

23. Scalar, or Dot, Product of Vectors

Exercise 23

1. Question

Find $\vec{a} \cdot \vec{b}$ when

i. $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$

ii. $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{j} + 4\hat{k}$

iii. $\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{k}$

Answer

i)

$$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$$

$$\vec{a} \cdot \vec{b} = (\hat{i} - 2\hat{j} + \hat{k}) \cdot (3\hat{i} - 4\hat{j} - 2\hat{k})$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (1 \times 3) + (-2 \times -4) + (1 \times -2)$$

$$\Rightarrow \vec{a} \cdot \vec{b} = 3 + 8 - 2 = 9$$

Ans: $\vec{a} \cdot \vec{b} = 9$

ii)

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b} = 0\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\vec{a} \cdot \vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (0\hat{i} - 2\hat{j} + 4\hat{k})$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (1 \times 0) + (2 \times -2) + (3 \times 4)$$

$$\Rightarrow \vec{a} \cdot \vec{b} = 0 - 4 + 12 = 8$$

Ans: $\Rightarrow \vec{a} \cdot \vec{b} = 8$

iii)

$$\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$$

$$\vec{b} = 3\hat{i} + 0\hat{j} - 2\hat{k}$$

$$\vec{a} \cdot \vec{b} = (\hat{i} - \hat{j} + 5\hat{k}) \cdot (3\hat{i} + 0\hat{j} - 2\hat{k})$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (1 \times 3) + (-1 \times 0) + (5 \times -2)$$

$$\Rightarrow \vec{a} \cdot \vec{b} = 3 - 0 - 10 = -7$$

Ans: $\Rightarrow \vec{a} \cdot \vec{b} = -7$

2. Question

Find the value of λ for which \vec{a} and \vec{b} are perpendicular, where

i. $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k})$



$$\text{ii. } \vec{a} = 3\hat{i} - \hat{j} + 4\hat{k} \text{ and } \vec{b} = -\lambda\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\text{iii. } \vec{a} = 2\hat{i} + 4\hat{j} - \hat{k} \text{ and } \vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$$

$$\text{iv. } \vec{a} = 3\hat{i} + 2\hat{j} - 5\hat{k} \text{ and } \vec{b} = -5\hat{j} + \lambda\hat{k}$$

Answer

i)

$$\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$$

$$\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$$

Since these two vectors are perpendicular, their dot product is zero.

$$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (2 \times 1) + (\lambda \times -2) + (1 \times 3) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = 2 - 2\lambda + 3 = 0$$

$$\Rightarrow 5 = 2\lambda$$

$$\Rightarrow \lambda = \frac{5}{2}$$

$$\text{Ans: } \lambda = \frac{5}{2}$$

ii)

$$\vec{a} = 3\hat{i} - \hat{j} + 4\hat{k}$$

$$\vec{b} = -\lambda\hat{i} + 3\hat{j} + 3\hat{k}$$

Since these two vectors are perpendicular, their dot product is zero.

$$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (3\hat{i} - \hat{j} + 4\hat{k}) \cdot (-\lambda\hat{i} + 3\hat{j} + 3\hat{k}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (3 \times -\lambda) + (-1 \times 3) + (4 \times 3) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = -3\lambda - 3 + 12 = 0$$

$$\Rightarrow 9 = 3\lambda$$

$$\Rightarrow \lambda = \frac{9}{3} = 3$$

$$\text{Ans: } \lambda = 3$$

iii)

$$\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$$

$$\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$$

Since these two vectors are perpendicular, their dot product is zero.

$$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$$



$$\Rightarrow \vec{a} \cdot \vec{b} = (2\hat{i} + 4\hat{j} - \hat{k}) \cdot (3\hat{i} - 2\hat{j} + \lambda\hat{k}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (2 \times 3) + (4 \times -2) + (-1 \times \lambda) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = -\lambda + 6 - 8 = 0$$

$$\Rightarrow -2 = \lambda$$

$$\Rightarrow \lambda = -2$$

Ans: $\lambda = -2$

iv)

$$\vec{a} = 3\hat{i} + 2\hat{j} - 5\hat{k}$$

$$\vec{b} = -5\hat{j} + \lambda\hat{k}$$

Since these two vectors are perpendicular, their dot product is zero.

$$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (3\hat{i} + 2\hat{j} - 5\hat{k}) \cdot (-5\hat{j} + \lambda\hat{k}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (3 \times 0) + (2 \times -5) + (-5 \times \lambda) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = -5\lambda + 0 - 10 = 0$$

$$\Rightarrow -10 = 5\lambda$$

$$\Rightarrow \lambda = \frac{-10}{5} = -2$$

Ans: $\lambda = -2$



3. Question

i. If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, show that $(\vec{a} + \vec{b})$ is perpendicular to $(\vec{a} - \vec{b})$.

ii. If $\vec{a} = (5\hat{i} - \hat{j} - 3\hat{k})$ and $\vec{b} = (\hat{i} + 3\hat{j} - 5\hat{k})$ then show that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal.

Answer

i)

$$\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$$

$$\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{a} + \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - \hat{j} + 2\hat{k}$$

$$\Rightarrow \vec{a} + \vec{b} = 4\hat{i} + \hat{j} - \hat{k}$$

$$\vec{a} - \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k} - (3\hat{i} - \hat{j} + 2\hat{k})$$

$$\Rightarrow \vec{a} - \vec{b} = -2\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\text{Now } (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (4\hat{i} + \hat{j} - \hat{k}) \cdot (-2\hat{i} + 3\hat{j} - 5\hat{k})$$

$$= (4 \times -2) + (1 \times 3) + (-1 \times -5) = -8 + 3 + 5 = 0$$

Since the dot product of these two vectors is 0, the vector $(\vec{a} + \vec{b})$ is perpendicular to $(\vec{a} - \vec{b})$.

Hence, proved.

ii)

$$\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$$

$$\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$$

$$\vec{a} + \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} + \hat{i} + 3\hat{j} - 5\hat{k}$$

$$\Rightarrow \vec{a} + \vec{b} = 6\hat{i} + 2\hat{j} - 8\hat{k}$$

$$\vec{a} - \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} - (\hat{i} + 3\hat{j} - 5\hat{k})$$

$$\Rightarrow \vec{a} - \vec{b} = 4\hat{i} - 4\hat{j} + 2\hat{k}$$

$$\text{Now } (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k}) \cdot (4\hat{i} - 4\hat{j} + 2\hat{k})$$

$$= (6 \times 4) + (2 \times -4) + (-8 \times 2) = 24 - 8 - 16 = 0$$

Since the dot product of these two vectors is 0, the vector $(\vec{a} + \vec{b})$ is perpendicular to $(\vec{a} - \vec{b})$.

Hence, proved that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal.

4. Question

If $\vec{a} = (\hat{i} - \hat{j} + 7\hat{k})$ and $\vec{b} = (5\hat{i} - \hat{j} + \lambda\hat{k})$ then find the value of λ so that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal vectors.

Answer

$$\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$$

$$\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$$

$$(\vec{a} + \vec{b}) = \hat{i} - \hat{j} + 7\hat{k} + 5\hat{i} - \hat{j} + \lambda\hat{k}$$

$$\Rightarrow \vec{a} + \vec{b} = 6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}$$

$$\vec{a} - \vec{b} = \hat{i} - \hat{j} + 7\hat{k} - (5\hat{i} - \hat{j} + \lambda\hat{k})$$

$$\Rightarrow \vec{a} - \vec{b} = -4\hat{i} + 0\hat{j} + (7 - \lambda)\hat{k}$$

$$\text{Now } (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}) \cdot (-4\hat{i} + 0\hat{j} + (7 - \lambda)\hat{k})$$

Since these two vectors are orthogonal, their dot product is zero.

$$\Rightarrow (6 \times -4) + (-2 \times 0) + ((7 + \lambda) \times (7 - \lambda)) = 0 \Rightarrow -24 + 0 + (49 - \lambda^2) = 0$$

$$\Rightarrow \lambda^2 = 25$$

$$\Rightarrow \lambda = \pm 5$$

Ans: $\lambda = \pm 5$

5. Question

Show that the vectors

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) \text{ and } \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

are mutually perpendicular unit vectors.

Answer

Let,

$$\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$$

$$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$$

We have to show that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{c} = 0$

L.H.S.

$$\vec{a} \cdot \vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) \cdot \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{1}{49}(6 - 18 + 12) = 0$$

$$\vec{b} \cdot \vec{c} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) \cdot \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k}) = \frac{1}{49}(18 - 12 - 6) = 0$$

$$\vec{a} \cdot \vec{c} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) \cdot \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k}) = \frac{1}{49}(12 + 6 - 18) = 0$$

= R.H.S.

Hence, showed that vectors are mutually perpendicular unit vectors.

6. Question

Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$.

Find a vector \vec{d} which is perpendicular to both \vec{a} and \vec{b} , and is such that $\vec{d} \cdot \vec{c} = 21$.

Answer

$$\vec{a} = (4\hat{i} + 5\hat{j} - \hat{k})$$

$$\vec{b} = (\hat{i} - 4\hat{j} + 5\hat{k})$$

$$\vec{c} = (3\hat{i} + \hat{j} - \hat{k})$$

$$\text{Let } \vec{d} = p\hat{i} + q\hat{j} + r\hat{k}$$

the vector \vec{d} which is perpendicular to both \vec{a} and \vec{b} ,

$$\Rightarrow \vec{d} \cdot \vec{a} = \vec{d} \cdot \vec{b} = 0$$

$$(p\hat{i} + q\hat{j} + r\hat{k}) \cdot (4\hat{i} + 5\hat{j} - \hat{k}) = 0$$

$$\Rightarrow 4p + 5q - r = 0 \dots (1)$$

$$(p\hat{i} + q\hat{j} + r\hat{k}) \cdot (\hat{i} - 4\hat{j} + 5\hat{k}) = 0$$

$$p - 4q + 5r = 0 \dots (2)$$

$$\vec{d} \cdot \vec{c} = 21.$$

$$(p\hat{i} + q\hat{j} + r\hat{k}) \cdot (3\hat{i} + \hat{j} - \hat{k}) = 21$$

$$\Rightarrow 3p + q - r = 21 \dots (3)$$

Solving equations 1,2,3 simultaneously we get

$$p = 7, q = -7, r = -7$$

$$\therefore \vec{d} = p\hat{i} + q\hat{j} + r\hat{k} = 7\hat{i} - 7\hat{j} - 7\hat{k} = 7(\hat{i} - \hat{j} - \hat{k})$$

$$\text{Ans: } \vec{d} = 7(\hat{i} - \hat{j} - \hat{k})$$

7. Question

$$\text{Let } \vec{a} = (2\hat{i} + 3\hat{j} + 2\hat{k}) \text{ and } \vec{b} = (\hat{i} + 2\hat{j} + \hat{k}).$$

Find the projection of (i) \vec{a} on \vec{b} and (ii) \vec{b} on \vec{a} .

Answer

$$\vec{a} = (2\hat{i} + 3\hat{j} + 2\hat{k})$$

$$\vec{b} = (\hat{i} + 2\hat{j} + \hat{k})$$

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{4 + 9 + 4} = \sqrt{17}$$

$$|\vec{b}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{17}}$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$$

$$\text{Projection of } \vec{a} \text{ on } \vec{b} \text{ is } \vec{a}\hat{b} = (2\hat{i} + 3\hat{j} + 2\hat{k}) \cdot \frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}} = \frac{2 + 6 + 2}{\sqrt{6}} = \frac{10}{\sqrt{6}} = \frac{5\sqrt{6}}{3}$$

$$\text{Projection of } \vec{b} \text{ on } \vec{a} \text{ is } \vec{b}\hat{a} = (\hat{i} + 2\hat{j} + \hat{k}) \cdot \frac{2\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{17}} = \frac{2 + 6 + 2}{\sqrt{17}} = \frac{10}{\sqrt{17}} = \frac{10\sqrt{17}}{17}$$

$$\text{Ans: i) } \frac{5\sqrt{6}}{3}$$

$$\text{ii) } \frac{10\sqrt{17}}{17}$$

8. Question

Find the projection of $(8\hat{i} + \hat{j})$ in the direction of $(\hat{i} + 2\hat{j} - 2\hat{k})$.

Answer

Let,

$$\vec{a} = (8\hat{i} + \hat{j})$$

$$\vec{b} = (\hat{i} + 2\hat{j} - 2\hat{k})$$

$$|\vec{b}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{i} + 2\hat{j} - 2\hat{k}}{3}$$

\therefore The projection of $(8\hat{i} + \hat{j})$ on $(\hat{i} + 2\hat{j} - 2\hat{k})$

$$\text{is: } (8\hat{i} + \hat{j}) \cdot \frac{\hat{i} + 2\hat{j} - 2\hat{k}}{3} = \frac{8 + 2 + 0}{3} = \frac{10}{3}$$

$$\text{Ans: } 10/3$$

9. Question

Write the projection of vector $(\hat{i} + \hat{j} + \hat{k})$ along the vector \hat{j} .

Answer

Let,

$$\vec{a} = (\hat{i} + \hat{j} + \hat{k})$$

$$\vec{b} = (\hat{j})$$

$$|\vec{b}| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{(\hat{j})}{1}$$

∴ The projection of $(\hat{i} + \hat{j} + \hat{k})$ on (\hat{j})

$$\text{is: } (\hat{i} + \hat{j} + \hat{k}) \cdot (\hat{j}) = 1$$

Ans:1

10. Question

i. Find the projection of \vec{a} on \vec{b} if $\vec{a} \cdot \vec{b} = 8$ and $\vec{b} = (2\hat{i} + 6\hat{j} + 3\hat{k})$.

ii. Write the projection of the vector $(\hat{i} + \hat{j})$ on the vector $(\hat{i} - \hat{j})$.

Answer

$$\text{i) } \vec{b} = (2\hat{i} + 6\hat{j} + 3\hat{k})$$

$$|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$$

Projection of \vec{a} on \vec{b}

$$= \vec{a} \cdot \frac{\vec{b}}{|\vec{b}|}$$

$$= \frac{8}{7}$$

ANS:8/7

ii) Sol:

Let,

$$\vec{a} = (\hat{i} + \hat{j})$$

$$\vec{b} = (\hat{i} - \hat{j})$$

$$|\vec{b}| = \sqrt{1^2 + (-1)^2} = \sqrt{1 + 1} = \sqrt{2}$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{i} - \hat{j}}{\sqrt{2}}$$

∴ The projection of $\hat{i} + \hat{j}$ on $(\hat{i} - \hat{j})$

$$\text{is: } (\hat{i} + \hat{j}) \cdot \frac{\hat{i} - \hat{j}}{\sqrt{2}} = \frac{1-1}{\sqrt{2}} = 0$$

Ans: 0

11. Question



Find the angle between the vectors \vec{a} and \vec{b} , when

i. $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

ii. $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

iii. $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$.

Answer

i) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$$\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$$

$$\vec{b} = (3\hat{i} - 2\hat{j} + \hat{k})$$

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$$

We know that ,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow (\hat{i} - 2\hat{j} + 3\hat{k})(3\hat{i} - 2\hat{j} + \hat{k}) = \sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow (3 + 4 + 3) = 14\cos\theta$$

$$\Rightarrow \cos\theta = 10/14$$

$$\Rightarrow \cos\theta = 5/7$$

$$\Rightarrow \theta = \cos^{-1}(5/7)$$

Ans: $\theta = \cos^{-1}(5/7)$

ii) $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

$$\vec{a} = (3\hat{i} + \hat{j} + 2\hat{k})$$

$$\vec{b} = (2\hat{i} - 2\hat{j} + 4\hat{k})$$

$$|\vec{a}| = \sqrt{3^2 + (1)^2 + 2^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{4 + 4 + 16} = \sqrt{24}$$

We know that ,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow (3\hat{i} + \hat{j} + 2\hat{k})(2\hat{i} - 2\hat{j} + 4\hat{k}) = \sqrt{14}\sqrt{24}\cos\theta$$

$$\Rightarrow (6 - 2 + 8) = \sqrt{336} \cos\theta$$

$$\Rightarrow \cos\theta = 12/\sqrt{336}$$

$$\Rightarrow \cos\theta = \sqrt{(144/336)}$$

$$\Rightarrow \theta = \cos^{-1}\sqrt{(3/7)}$$

Ans: $\theta = \cos^{-1}\sqrt{(3/7)}$

iii. $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$.



Ans:

$$\vec{a} = (\hat{i} - \hat{j})$$

$$\vec{b} = (\hat{j} + \hat{k})$$

$$|\vec{a}| = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$|\vec{b}| = \sqrt{(1)^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$

We know that ,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow (\hat{i} - \hat{j})(\hat{j} + \hat{k}) = \sqrt{2}\sqrt{2}\cos\theta$$

$$\Rightarrow (-1) = 2 \cos\theta$$

$$\Rightarrow \cos\theta = -1/2$$

$$\Rightarrow \theta = \cos^{-1}(-1/2)$$

$$\Rightarrow \theta = 120^\circ$$

Ans: $\theta = 120^\circ$

12. Question

If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$ then calculate the angle between $(2\vec{a} + \vec{b})$ and $(\vec{a} + 2\vec{b})$.

Answer

$$\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$$

$$\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$$

$$\vec{a} + 2\vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k}) + 2(3\hat{i} - \hat{j} + 2\hat{k}) = 7\hat{i} + \hat{k}$$

$$2\vec{a} + \vec{b} = 2(\hat{i} + 2\hat{j} - 3\hat{k}) + (3\hat{i} - \hat{j} + 2\hat{k}) = 5\hat{i} + 3\hat{j} - 4\hat{k}$$

$$|\vec{a} + 2\vec{b}| = \sqrt{7^2 + (1)^2} = \sqrt{49 + 1} = \sqrt{50}$$

$$|2\vec{a} + \vec{b}| = \sqrt{5^2 + (3)^2 + (-4)^2} = \sqrt{25 + 9 + 16} = \sqrt{50}$$

We know that ,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow (7\hat{i} + \hat{k})(5\hat{i} + 3\hat{j} - 4\hat{k}) = \sqrt{50}\sqrt{50}\cos\theta$$

$$\Rightarrow (35 - 4) = 50 \cos\theta$$

$$\Rightarrow \cos\theta = 31/50$$

$$\Rightarrow \theta = \cos^{-1}(31/50)$$

Ans: $\theta = \cos^{-1}(31/50)$

13. Question

If \vec{a} is a unit vector such that $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$, find $|\vec{x}|$.

Answer

If \vec{a} is a unit vector



$$\Rightarrow |\vec{a}| = 1$$

$$\Rightarrow (\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 8$$

$$\Rightarrow |\vec{x}|^2 = 8 + 1 = 9$$

$$\Rightarrow |\vec{x}| = 3$$

Ans: $|\vec{x}| = 3$

14. Question

Find the angles which the vector $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ makes with the coordinate axes.

Answer

If we have a vector $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$

then the angle with the x - axis $= \alpha = \cos^{-1} \frac{a}{\sqrt{a^2 + b^2 + c^2}}$

the angle with the y - axis $= \beta = \cos^{-1} \frac{b}{\sqrt{a^2 + b^2 + c^2}}$

the angle with the z - axis $= \gamma = \cos^{-1} \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

Here, $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

then the angle with the x - axis $= \alpha = \cos^{-1} \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \cos^{-1} \frac{3}{7}$

the angle with the y - axis $= \beta = \cos^{-1} \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \cos^{-1} \frac{-6}{7}$

the angle with the z - axis $= \gamma = \cos^{-1} \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \cos^{-1} \frac{2}{7}$

Ans:

$$\cos^{-1} \frac{3}{7}, \cos^{-1} \frac{-6}{7}, \cos^{-1} \frac{2}{7}$$

15. Question

Show that the vector $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ is equally inclined to the coordinate axes.

Answer

If we have a vector $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$

then the angle with the x - axis $= \alpha = \cos^{-1} \frac{a}{\sqrt{a^2 + b^2 + c^2}}$

the angle with the y - axis $= \beta = \cos^{-1} \frac{b}{\sqrt{a^2 + b^2 + c^2}}$

the angle with the z - axis $= \gamma = \cos^{-1} \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{1^2 + (1)^2 + 1^2} = \sqrt{1 + 1 + 1} = \sqrt{3}$$

then the angle with the x - axis $= \alpha = \cos^{-1} \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \cos^{-1} \frac{1}{\sqrt{3}}$

$$\text{the angle with the } y\text{-axis} = \beta = \cos^{-1} \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \cos^{-1} \frac{1}{\sqrt{3}}$$

$$\text{the angle with the } z\text{-axis} = \gamma = \cos^{-1} \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \cos^{-1} \frac{1}{\sqrt{3}}$$

Now since, $\alpha = \beta = \gamma$

\therefore the vector $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$ is equally inclined to the coordinate axes.

Hence, proved.

16. Question

Find a vector \vec{a} of magnitude $5\sqrt{2}$, making an angle $\pi/4$ with x - axis, $\pi/2$ with y - axis and an acute angle θ with z - axis.

Answer

$$|\vec{a}| = 5\sqrt{2}$$

$$l = \cos \alpha = \cos \pi/4 = 1/\sqrt{2}$$

$$m = \cos \beta = \cos \pi/2 = 0$$

$$n = \cos \theta$$

we know that

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \frac{1}{2} + 0^2 + n^2 = 1$$

$$\Rightarrow n^2 = 1 - \frac{1}{2}$$

$$\Rightarrow n^2 = \frac{1}{2}$$

$$\Rightarrow n = \pm \frac{1}{\sqrt{2}}$$

since the vector makes an acute angle with the z axis

$$\therefore n = + \frac{1}{\sqrt{2}}$$

$$\therefore \vec{a} = |\vec{a}|(l\hat{i} + m\hat{j} + n\hat{k})$$

$$\therefore \vec{a} = 5\sqrt{2}(1/\sqrt{2}\hat{i} + 1/\sqrt{2}\hat{k})$$

$$\therefore \vec{a} = 5(\hat{i} + \hat{k})$$

$$\text{Ans: } \vec{a} = 5(\hat{i} + \hat{k})$$

17. Question

Find the angle between $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, if $\vec{a} = (2\hat{i} - \hat{j} + 3\hat{k})$ and $\vec{b} = (3\hat{i} + \hat{j} + 2\hat{k})$.

Answer

$$\vec{a} = (2\hat{i} - \hat{j} + 3\hat{k})$$

$$\vec{b} = (3\hat{i} + \hat{j} + 2\hat{k})$$

$$\vec{a} + \vec{b} = (2\hat{i} - \hat{j} + 3\hat{k}) + (3\hat{i} + \hat{j} + 2\hat{k}) = 5\hat{i} + 5\hat{k}$$



$$\vec{a} - \vec{b} = (2\hat{i} - \hat{j} + 3\hat{k}) - (3\hat{i} + \hat{j} + 2\hat{k}) = -\hat{i} - 2\hat{j} + \hat{k}$$

$$|\vec{a} + \vec{b}| = \sqrt{5^2 + (5)^2} = \sqrt{25 + 25} = \sqrt{50}$$

$$|\vec{a} - \vec{b}| = \sqrt{(-1)^2 + (-2)^2 + (1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

We know that ,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow (5\hat{i} + 5\hat{k})(-\hat{i} - 2\hat{j} + \hat{k}) = \sqrt{50}\sqrt{6}\cos\theta$$

$$\Rightarrow (-5 + 5) = \sqrt{300}\cos\theta$$

$$\Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = \cos^{-1}(0) = \pi/2$$

Ans: $\theta = \pi/2$

18. Question

Express the vector $\vec{a} = (6\hat{i} - 3\hat{j} - 6\hat{k})$ as sum of two vectors such that one is parallel to the vector $\vec{b} = (\hat{i} + \hat{j} + \hat{k})$ and the other is perpendicular to \vec{b} .

Answer

$$\vec{a} = (6\hat{i} - 3\hat{j} - 6\hat{k})$$

$$\vec{b} = (\hat{i} + \hat{j} + \hat{k})$$

$$\Rightarrow \vec{c} \parallel \vec{b} \text{ \& } \vec{d} \perp \vec{b}$$

$$\therefore \vec{a} = \vec{c} + \vec{d}$$

$$\vec{c} = \lambda\vec{b} \text{ \& } \vec{b} \cdot \vec{d} = 0$$

$$\Rightarrow \vec{b} \cdot \vec{a} = \vec{b} \cdot (\vec{c} + \vec{d})$$

$$\Rightarrow (\hat{i} + \hat{j} + \hat{k}) \cdot (6\hat{i} - 3\hat{j} - 6\hat{k}) = \vec{b} \cdot \lambda\vec{b} + 0$$

$$\Rightarrow 6 - 3 - 6 = \lambda(|\vec{b}|^2) = 3\lambda$$

$$\Rightarrow \lambda = -1$$

$$\vec{c} = \lambda\vec{b} = -1(\hat{i} + \hat{j} + \hat{k}) = -(\hat{i} + \hat{j} + \hat{k})$$

$$\therefore \vec{a} = \vec{c} + \vec{d}$$

$$\Rightarrow (6\hat{i} - 3\hat{j} - 6\hat{k}) = -(\hat{i} + \hat{j} + \hat{k}) + \vec{d}$$

$$\Rightarrow \vec{d} = 7\hat{i} - 2\hat{j} - 5\hat{k}$$

$$\Rightarrow \vec{a} = \vec{c} + \vec{d}$$

$$\Rightarrow \vec{a} = -(\hat{i} + \hat{j} + \hat{k}) + (7\hat{i} - 2\hat{j} - 5\hat{k})$$

Ans: $\vec{a} = -(\hat{i} + \hat{j} + \hat{k}) + (7\hat{i} - 2\hat{j} - 5\hat{k})$

19. Question

Prove that $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 \Leftrightarrow \vec{a} \perp \vec{b}$, where $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$.



Answer

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$$

$$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$$

$$\Rightarrow |\vec{b}| = 0$$

Which is not possible hence

$$(\vec{a}) \perp (\vec{b})$$

20. Question

If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between \vec{a} and \vec{b} .

Answer

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$$

$$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = -\vec{c} \cdot -\vec{c}$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = |\vec{c}|^2$$

$$\Rightarrow 3^2 + 5^2 + 2 \times 3 \times 5 \cos\theta = 7^2$$

$$\Rightarrow 2 \times 3 \times 5 \cos\theta = 49 - 9 - 25$$

$$\Rightarrow 30 \cos\theta = 15$$

$$\Rightarrow \cos\theta = \frac{15}{30} = \frac{1}{2}$$

$$\Rightarrow \theta = \cos^{-1}\frac{1}{2} = 60^\circ$$

$$\text{Ans: } \theta = 60^\circ = \frac{\pi}{3}$$

**21. Question**

Find the angle between \vec{a} and \vec{b} , when

i. $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = \sqrt{3}$

ii. $|\vec{a}| = |\vec{b}| = \sqrt{2}$ and $\vec{a} \cdot \vec{b} = -1$

Answer

i)

We know that ,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \sqrt{3} = 2 \times 1 \cos\theta$$

$$\Rightarrow \sqrt{3} = 2 \cos\theta$$

$$\Rightarrow \cos\theta = \sqrt{3}/2$$

$$\Rightarrow \theta = \cos^{-1}(\sqrt{3}/2) = 30^\circ = \frac{\pi}{6}$$

$$\text{Ans: } \theta = \cos^{-1}(\sqrt{3}/2) = 30^\circ = \frac{\pi}{6}$$

ii)

We know that ,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow -1 = \sqrt{2} \times \sqrt{2}\cos\theta$$

$$\Rightarrow -1 = 2\cos\theta$$

$$\Rightarrow \cos\theta = -1/2$$

$$\Rightarrow \theta = \cos^{-1}(-1/2) = 120^\circ = \frac{2\pi}{3}$$

$$\text{Ans: } \theta = \cos^{-1}(-1/2) = 120^\circ = \frac{2\pi}{3}$$

22. Question

If $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$, find $|\vec{a} - \vec{b}|$.

Answer

We know that ,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow 4 = 2 \times 3\cos\theta$$

$$\Rightarrow 4 = 6\cos\theta$$

$$\Rightarrow \cos\theta = 4/6$$

$$\Rightarrow \cos\theta = 2/3$$

$$\Rightarrow |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow |\vec{a} - \vec{b}|^2 = 2^2 + 3^2 - (2 \times 2 \times 3) \times \frac{2}{3}$$

$$\Rightarrow |\vec{a} - \vec{b}|^2 = 4 + 9 - 8 = 5$$

$$\Rightarrow |\vec{a} - \vec{b}| = \sqrt{5}$$

Ans: $\sqrt{5}$

23. Question

If $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$, find $|\vec{a}|$ and $|\vec{b}|$.

Answer

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$$

$$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$$

$$\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8$$

$$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8$$

$$\Rightarrow 63|\vec{b}|^2 = 8$$



$$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}}$$

$$\Rightarrow |\vec{a}| = 8|\vec{b}| = 8\sqrt{\frac{8}{63}}$$

$$\text{Ans: } |\vec{a}| = 8\sqrt{\frac{8}{63}}, |\vec{b}| = \sqrt{\frac{8}{63}}$$

24. Question

If \hat{a} and \hat{b} are unit vectors inclined at an angle θ then prove that:

$$\text{i. } \cos \frac{\theta}{2} = \frac{1}{2} |\hat{a} + \hat{b}|$$

$$\text{ii. } \tan \frac{\theta}{2} = \frac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}$$

Answer

R.H.S:

$$\left(\frac{1}{2}\right) (|\hat{a} + \hat{b}|) = \frac{1}{2} (\sqrt{|\hat{a}|^2 + |\hat{b}|^2 + 2|\hat{a}||\hat{b}|\cos\theta})$$

$$\Rightarrow \frac{1}{2} (\sqrt{1^2 + 1^2 + 2 \times 1 \times 1 \cos\theta})$$

$$\Rightarrow \frac{1}{2} (\sqrt{1 + 1 + 2\cos\theta})$$

$$\Rightarrow \sqrt{\frac{2 + 2\cos\theta}{4}}$$

$$\Rightarrow \sqrt{\frac{2(1 + \cos\theta)}{4}}$$

$$\Rightarrow \sqrt{\frac{(1 + \cos\theta)}{2}}$$

$$\Rightarrow \sqrt{\cos^2 \frac{\theta}{2}}$$

$$\Rightarrow \cos \frac{\theta}{2} = \text{L.H.S}$$

Hence, proved

ii)

$$\text{R.H.S.} = \frac{(|\hat{a} - \hat{b}|)}{(|\hat{a} + \hat{b}|)}$$

$$\Rightarrow \frac{\sqrt{|\hat{a}|^2 + |\hat{b}|^2 - 2|\hat{a}||\hat{b}|\cos\theta}}{\sqrt{|\hat{a}|^2 + |\hat{b}|^2 + 2|\hat{a}||\hat{b}|\cos\theta}}$$

$$\Rightarrow \frac{\sqrt{1^2 + 1^2 - 2 \times 1 \times 1 \cos\theta}}{\sqrt{1^2 + 1^2 + 2 \times 1 \times 1 \cos\theta}}$$

$$\Rightarrow \frac{\sqrt{1 + 1 - 2\cos\theta}}{\sqrt{1 + 1 + 2\cos\theta}}$$

$$\Rightarrow \frac{\sqrt{1 - \cos\theta}}{\sqrt{1 + \cos\theta}}$$



$$\Rightarrow \sqrt{\frac{\sin^2 \theta}{\cos^2 \frac{\theta}{2}}}$$

$$\Rightarrow \sqrt{\tan^2 \frac{\theta}{2}}$$

$$\Rightarrow \tan \theta/2 = \text{L.H.S}$$

Hence, proved.

25. Question

The dot products of a vector with the vector $(\hat{i} + \hat{j} - 3\hat{k})$, $(\hat{i} + 3\hat{j} - 2\hat{k})$ and $(2\hat{i} + \hat{j} + 4\hat{k})$ are 0, 5 and 8 respectively. Find the vector.

Answer

Let the unknown vector be: $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$

$$\therefore (a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + \hat{j} - 3\hat{k}) = 0$$

$$\Rightarrow a + b - 3c = 0 \dots(1)$$

$$\therefore (a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + 3\hat{j} - 2\hat{k}) = 5$$

$$\Rightarrow a + 3b - 2c = 5 \dots(2)$$

$$\therefore (a\hat{i} + b\hat{j} + c\hat{k}) \cdot (2\hat{i} + \hat{j} + 4\hat{k}) = 8$$

$$\Rightarrow 2a + b + 4c = 8 \dots(3)$$

Solving equations 1, 2, 3, simultaneously we get:

$$a = 1, b = 2, c = 1$$

$$\therefore \vec{a} = \hat{i} + 2\hat{j} + \hat{k}$$

$$\text{Ans: } \vec{a} = \hat{i} + 2\hat{j} + \hat{k}$$

26. Question

If $\vec{AB} = (3\hat{i} - \hat{j} + 2\hat{k})$ and the coordinates of A are (0, -2, -1), find the coordinates of B.

Answer

$$\vec{AB} = \vec{B} - \vec{A} = 3\hat{i} - \hat{j} + 2\hat{k}$$

$$\Rightarrow \vec{B} - (0\hat{i} - 2\hat{j} - \hat{k}) = 3\hat{i} - \hat{j} + 2\hat{k}$$

$$\Rightarrow \vec{B} = (0\hat{i} - 2\hat{j} - \hat{k}) + 3\hat{i} - \hat{j} + 2\hat{k}$$

$$\Rightarrow \vec{B} = 3\hat{i} - 3\hat{j} + \hat{k}$$

$$\therefore B(3, -3, 1)$$

$$\text{Ans: } B(3, -3, 1)$$

27. Question

If A(2, 3, 4), B(5, 4, -1), C(3, 6, 2) and D(1, 2, 0) be four points, show that \vec{AB} is perpendicular to \vec{CD} .

Answer

$$\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{B} = 5\hat{i} + 4\hat{j} - \hat{k}$$

$$\vec{C} = 3\hat{i} + 6\hat{j} + 2\hat{k}$$

$$\vec{D} = \hat{i} + 2\hat{j} + 0\hat{k}$$

$$\vec{AB} = \vec{B} - \vec{A} = 5\hat{i} + 4\hat{j} - \hat{k} - (2\hat{i} + 3\hat{j} + 4\hat{k}) = 3\hat{i} + \hat{j} - 5\hat{k}$$

$$\vec{CD} = \vec{D} - \vec{C} = \hat{i} + 2\hat{j} + 0\hat{k} - (3\hat{i} + 6\hat{j} + 2\hat{k}) = -2\hat{i} - 4\hat{j} - 2\hat{k}$$

$$\vec{AB} \cdot \vec{CD} = (3\hat{i} + \hat{j} - 5\hat{k}) \cdot (-2\hat{i} - 4\hat{j} - 2\hat{k}) = -6 - 4 + 10 = 0$$

Hence, $\vec{AB} \perp \vec{CD}$

28. Question

Find the value of λ for which the vectors $(2\hat{i} + \lambda\hat{j} + 3\hat{k})$ and $(3\hat{i} + 2\hat{j} - 4\hat{k})$ are perpendicular to each other.

Answer

$$\vec{a} = 2\hat{i} + \lambda\hat{j} + 3\hat{k}$$

$$\vec{b} = 3\hat{i} + 2\hat{j} - 4\hat{k}$$

Since these two vectors are perpendicular, their dot product is zero.

$$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (2\hat{i} + \lambda\hat{j} + 3\hat{k}) \cdot (3\hat{i} + 2\hat{j} - 4\hat{k}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = (2 \times 3) + (\lambda \times 2) + (3 \times -4) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = 6 + 2\lambda - 12 = 0$$

$$\Rightarrow 6 = 2\lambda$$

$$\Rightarrow \lambda = \frac{6}{2} = 3$$

Ans: $\lambda = 3$

29. Question

Show that the vectors $\vec{a} = (3\hat{i} - 2\hat{j} + \hat{k})$, $\vec{b} = (\hat{i} - 3\hat{j} + 5\hat{k})$ and $\vec{c} = (2\hat{i} + \hat{j} - 4\hat{k})$ form a right - angled triangle.

Answer

$$\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$$

$$\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$$

$$|\vec{a}| = \sqrt{9 + 4 + 1} = \sqrt{14}$$

$$|\vec{c}| = \sqrt{4 + 1 + 16} = \sqrt{21}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{c}}{|\vec{a}||\vec{c}|} = \frac{(3\hat{i} - 2\hat{j} + \hat{k}) \cdot (2\hat{i} + \hat{j} - 4\hat{k})}{\sqrt{14}\sqrt{21}} = \frac{6 - 2 - 4}{\sqrt{14}\sqrt{21}} = 0$$

$$\Rightarrow \theta = \cos^{-1}0 = \frac{\pi}{2}$$

Hence, the triangle is a right angled triangle at c

30. Question

Three vertices of a triangle are A(0, - 1, - 2), B(3, 1, 4) and C(5, 7, 1). Show that it is a right - angled triangle. Also, find its other two angles.

Answer

$$\vec{a} = 0\hat{i} - \hat{j} - 2\hat{k}$$

$$\vec{b} = 3\hat{i} + \hat{j} + 4\hat{k}$$

$$\vec{c} = 5\hat{i} + 7\hat{j} + \hat{k}$$

$$|\overline{AB}| = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$$

$$|\overline{BC}| = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$$

$$|\overline{CA}| = \sqrt{25 + 64 + 9} = \sqrt{98} = 7\sqrt{2}$$

$$\overline{AB} = \vec{B} - \vec{A} = 3\hat{i} + \hat{j} + 4\hat{k} - (0\hat{i} - \hat{j} - 2\hat{k}) = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\overline{BC} = \vec{C} - \vec{B} = 5\hat{i} + 7\hat{j} + \hat{k} - (3\hat{i} + \hat{j} + 4\hat{k}) = 2\hat{i} + 6\hat{j} - 3\hat{k}$$

$$\overline{CA} = \vec{A} - \vec{C} = 0\hat{i} - \hat{j} - 2\hat{k} - (5\hat{i} + 7\hat{j} + \hat{k}) = -5\hat{i} - 8\hat{j} - 3\hat{k}$$

$$\cos \theta = \frac{\overline{AB} \cdot \overline{BC}}{|\overline{AB}| |\overline{BC}|} = \frac{(3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (2\hat{i} + 6\hat{j} - 3\hat{k})}{7 \times 7} = \frac{6 + 12 - 18}{49} = 0$$

$$\therefore \theta = \frac{\pi}{2}$$

$$\cos \alpha = \frac{\overline{CA} \cdot \overline{BC}}{|\overline{CA}| |\overline{BC}|} = \frac{(-5\hat{i} - 8\hat{j} - 3\hat{k}) \cdot (2\hat{i} + 6\hat{j} - 3\hat{k})}{7\sqrt{2} \times 7} = \frac{-10 - 48 + 9}{49\sqrt{2}}$$

$$= \left| \frac{-1}{\sqrt{2}} \right|$$

$$\therefore \theta = \frac{\pi}{4} = 45^\circ$$

$$\cos \alpha = \frac{\overline{CA} \cdot \overline{AB}}{|\overline{CA}| |\overline{AB}|} = \frac{(-5\hat{i} - 8\hat{j} - 3\hat{k}) \cdot (3\hat{i} + 2\hat{j} + 6\hat{k})}{7\sqrt{2} \times 7} = \frac{-15 - 16 + 18}{49\sqrt{2}}$$

$$= \left| \frac{-1}{\sqrt{2}} \right|$$

$$\therefore \theta = \frac{\pi}{4} = 45^\circ$$

Ans: 45°, 90°, 45°

31. Question

If the position vectors of the vertices A, B and C of a ΔABC be (1, 2, 3), (- 1, 0, 0) and (0, 1, 2) respectively then find $\angle ABC$.

Answer

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b} = -\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\vec{c} = 0\hat{i} + \hat{j} + 2\hat{k}$$

$$|\overline{AB}| = \sqrt{4 + 4 + 9} = \sqrt{17}$$

$$|\overline{BC}| = \sqrt{1 + 1 + 4} = \sqrt{6}$$

$$|\overline{CA}| = \sqrt{1 + 1 + 1} = \sqrt{3}$$

$$\overline{AB} = \vec{B} - \vec{A} = -\hat{i} + 0\hat{j} + 0\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k}) = -2\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\overline{BC} = \vec{C} - \vec{B} = 0\hat{i} + 1\hat{j} + 2\hat{k} - (-\hat{i} + 0\hat{j} + 0\hat{k}) = \hat{i} + \hat{j} + 2\hat{k}$$

$$\overline{CA} = \vec{A} - \vec{C} = \hat{i} + 2\hat{j} + 3\hat{k} - (0\hat{i} + 1\hat{j} + 2\hat{k}) = \hat{i} + \hat{j} + \hat{k}$$

$$\cos\theta = \frac{\overline{AB} \cdot \overline{BC}}{|\overline{AB}| |\overline{BC}|} = \frac{(-2\hat{i} - 2\hat{j} - 3\hat{k}) \cdot (\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{17} \times \sqrt{6}} = \frac{-2 - 2 - 6}{\sqrt{102}} = \left| \frac{-10}{\sqrt{102}} \right|$$

$$\therefore \theta = \cos^{-1} \frac{10}{\sqrt{102}}$$

$$\text{Ans: } \theta = \cos^{-1} \frac{10}{\sqrt{102}} = \angle ABC$$

32. Question

If \vec{a} and \vec{b} are two unit vectors such that $|\vec{a} + \vec{b}| = \sqrt{3}$, find $(2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b})$.

Answer

$$|\vec{a}| = |\vec{b}| = 1$$

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow 3 = 1 + 1 + 2\cos\theta$$

$$\Rightarrow \cos\theta = 1/2$$

$$\therefore (2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b}) = 6|\vec{a}|^2 - 5|\vec{b}|^2 - 13\vec{a} \cdot \vec{b}$$

$$\Rightarrow (2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b}) = 6 - 5 - 13|\vec{a}||\vec{b}|\cos\theta = 1 - 13 \times 1 \times 1 \times (1/2)$$

$$\Rightarrow (2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b}) = 1 - \frac{13}{2} = \frac{-11}{2}$$

$$\text{Ans: } (2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b}) = \frac{-11}{2}$$

33. Question

If \vec{a} and \vec{b} are two vectors such that $|\vec{a} + \vec{b}| = |\vec{a}|$ then prove that vector $(2\vec{a} + \vec{b})$ is perpendicular to the vector \vec{b} .

Answer

$$|\vec{a} + \vec{b}| = |\vec{a}|$$

$$\Rightarrow |\vec{a} + \vec{b}|^2 = |\vec{a}|^2$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = |\vec{a}|^2$$

$$\Rightarrow |\vec{b}| = -2|\vec{a}|\cos\theta$$

NOW,

$$(2\vec{a} + \vec{b}) \cdot (\vec{b}) = 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

$$\Rightarrow (2\vec{a} + \vec{b}) \cdot (\vec{b}) = 2|\vec{a}||\vec{b}|\cos\theta + ((2|\vec{a}|\cos\theta)^2)$$

$$\Rightarrow (2\vec{a} + \vec{b}) \cdot (\vec{b}) = 2|\vec{a}|(-2|\vec{a}|\cos\theta)\cos\theta + ((2|\vec{a}|\cos\theta)^2) = 0$$

Hence, $(2\vec{a} + \vec{b}) \perp (\vec{b})$

34. Question

If $\vec{a} = (3\hat{i} - \hat{j})$ and $\vec{b} = (2\hat{i} + \hat{j} - 3\hat{k})$ then express \vec{b} in the form $\vec{b} = (\vec{b}_1 + \vec{b}_2)$, where $\vec{b}_1 \parallel \vec{a}$ and $\vec{b}_2 \perp \vec{a}$.

Answer

Let $b_1 = c$ and $b_2 = d$

$$\vec{a} = (3\hat{i} - \hat{j})$$

$$\vec{b} = (2\hat{i} + \hat{j} - 3\hat{k})$$

$$\Rightarrow \vec{c} \parallel \vec{a} \text{ \& } \vec{d} \perp \vec{a}$$

$$\therefore \vec{b} = \vec{c} + \vec{d}$$

$$\vec{c} = \lambda\vec{a} \text{ \& } \vec{a} \cdot \vec{d} = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = \vec{a} \cdot (\vec{c} + \vec{d})$$

$$\Rightarrow (3\hat{i} - \hat{j}) \cdot (2\hat{i} + \hat{j} - 3\hat{k}) = \vec{a} \cdot \lambda\vec{a} + 0$$

$$\Rightarrow 6 - 1 = \lambda(|\vec{a}|^2) = 10\lambda$$

$$\Rightarrow \lambda = 5/10 = 1/2$$

$$\vec{c} = \lambda\vec{a} = (1/2)(3\hat{i} - \hat{j}) = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right)$$

$$\therefore \vec{b} = \vec{c} + \vec{d}$$

$$\Rightarrow (2\hat{i} + \hat{j} - 3\hat{k}) = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \vec{d}$$

$$\Rightarrow \vec{d} = \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}\right) - 3\hat{k}$$

$$\Rightarrow \vec{b} = b_1 + b_2$$

$$\Rightarrow \vec{b} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \left(\left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}\right) - 3\hat{k}\right)$$

$$\text{Ans: } \vec{b} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \left(\left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}\right) - 3\hat{k}\right)$$

