \tan 3x$$

**Solution:**

Let us consider LHS:

$$\begin{aligned} &\tan x + \tan\left(\frac{\pi}{3} + x\right) - \tan\left(\frac{\pi}{3} - x\right) \\ &= \tan x + \left(\frac{\tan \frac{\pi}{3} + \tan x}{1 - \tan x \tan \frac{\pi}{3}}\right) - \left(\frac{\tan \frac{\pi}{3} - \tan x}{1 + \tan x \tan \frac{\pi}{3}}\right) \end{aligned}$$

We know that,

$$\tan(A + B) = \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \text{ and } \tan(A - B) = \left(\frac{\tan A - \tan B}{1 + \tan A \tan B}\right)$$

So,

$$\begin{aligned} &= \tan x + \left(\frac{\sqrt{3} + \tan x}{1 - \sqrt{3} \tan x}\right) - \left(\frac{\sqrt{3} - \tan x}{1 + \sqrt{3} \tan x}\right) \\ &= \tan x + \left(\frac{(1 + \sqrt{3} \tan x)(\sqrt{3} + \tan x) - (1 - \sqrt{3} \tan x)(\sqrt{3} - \tan x)}{(1 - \tan x(\sqrt{3}))(1 + \tan x(\sqrt{3}))}\right) \end{aligned}$$

Simplify and cancel the similar terms of different sign in the above expression we get,

$$= \tan x + \left( \frac{(0 + 6 \tan x + 2 \tan x + 0)}{(1 - 3 \tan^2 x)} \right)$$

$$= \tan x + \left( \frac{8 \tan x}{(1 - 3 \tan^2 x)} \right)$$

$$= \left( \frac{\tan x (1 - 3 \tan^2 x) + 8 \tan x}{(1 - 3 \tan^2 x)} \right)$$

$$= \left( \frac{(\tan x - 3 \tan^3 x) + 8 \tan x}{(1 - 3 \tan^2 x)} \right)$$

$$= \left( \frac{9 \tan x - 3 \tan^3 x}{(1 - 3 \tan^2 x)} \right)$$

$$= 3 \left( \frac{3 \tan x - \tan^3 x}{(1 - 3 \tan^2 x)} \right)$$

$$= 3 \tan 3x \text{ (since, } \tan 3x = (3 \tan x - \tan^3 x) / (1 - 3 \tan^2 x) \text{)}$$

= RHS

Hence proved.

