EXERCISE 19.15

1. $\int \frac{1}{4x^2+12x+5}dx$

Solution:

Let

$$I=\int\frac{1}{4x^2+12x+5}dx$$

Taking out ¼ as common, then we get

$$=\frac{1}{4}\int\frac{1}{x^2+3x+\frac{5}{4}}dx$$

Adding and subtracting $(3/2)^2$ to the denominator

$$=\frac{1}{4}\int\frac{1}{x^2+2x\times\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+\frac{5}{4}}dx$$

The above equation can be written as

$$=\frac{1}{4}\int\frac{1}{\left(x+\frac{3}{2}\right)^2-1}dx$$

Let

$\left(x+\frac{3}{2}\right)=t$ ..... (i)

$\Rightarrow dx = dt$

So, substituting the t values we get

$$I=\frac{1}{4}\int\frac{1}{t^2-(1)^2}dt$$

$$I=\frac{1}{4}\times\frac{1}{2\times1}\log\left|\frac{t-1}{t+1}\right|+c$$

[since, $\int \frac{1}{x^2-(a)^2}dx = \frac{1}{2\times a}\log\left|\frac{x-a}{x+a}\right| + c$

$I = \frac{1}{8}\log\left|\frac{x+\frac{3}{2}-1}{x+\frac{3}{2}+1}\right| + c$ [Using (i)]

2. $\int \frac{1}{x^2-10x+34}\,dx$

Solution:

Let

$I = \int \frac{1}{x^2-10x+34}dx$

$I = \int \frac{1}{x^2-10x+34}dx$

Adding and subtracting 5^2 to both sides

$= \int \frac{1}{x^2-2x\times 5+(5)^2-(5)^2+34}dx$

The above equation can be written as

$= \int \frac{1}{(x-5)^2+9}dx$

Let $(x-5) = t$ ..... (i)

⇒ dx = dt

So, substituting the values of t we get

$I = \int \frac{1}{t^2+(3)^2}dt$

$I = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + c$

[since, $\int \frac{1}{x^2+(a)^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$]

$I = \frac{1}{3}\tan^{-1}(\frac{x-5}{3})+c$ [Using (i)]

$$I = \frac{1}{3}\tan^{-1}(\frac{x-5}{3})+c$$

3. $\int \frac{1}{1+x-x^2}dx$

Solution:

Let $I = \int \frac{1}{1+x-x^2}dx = \int \frac{1}{-(x^2-x-1)}dx$

The above equation can be written as

$$= \int \frac{1}{-(x^2-x-1)}dx$$

Add and subtract ¼ to both sides

$$= \int \frac{1}{-(x^2-x-\frac{1}{4}-1+\frac{1}{4})}dx$$

The above equation can be written as

$$= \int \frac{1}{-\left(\left(x-\frac{1}{2}\right)^2-\frac{5}{4}\right)}dx$$

On computing we get

$$= \int \frac{1}{\left(\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2\right)}dx$$

$$I = \frac{1}{2\times\frac{\sqrt{5}}{2}}\log\left|\frac{\frac{\sqrt{5}}{2}+(x-\frac{1}{2})}{\frac{\sqrt{5}}{2}-(x-\frac{1}{2})}\right|+c$$

By using, $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + c$

$$I = \frac{1}{\sqrt{5}} \log \left|\frac{\sqrt{5} + 2x - 1}{\sqrt{5} - 2x + 1}\right| + c$$

$$I = \frac{1}{\sqrt{5}} \log \left|\frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x}\right| + c$$

4. $\int \frac{1}{2x^2 - x - 1} dx$

Solution:

Let $I = \int \frac{1}{2x^2 - x - 1} dx$

Taking out ½ as common we get

$$= \frac{1}{2} \int \frac{1}{x^2 - \frac{x}{2} - \frac{1}{2}} dx$$

Again adding and subtracting $(¼)^2$ to the denominator we get

$$= \frac{1}{2} \int \frac{1}{x^2 - 2x \times \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \frac{1}{2}} dx$$

The above equation can be written as

$$= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}} dx$$

Let $\left(x - \frac{1}{4}\right) = t$ …… (i)

⇒ dx = dt

So, $I = \frac{1}{2} \int \frac{1}{t^2 - \left(\frac{3}{4}\right)^2} dt$

$$I = \frac{1}{2} \times \frac{1}{2 \times \frac{3}{4}} \log \left| \frac{t - \frac{3}{4}}{t + \frac{3}{4}} \right| + c$$

[since, $\int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x - a}{x + a} \right| + c$]

$$I = \frac{1}{3} \log \left| \frac{x - \frac{1}{4} - \frac{3}{4}}{x - \frac{1}{4} + \frac{3}{4}} \right| + c$$ [Using (i)]

$$I = \frac{1}{3} \log \left| \frac{2x - 2}{2x + 1} \right| + c$$

5. $\int \frac{1}{x^2 + 6x + 13} dx$

Solution:

In the denominator we have, and it can be written as

$x^2 + 6x + 13 = x^2 + 6x + 3^2 - 3^2 + 13$

The above equation can be written as

$= (x + 3)^2 + 4$

Substituting these values we get

So, $\int \frac{1}{x^2+6x+13} dx = \int \frac{1}{(x+3)^2+2^2} dx$

Let x+3 =t

Then dx = dt

$$\int \frac{1}{(t)^2 + 2^2} dt = \frac{1}{2} \tan^{-1} \frac{t}{2} + c$$

[since, $\int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) + c$]

$$\frac{1}{2} \tan^{-1} \frac{x + 3}{2} + c$$