

EXERCISE 10.1

1. If in a ΔABC , $\angle A = 45^\circ$, $\angle B = 60^\circ$, and $\angle C = 75^\circ$; find the ratio of its sides.

Solution:

Given: In ΔABC , $\angle A = 45^\circ$, $\angle B = 60^\circ$, and $\angle C = 75^\circ$

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Now by substituting the values we get,

$$\frac{a}{\sin 45^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 75^\circ}$$

$$\frac{a}{\sin 45^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin(30^\circ + 45^\circ)}$$

$$\frac{a}{\sin 45^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 30^\circ \cos 45^\circ + \sin 45^\circ \cos 30^\circ}$$

We know, $\sin(a + b) = \sin a \cos b + \sin b \cos a$

Now by substituting the corresponding values, we get,

$$\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}}$$

$$\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1 + \sqrt{3}}{2\sqrt{2}}}$$

$$a : b : c = \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

Multiply the above expression by $2\sqrt{2}$, we get

$$a : b : c = 2 : \sqrt{6} : (1 + \sqrt{3})$$

Hence the ratio of the sides of the given triangle is $a : b : c = 2 : \sqrt{6} : (1 + \sqrt{3})$

2. If in any ΔABC , $\angle C = 105^\circ$, $\angle B = 45^\circ$, $a = 2$, then find b .

Solution:

Given: In ΔABC , $\angle C = 105^\circ$, $\angle B = 45^\circ$, $a = 2$

We know in a triangle,

$$\angle A + \angle B + \angle C = 180^\circ$$

$$\angle A = 180^\circ - \angle B - \angle C$$

Substituting the given values, we get

$$\angle A = 180^\circ - 45^\circ - 105^\circ$$

$$\angle A = 30^\circ$$

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Now by substituting the corresponding values we get,

$$\frac{2}{\sin 30^\circ} = \frac{b}{\sin 45^\circ}$$

Substitute the equivalent values of the sine, we get

$$\frac{2}{\frac{1}{2}} = \frac{b}{\frac{1}{\sqrt{2}}}$$

$$4 = b\sqrt{2}$$

$$b = \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

Hence the value of b is $2\sqrt{2}$ units.

3. In $\triangle ABC$, if $a = 18$, $b = 24$ and $c = 30$ and $\angle C = 90^\circ$, find $\sin A$, $\sin B$ and $\sin C$.

Solution:

Given: In $\triangle ABC$, $a = 18$, $b = 24$ and $c = 30$ and $\angle C = 90^\circ$

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Now by substituting the given values we get,

$$\frac{18}{\sin A} = \frac{30}{\sin 90^\circ}$$

$$\sin A = \frac{18 \times \sin 90^\circ}{30}$$

$$\sin A = \frac{18 \times 1}{30}$$

$$\sin A = \frac{3}{5}$$

Similarly,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Substitute the given values, we get

$$\frac{24}{\sin B} = \frac{30}{\sin 90^\circ}$$

$$\sin B = \frac{24 \times \sin 90^\circ}{30}$$

$$\sin B = \frac{24 \times 1}{30}$$

$$\sin B = \frac{4}{5}$$

And given, $\angle C = 90^\circ$, so $\sin C = \sin 90^\circ = 1$.

Hence the values of $\sin A = 3/5$, $\sin B = 4/5$ and $\sin C = 1$ respectively.

In any triangle ABC, prove the following:

$$4. \frac{a - b}{a + b} = \frac{\tan \left(\frac{A - B}{2} \right)}{\tan \left(\frac{A + B}{2} \right)}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, $a = k \sin A$

Similarly, $b = k \sin B$

And $c = k \sin C$

We know,

$$a - b = k (\sin A - \sin B)$$

$$a + b = k (\sin A + \sin B)$$

Now let us consider LHS:

$$\begin{aligned} \frac{a-b}{a+b} &= \frac{k(\sin A - \sin B)}{k(\sin A + \sin B)} \\ &= \frac{(\sin A - \sin B)}{(\sin A + \sin B)} \dots (i) \end{aligned}$$

We know,

$$\sin A - \sin B = 2 \sin \frac{(A-B)}{2} \cos \frac{(A+B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

Substituting the above formulas in equation (i), we get

$$\frac{a-b}{a+b} = \frac{\left(2 \sin \left(\frac{A-B}{2}\right) \cos \left(\frac{A+B}{2}\right)\right)}{\left(2 \sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)\right)}$$

Upon rearranging we get,

$$= \frac{\left(\sin \left(\frac{A-B}{2}\right)\right)}{\left(\cos \left(\frac{A-B}{2}\right)\right)} \times \frac{\cos \left(\frac{A+B}{2}\right)}{\sin \left(\frac{A+B}{2}\right)}$$

$$= \frac{\left(\tan \left(\frac{A-B}{2}\right)\right)}{1} \times \frac{1}{\tan \left(\frac{A+B}{2}\right)}$$

$$= \frac{\left(\tan \left(\frac{A-B}{2}\right)\right)}{\left(\tan \left(\frac{A+B}{2}\right)\right)}$$

$$= \text{RHS}$$

Hence proved.

5. (a - b) cos C/2 = C sin (A - B)/2

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, $a = k \sin A$

Similarly, $b = k \sin B$

We know,

So the above equation becomes,

$$\begin{aligned}
 &= k \sin\left(\frac{A-B}{2}\right) (\sin(A+B)) \\
 &= k \sin\left(\frac{A-B}{2}\right) (\sin(\pi-C)) \quad [\text{since, } \pi = A+B+C, \text{ where, } A+B = \pi-C] \\
 &= k \sin(C) \sin\left(\frac{A-B}{2}\right) \quad [\text{since, } \sin(\pi-A) = \sin A]
 \end{aligned}$$

From the sine rule,

$$\frac{c}{\sin C} = k \Rightarrow c = k \sin C$$

So the above equation becomes,

$$= c \sin\left(\frac{A-B}{2}\right)$$

$$a - b = k (\sin A - \sin B) \dots (i)$$

Now let us consider LHS:

$$(a - b) \cos \frac{C}{2}$$

Substituting equation (i) in above equation, we get

$$(k(\sin A - \sin B)) \cos \frac{C}{2} \dots (ii)$$

We know,

$$\sin A - \sin B = 2 \sin \frac{(A-B)}{2} \cos \frac{(A+B)}{2}$$

Substituting the above formulas in equation (ii), we get

$$\begin{aligned}
 (a - b) \cos \frac{C}{2} &= \left(k \left(2 \sin \left(\frac{A-B}{2} \right) \cos \left(\frac{A+B}{2} \right) \right) \right) \cos \frac{C}{2} \\
 &= \left(k \left(2 \sin \left(\frac{A-B}{2} \right) \cos \left(\frac{A+B}{2} \right) \right) \right) \cos \frac{(\pi - (A+B))}{2} \\
 &= \left(2k \sin \left(\frac{A-B}{2} \right) \cos \left(\frac{A+B}{2} \right) \right) \sin \left(\frac{A+B}{2} \right) \quad [\text{since, } \cos(\pi/2 - A) = \sin A]
 \end{aligned}$$

Upon rearranging we get,

$$= k \sin\left(\frac{A-B}{2}\right) \left(2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A+B}{2}\right) \right)$$

$$\begin{aligned}
 \text{We know, } \sin A &= 2 \cos(A/2) \sin(A/2) \\
 &= \text{RHS}
 \end{aligned}$$

Hence proved.

$$6. \frac{c}{a-b} = \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right) - \tan\left(\frac{B}{2}\right)}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

$$\text{So, } a = k \sin A$$

$$\text{Similarly, } b = k \sin B$$

$$\text{And } c = k \sin C \dots \text{(i)}$$

We know,

$$a - b = k(\sin A - \sin B) \dots \text{(ii)}$$

Now let us consider LHS:

$$\frac{c}{a-b}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k \sin C}{k(\sin A - \sin B)} = \frac{\sin C}{(\sin A - \sin B)} \dots \text{(iii)}$$

By applying half angle rule,

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2} \dots \text{(iv)}$$

And we know,

$$\sin A - \sin B = 2 \sin \frac{(A-B)}{2} \cos \frac{(A+B)}{2} \dots \text{(v)}$$

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\begin{aligned} \frac{c}{(a-b)} &= \frac{2 \sin \frac{C}{2} \cos \frac{C}{2}}{2 \sin \left(\frac{A-B}{2}\right) \cos \left(\frac{A+B}{2}\right)} \\ &= \frac{\sin \left(\frac{\pi - (A+B)}{2}\right) \cos \left(\frac{C}{2}\right)}{\sin \left(\frac{A-B}{2}\right) \cos \left(\frac{A+B}{2}\right)} \quad [\text{since, } \pi = A+B+C, \text{ where, } C = \pi - (A+B)] \end{aligned}$$

$$= \frac{\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)} \quad [\text{since, } \sin(\pi/2 - A) = \cos A]$$

Upon simplification we get,

$$\begin{aligned} &= \frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A-B}{2}\right)} \\ &= \frac{\cos\left(\frac{\pi - (A + B)}{2}\right)}{\sin\left(\frac{A-B}{2}\right)} \quad [\text{since, } \pi = A+B+C, \text{ where, } C = \pi - (A+B)] \\ &= \frac{\sin\left(\frac{(A + B)}{2}\right)}{\sin\left(\frac{A-B}{2}\right)} \quad \dots \text{(v)} \quad [\text{since, } \cos(\pi/2 - A) = \sin A] \end{aligned}$$

We know,

$$\sin(A + B)/2 = \sin(A/2 + B/2) = \sin A/2 \cos B/2 + \cos A/2 \sin B/2$$

$$\sin(A - B)/2 = \sin(A/2 - B/2) = \sin A/2 \cos B/2 - \cos A/2 \sin B/2$$

Substituting the above equations in equation (v) we get,

$$= \frac{\sin \frac{A}{2} \cos \frac{B}{2} + \cos \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\sin \frac{A}{2} \cos \frac{B}{2} - \cos \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}$$

Let us divide the numerator and denominator by $\cos A/2 \cos B/2$, we get

$$\begin{aligned} &\frac{\sin \frac{A}{2} \cos \frac{B}{2} + \cos \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ &= \frac{\sin \frac{A}{2} \cos \frac{B}{2} - \cos \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ &= \frac{\frac{\sin \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} + \frac{\cos \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}}}{\frac{\sin \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} - \frac{\cos \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}}} \end{aligned}$$

Upon simplification we get,

$$\begin{aligned} & \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} + \frac{\sin \left(\frac{B}{2}\right)}{\cos \frac{B}{2}} \\ &= \frac{\sin \frac{A}{2} \cos \frac{B}{2} + \sin \left(\frac{B}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ &= \frac{\tan \frac{A}{2} \cos \frac{B}{2} + \cos \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ &= \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{\tan \frac{A}{2} - \tan \frac{B}{2}} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$7. \frac{c}{a+b} = \frac{1 - \tan \left(\frac{A}{2}\right) \tan \left(\frac{B}{2}\right)}{1 + \tan \left(\frac{A}{2}\right) \tan \left(\frac{B}{2}\right)}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, $a = k \sin A$

Similarly, $b = k \sin B$

And $c = k \sin C \dots$ (i)

We know,

$$a + b = k (\sin A + \sin B) \dots$$
 (ii)

Now let us consider LHS:

$$\frac{c}{a+b}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k \sin C}{k(\sin A + \sin B)} = \frac{\sin C}{(\sin A + \sin B)} \dots$$
 (iii)

By applying half angle rule,

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2} \dots \text{(iv)}$$

And we know,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2} \dots \text{(v)}$$

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\begin{aligned} \frac{c}{(a+b)} &= \frac{2 \sin \frac{C}{2} \cos \frac{C}{2}}{2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)} \\ &= \frac{\sin \left(\frac{\pi - (A+B)}{2} \right) \cos \left(\frac{\pi - (A+B)}{2} \right)}{\sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)} \end{aligned} \quad \text{[Since, } \pi = A+B+C, \text{ where,}$$

$$C = \pi - (A+B)]$$

$$= \frac{\cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A+B}{2} \right)}{\sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)} \quad \text{[Since, } \sin(\pi/2 - A) = \cos A, \cos(\pi/2 - A) =$$

$\sin A]$

Upon simplification we get,

$$= \frac{\cos \left(\frac{A+B}{2} \right)}{\cos \left(\frac{A-B}{2} \right)} \dots \text{(vi)}$$

We know,

$$\cos \frac{(A+B)}{2} = \cos \left(\frac{A}{2} + \frac{B}{2} \right) = \cos \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2}$$

$$\cos \frac{(A-B)}{2} = \cos \left(\frac{A}{2} - \frac{B}{2} \right) = \cos \frac{A}{2} \cos \frac{B}{2} + \sin \frac{A}{2} \sin \frac{B}{2}$$

Substituting the above equations in equation (vi) we get,

$$= \frac{\cos \frac{A}{2} \cos \frac{B}{2} - \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2} + \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}$$

Let us divide the numerator and denominator by $\cos \frac{A}{2} \cos \frac{B}{2}$, we get

$$= \frac{\frac{\cos \frac{A}{2} \cos \frac{B}{2} - \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}}}{\frac{\cos \frac{A}{2} \cos \frac{B}{2} + \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}}}$$

$$\begin{aligned} & \frac{\cos \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} + \frac{\sin \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ = & \frac{\cos \frac{A}{2} \cos \frac{B}{2} - \sin \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \end{aligned}$$

Upon simplification we get,

$$\begin{aligned} & 1 + \frac{\sin \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ = & \frac{1 - \frac{\sin \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}}}{1 - \frac{\sin \left(\frac{A}{2}\right) \sin \left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}}} \end{aligned}$$

$$= \frac{1 + \tan \frac{A}{2} \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}}$$

= RHS

Hence proved.

8. $\frac{a+b}{c} = \frac{\cos \left(\frac{A-B}{2}\right)}{\sin \frac{C}{2}}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, $a = k \sin A$

Similarly, $b = k \sin B$

And $c = k \sin C \dots$ (i)

We know,

$$a + b = k (\sin A + \sin B) \dots$$
 (ii)

Now let us consider LHS:

$$\frac{a + b}{c}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k(\sin A + \sin B)}{k(\sin C)} = \frac{(\sin A + \sin B)}{(\sin C)} \dots \text{(iii)}$$

By applying half angle rule,

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2} \dots \text{(iv)}$$

And we know,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2} \dots \text{(v)}$$

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\begin{aligned} \frac{a + b}{c} &= \frac{2 \sin \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right)}{2 \sin \left(\frac{C}{2} \right) \cos \left(\frac{C}{2} \right)} \\ &= \frac{\sin \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right)}{\sin \left(\frac{C}{2} \right) \cos \left(\frac{\pi - (A + B)}{2} \right)} \quad [\text{Since, } \pi = A+B+C, \text{ where, } C = \pi - \\ & \quad (A+B)] \\ &= \frac{\sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)}{\sin \left(\frac{C}{2} \right) \sin \left(\frac{A+B}{2} \right)} \quad [\text{Since, } \sin (\pi/2 - A) = \cos A, \cos (\pi/2 - A) = \sin \\ & \quad A] \end{aligned}$$

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Upon simplification we get,

$$\begin{aligned} &= \frac{\cos \left(\frac{A-B}{2} \right)}{\sin \left(\frac{C}{2} \right)} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$9. \sin \left(\frac{B - C}{2} \right) = \frac{b - c}{a} \cos \frac{A}{2}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, $c = k \sin C$

Similarly, $b = k \sin B$

We know,

$$b - c = k (\sin B - \sin C) \dots (i)$$

Now let us consider RHS:

$$\frac{b - c}{a} \cos \frac{A}{2}$$

Substituting equation (i) in above equation, we get

$$\frac{(k(\sin B - \sin C))}{k \sin A} \cos \frac{A}{2} = \frac{(\sin B - \sin C)}{\sin A} \cos \frac{A}{2} \dots (ii)$$

And we know,

$$\sin B - \sin C = 2 \sin \frac{(B-C)}{2} \cos \frac{(B+C)}{2} \dots (iii)$$

Substituting the above equation (iii) in equation (ii), we get

$$\frac{b - c}{a} \cos \frac{A}{2} = \frac{2 \sin \left(\frac{B - C}{2} \right) \cos \left(\frac{B + C}{2} \right)}{\sin A} \cos \left(\frac{\pi - (B + C)}{2} \right) \quad [\text{Since, } \pi = A + B + C]$$

$A + B + C$, where, $C = \pi - (A + B)$

$$= \frac{2 \sin \left(\frac{B - C}{2} \right) \cos \left(\frac{B + C}{2} \right)}{\sin A} \sin \left(\frac{(B + C)}{2} \right) \quad [\text{Since, } \cos (\pi/2 - A) = \sin A]$$

$= \sin A$

Upon rearranging the above equation we get,

$$= \frac{\sin \left(\frac{B - C}{2} \right) \left(2 \sin \left(\frac{(B + C)}{2} \right) \cos \left(\frac{B + C}{2} \right) \right)}{\sin A}$$

We know $\sin A = 2 \cos (A/2) \sin (A/2)$

So,

$$= \frac{\sin \left(\frac{B - C}{2} \right) (\sin(B + C))}{\sin A}$$

$$\begin{aligned}
 &= \frac{\sin\left(\frac{B-C}{2}\right) (\sin(\pi - A))}{\sin A} \quad [\text{Since, } \pi = A+B+C, \text{ where, } A+B = \pi-C] \\
 &= \frac{\sin\left(\frac{B-C}{2}\right) \sin A}{\sin A} \quad [\text{Since, } \sin(\pi - A) = \sin A]
 \end{aligned}$$

Upon simplification we get,

$$\begin{aligned}
 &= \sin\left(\frac{B-C}{2}\right) \\
 &= \text{LHS}
 \end{aligned}$$

Hence proved.

$$10. \frac{a^2 - c^2}{b^2} = \frac{\sin(A - C)}{\sin(A + C)}$$

Solution:

By using the sine rule we know,

$$\begin{aligned}
 \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = k \\
 \frac{c}{\sin C} &= k
 \end{aligned}$$

So, $c = k \sin C$

Similarly, $a = k \sin A$

And $b = k \sin B$

So, $a - c = k (\sin A - \sin C) \dots (i)$

We know,

Now let us consider LHS:

$$\frac{a^2 - c^2}{b^2}$$

Substituting the values in the above equation, we get

$$\frac{(k \sin A)^2 - (k \sin C)^2}{(k \sin B)^2} = \frac{k^2 (\sin^2 A - \sin^2 C)}{k^2 \sin^2 B} \dots (ii)$$

And we know,

$\sin^2 A - \sin^2 C = \sin(A + C) \sin(A - C) \dots (iii)$

Substituting the above equation (iii) in equation (ii), we get

$$\begin{aligned} \frac{a^2 - c^2}{b^2} &= \frac{\sin(A + C) \sin(A - C)}{\sin^2(\pi - (A + C))} \quad [\text{Since, } \pi = A+B+C, \text{ where, } C = \pi - (A+B)] \\ &= \frac{\sin(A + C) \sin(A - C)}{\sin^2((A + C))} \quad [\text{Since, } \sin(\pi - A) = \sin A] \\ &= \frac{\sin(A - C)}{\sin(A + C)} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

11. $b \sin B - c \sin C = a \sin (B - C)$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, $c = k \sin C$

Similarly, $a = k \sin A$

And $b = k \sin B$

We know,

Now let us consider LHS:

$$b \sin B - c \sin C$$

Substituting the values of b and c in the above equation, we get

$$k \sin B \sin B - k \sin C \sin C = k (\sin^2 B - \sin^2 C) \dots\dots\dots (i)$$

We know,

$$\sin^2 B - \sin^2 C = \sin (B + C) \sin (B - C),$$

Substituting the above values in equation (i), we get

$$k (\sin^2 B - \sin^2 C) = k (\sin (B + C) \sin (B - C)) \quad [\text{since, } \pi = A + B + C \Rightarrow B + C = \pi - A]$$

The above equation becomes,

$$\begin{aligned} &= k (\sin (\pi - A) \sin (B - C)) \quad [\text{since, } \sin (\pi - \theta) = \sin \theta] \\ &= k (\sin (A) \sin (B - C)) \end{aligned}$$

From sine rule, $a = k \sin A$, so the above equation becomes,

$$\begin{aligned} &= a \sin (B - C) \\ &= \text{RHS} \end{aligned}$$

Hence proved.

12. $a^2 \sin (B - C) = (b^2 - c^2) \sin A$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, $c = k \sin C$

Similarly, $a = k \sin A$

And $b = k \sin B$

We know,

Now let us consider RHS:

$$(b^2 - c^2) \sin A \dots$$

Substituting the values of b and c in the above equation, we get

$$\begin{aligned} (b^2 - c^2) \sin A &= [(k \sin B)^2 - (k \sin C)^2] \sin A \\ &= k^2 (\sin^2 B - \sin^2 C) \sin A \dots \dots \dots (i) \end{aligned}$$

We know,

$$\sin^2 B - \sin^2 C = \sin (B + C) \sin (B - C),$$

Substituting the above values in equation (i), we get

$$\begin{aligned} &= k^2 (\sin (B + C) \sin (B - C)) \sin A \text{ [since, } \pi = A + B + C \Rightarrow B + C = \pi - A] \\ &= k^2 (\sin (\pi - A) \sin (B - C)) \sin A \\ &= k^2 (\sin A) \sin (B - C) \sin A \text{ [since, } \sin (\pi - \theta) = \sin \theta] \end{aligned}$$

Rearranging the above equation we get

$$= (k \sin A) (\sin (B - C)) (k \sin A)$$

From sine rule, $a = k \sin A$, so the above equation becomes,

$$\begin{aligned} &= a^2 \sin (B - C) \\ &= \text{RHS} \end{aligned}$$

Hence proved.

13. $\frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} + \sqrt{\sin B}} = \frac{a + b - 2\sqrt{ab}}{a - b}$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\sin A = \frac{a}{k}, \sin B = \frac{b}{k}, \sin C = \frac{c}{k}$$

Let us consider LHS,

$$\frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} + \sqrt{\sin B}}$$

Let us multiply and divide the above expression by $\frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} - \sqrt{\sin B}}$ we get,

$$\begin{aligned} \frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} + \sqrt{\sin B}} \times \frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} - \sqrt{\sin B}} &= \frac{(\sqrt{\sin A} - \sqrt{\sin B})^2}{(\sqrt{\sin A})^2 - (\sqrt{\sin B})^2} \\ &= \frac{(\sqrt{\sin A})^2 + (\sqrt{\sin B})^2 - (2\sqrt{\sin A} \times \sqrt{\sin B})}{\sin A - \sin B} \\ &= \frac{\sin A + \sin B - (2\sqrt{\sin A} \times \sin B)}{\sin A - \sin B} \end{aligned}$$

Substituting the values of a and b from sine rule in the above equation, we get

$$\begin{aligned} &= \frac{\frac{a}{k} + \frac{b}{k} - (2\sqrt{\frac{a}{k} \times \frac{b}{k}})}{\frac{a}{k} - \frac{b}{k}} \\ &= \frac{\frac{1}{k}(a + b - 2\sqrt{ab})}{\frac{1}{k}(a - b)} \\ &= \frac{a + b - 2\sqrt{ab}}{a - b} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

14. $a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$a = k \sin A, b = k \sin B, c = k \sin C$$

Let us consider LHS:

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B)$$

Substituting the values of a, b, c from sine rule in above equation, we get

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = k \sin A (\sin B - \sin C) + k \sin B (\sin C - \sin A) + k \sin C (\sin A - \sin B)$$

$$= k \sin A \sin B - k \sin A \sin C + k \sin B \sin C - k \sin B \sin A + k \sin C \sin A - k \sin C \sin B$$

Upon simplification, we get

$$= 0 \\ = \text{RHS}$$

Hence proved.

$$15. \frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C} = 0$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$a = k \sin A, b = k \sin B, c = k \sin C$$

Let us consider LHS:

$$\frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C}$$

Substituting the values of a, b and c from sine rule in the above equation, we get

$$= \frac{(k \sin A)^2 \sin(B - C)}{\sin A} + \frac{(k \sin B)^2 \sin(C - A)}{\sin B} + \frac{(k \sin C)^2 \sin(A - B)}{\sin C} \\ = \frac{k^2 \sin^2 A \sin(B - C)}{\sin A} + \frac{k^2 \sin^2 B \sin(C - A)}{\sin B} + \frac{k^2 \sin^2 C \sin(A - B)}{\sin C}$$

Upon simplification we get,

$$= k^2 [\sin A \sin(B - C) + \sin B \sin(C - A) + \sin C \sin(A - B)]$$

We know, $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$\sin(B - C) = \sin B \cos C - \cos B \sin C$$

$$\sin(C - A) = \sin C \cos A - \cos C \sin A$$

So the above equation becomes,

$$= k^2 [\sin A (\sin B \cos C - \cos B \sin C) + \sin B (\sin C \cos A - \cos C \sin A) + \sin C (\sin A \cos B - \cos A \sin B)]$$

$$= k^2 [\sin A \sin B \cos C - \sin A \cos B \sin C + \sin B \sin C \cos A - \sin B \cos C \sin A + \sin C \sin A \cos B - \sin C \cos A \sin B]$$

Upon simplification we get,

$$= 0$$

= RHS

Hence proved.

