EXERCISE 4.7

1. Evaluate each of the following:
(i) $\sin^{-1}(\sin \pi/6)$
(ii) $\sin^{-1}(\sin 7\pi/6)$
(iii) $\sin^{-1}(\sin 5\pi/6)$
(iv) $\sin^{-1}(\sin 13\pi/7)$
(v) $\sin^{-1}(\sin 17\pi/8)$
(vi) $\sin^{-1}\{(\sin - 17\pi/8)\}$
(vii) $\sin^{-1}(\sin 3)$
(viii) $\sin^{-1}(\sin 4)$
(ix) $\sin^{-1}(\sin 12)$
(x) $\sin^{-1}(\sin 2)$

Solution:
(i) Given $\sin^{-1}(\sin \pi/6)$
We know that the value of $\sin \pi/6$ is ½
By substituting this value in $\sin^{-1}(\sin \pi/6)$
We get, $\sin^{-1}(1/2)$
Now let $y = \sin^{-1}(1/2)$
$\sin(\pi/6) = ½$
The range of principal value of $\sin^{-1}(-\pi/2, \pi/2)$ and $\sin(\pi/6) = ½$
Therefore $\sin^{-1}(\sin \pi/6) = \pi/6$

(ii) Given $\sin^{-1}(\sin 7\pi/6)$
But we know that $\sin 7\pi/6 = - ½$
By substituting this in $\sin^{-1}(\sin 7\pi/6)$ we get,
$\sin^{-1}(-1/2)$
Now let $y = \sin^{-1}(-1/2)$
$- \sin y = ½$
$- \sin(\pi/6) = ½$
$- \sin(\pi/6) = \sin(- \pi/6)$
The range of principal value of $\sin^{-1}(-\pi/2, \pi/2)$ and $\sin(- \pi/6) = - ½$
Therefore $\sin^{-1}(\sin 7\pi/6) = - \pi/6$

(iii) Given $\sin^{-1}(\sin 5\pi/6)$

We know that the value of sin 5π/6 is ½
By substituting this value in $\sin^{-1}$(sin 5π/6)
We get, $\sin^{-1}$ (1/2)
Now let y = $\sin^{-1}$ (1/2)
Sin (π/6) = ½
The range of principal value of $\sin^{-1}$(-π/2, π/2) and sin (π/6) = ½
Therefore $\sin^{-1}$(sin 5π/6) = π/6

(iv) Given $\sin^{-1}$(sin 13π/7)
Given question can be written as sin (2π – π/7)
Sin (2π – π/7) can be written as sin (-π/7) [since sin (2π – θ) = sin (-θ)]
By substituting these values in $\sin^{-1}$(sin 13π/7) we get $\sin^{-1}$(sin - π/7)
As $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2]
Therefore $\sin^{-1}$(sin 13π/7) = - π/7

(v) Given $\sin^{-1}$(sin 17π/8)
Given question can be written as sin (2π + π/8)
Sin (2π + π/8) can be written as sin (π/8)
By substituting these values in $\sin^{-1}$(sin 17π/8) we get $\sin^{-1}$(sin π/8)
As $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2]
Therefore $\sin^{-1}$(sin 17π/8) = π/8

(vi) Given $\sin^{-1}${(sin - 17π/8)}
But we know that – sin θ = sin (-θ)
Therefore (sin -17π/8) = - sin 17π/8
- Sin 17π/8 = - sin (2π + π/8) [since sin (2π – θ) = -sin (θ)]
It can also be written as – sin (π/8)
– Sin (π/8) = sin (-π/8) [since – sin θ = sin (-θ)]
By substituting these values in $\sin^{-1}${(sin - 17π/8)} we get,
Sin^{-1}(sin - π/8)
As $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2]
Therefore $\sin^{-1}$(sin -π/8) = - π/8

(vii) Given $\sin^{-1}$(sin 3)
We know that $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2] which is approximately equal to [-1.57, 1.57]
But here x = 3, which does not lie on the above range,

Therefore we know that sin (π – x) = sin (x)
Hence sin (π – 3) = sin (3) also π – 3 ∈ [-π/2, π/2]
Sin^{-1}(sin 3) = π – 3

(viii) Given $\sin^{-1}$(sin 4)
We know that $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2] which is approximately equal to [-1.57, 1.57]
But here x = 4, which does not lie on the above range,
Therefore we know that sin (π – x) = sin (x)
Hence sin (π – 4) = sin (4) also π – 4 ∈ [-π/2, π/2]
Sin^{-1}(sin 4) = π – 4

(ix) Given $\sin^{-1}$(sin 12)
We know that $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2] which is approximately equal to [-1.57, 1.57]
But here x = 12, which does not lie on the above range,
Therefore we know that sin (2nπ – x) = sin (-x)
Hence sin (2nπ – 12) = sin (-12)
Here n = 2 also 12 – 4π ∈ [-π/2, π/2]
Sin^{-1}(sin 12) = 12 – 4π

(x) Given $\sin^{-1}$(sin 2)
We know that $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2] which is approximately equal to [-1.57, 1.57]
But here x = 2, which does not lie on the above range,
Therefore we know that sin (π – x) = sin (x)
Hence sin (π – 2) = sin (2) also π – 2 ∈ [-π/2, π/2]
Sin^{-1}(sin 2) = π – 2

2. Evaluate each of the following:
(i) $\cos^{-1}$\{cos (-π/4)\}
(ii) $\cos^{-1}$(cos 5π/4)
(iii) $\cos^{-1}$(cos 4π/3)
(iv) $\cos^{-1}$(cos 13π/6)
(v) $\cos^{-1}$(cos 3)
(vi) $\cos^{-1}$(cos 4)
(vii) $\cos^{-1}$(cos 5)

(viii) $\cos^{-1}(\cos 12)$

Solution:

(i) Given $\cos^{-1}\{\cos(-\pi/4)\}$

We know that $\cos(-\pi/4) = \cos(\pi/4)$ [since $\cos(-\theta) = \cos\theta$

Also know that $\cos(\pi/4) = 1/\sqrt{2}$

By substituting these values in $\cos^{-1}\{\cos(-\pi/4)\}$ we get,

$\text{Cos}^{-1}(1/\sqrt{2})$

Now let $y = \cos^{-1}(1/\sqrt{2})$

Therefore $\cos y = 1/\sqrt{2}$

Hence range of principal value of $\cos^{-1}$ is $[0, \pi]$ and $\cos(\pi/4) = 1/\sqrt{2}$

Therefore $\cos^{-1}\{\cos(-\pi/4)\} = \pi/4$

(ii) Given $\cos^{-1}(\cos 5\pi/4)$

But we know that $\cos(5\pi/4) = -1/\sqrt{2}$

By substituting these values in $\cos^{-1}\{\cos(5\pi/4)\}$ we get,

$\text{Cos}^{-1}(-1/\sqrt{2})$

Now let $y = \cos^{-1}(-1/\sqrt{2})$

Therefore $\cos y = -1/\sqrt{2}$

$-\text{Cos}(\pi/4) = 1/\sqrt{2}$

$\text{Cos}(\pi - \pi/4) = -1/\sqrt{2}$

$\text{Cos}(3\pi/4) = -1/\sqrt{2}$

Hence range of principal value of $\cos^{-1}$ is $[0, \pi]$ and $\cos(3\pi/4) = -1/\sqrt{2}$

Therefore $\cos^{-1}\{\cos(5\pi/4)\} = 3\pi/4$

(iii) Given $\cos^{-1}(\cos 4\pi/3)$

But we know that $\cos(4\pi/3) = -1/2$

By substituting these values in $\cos^{-1}\{\cos(4\pi/3)\}$ we get,

$\text{Cos}^{-1}(-1/2)$

Now let $y = \cos^{-1}(-1/2)$

Therefore $\cos y = -1/2$

$-\text{Cos}(\pi/3) = 1/2$

$\text{Cos}(\pi - \pi/3) = -1/2$

$\text{Cos}(2\pi/3) = -1/2$

Hence range of principal value of $\cos^{-1}$ is $[0, \pi]$ and $\cos(2\pi/3) = -1/2$

Therefore $\cos^{-1}\{\cos(4\pi/3)\} = 2\pi/3$

(iv) Given $\cos^{-1}(\cos 13\pi/6)$
But we know that $\cos (13\pi/6) = \sqrt{3}/2$
By substituting these values in $\cos^{-1}\{\cos (13\pi/6)\}$ we get,
$\cos^{-1}(\sqrt{3}/2)$
Now let $y = \cos^{-1}(\sqrt{3}/2)$
Therefore $\cos y = \sqrt{3}/2$
$\cos (\pi/6) = \sqrt{3}/2$
Hence range of principal value of $\cos^{-1}$ is $[0, \pi]$ and $\cos (\pi/6) = \sqrt{3}/2$
Therefore $\cos^{-1}\{\cos (13\pi/6)\} = \pi/6$

(v) Given $\cos^{-1}(\cos 3)$
We know that $\cos^{-1}(\cos \theta) = \theta$ if $0 \le \theta \le \pi$
Therefore by applying this in given question we get,
$\cos^{-1}(\cos 3) = 3, 3 \in [0, \pi]$

(vi) Given $\cos^{-1}(\cos 4)$
We have $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi] \approx [0, 3.14]$
And here $x = 4$ which does not lie in the above range.
We know that $\cos (2\pi - x) = \cos(x)$
Thus, $\cos (2\pi - 4) = \cos (4)$ so $2\pi - 4$ belongs in $[0, \pi]$
Hence $\cos^{-1}(\cos 4) = 2\pi - 4$

(vii) Given $\cos^{-1}(\cos 5)$
We have $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi] \approx [0, 3.14]$
And here $x = 5$ which does not lie in the above range.
We know that $\cos (2\pi - x) = \cos(x)$
Thus, $\cos (2\pi - 5) = \cos (5)$ so $2\pi - 5$ belongs in $[0, \pi]$
Hence $\cos^{-1}(\cos 5) = 2\pi - 5$

(viii) Given $\cos^{-1}(\cos 12)$
$\cos^{-1}(\cos x) = x$ if $x \in [0, \pi] \approx [0, 3.14]$
And here $x = 12$ which does not lie in the above range.
We know $\cos (2n\pi - x) = \cos (x)$
$\cos (2n\pi - 12) = \cos (12)$
Here $n = 2$.
Also $4\pi - 12$ belongs in $[0, \pi]$
$\therefore \cos^{-1}(\cos 12) = 4\pi - 12$

3. Evaluate each of the following:

(i) $\tan^{-1}(\tan \pi/3)$

(ii) $\tan^{-1}(\tan 6\pi/7)$

(iii) $\tan^{-1}(\tan 7\pi/6)$

(iv) $\tan^{-1}(\tan 9\pi/4)$

(v) $\tan^{-1}(\tan 1)$

(vi) $\tan^{-1}(\tan 2)$

(vii) $\tan^{-1}(\tan 4)$

(viii) $\tan^{-1}(\tan 12)$

Solution:

(i) Given $\tan^{-1}(\tan \pi/3)$

As $\tan^{-1}(\tan x) = x$ if $x \in [-\pi/2, \pi/2]$

By applying this condition in the given question we get,

$\text{Tan}^{-1}(\tan \pi/3) = \pi/3$

(ii) Given $\tan^{-1}(\tan 6\pi/7)$

We know that tan $6\pi/7$ can be written as $(\pi - \pi/7)$

$\text{Tan}(\pi - \pi/7) = -\tan \pi/7$

We know that $\tan^{-1}(\tan x) = x$ if $x \in [-\pi/2, \pi/2]$

$\text{Tan}^{-1}(\tan 6\pi/7) = -\pi/7$

(iii) Given $\tan^{-1}(\tan 7\pi/6)$

We know that $\tan 7\pi/6 = 1/\sqrt{3}$

By substituting this value in $\tan^{-1}(\tan 7\pi/6)$ we get,

$\text{Tan}^{-1}(1/\sqrt{3})$

Now let $\tan^{-1}(1/\sqrt{3}) = y$

$\text{Tan } y = 1/\sqrt{3}$

$\text{Tan}(\pi/6) = 1/\sqrt{3}$

The range of the principal value of $\tan^{-1}$ is $(-\pi/2, \pi/2)$ and $\tan(\pi/6) = 1/\sqrt{3}$

Therefore $\tan^{-1}(\tan 7\pi/6) = \pi/6$

(iv) Given $\tan^{-1}(\tan 9\pi/4)$

We know that $\tan 9\pi/4 = 1$

By substituting this value in $\tan^{-1}(\tan 9\pi/4)$ we get,

$\text{Tan}^{-1}(1)$

Now let $\tan^{-1}(1) = y$

Tan y = 1
Tan (π/4) = 1
The range of the principal value of $\tan^{-1}$ is (-π/2, π/2) and tan (π/4) = 1
Therefore $\tan^{-1}$(tan 9π/4) = π/4

(v) Given $\tan^{-1}$(tan 1)
But we have $\tan^{-1}$(tan x) = x if x ϵ [-π/2, π/2]
By substituting this condition in given question
Tan^{-1}(tan 1) = 1

(vi) Given $\tan^{-1}$(tan 2)
As $\tan^{-1}$(tan x) = x if x ϵ [-π/2, π/2]
But here x = 2 which does not belongs to above range
We also have tan (π – θ) = –tan (θ)
Therefore tan (θ – π) = tan (θ)
Tan (2 – π) = tan (2)
Now 2 – π is in the given range
Hence $\tan^{-1}$ (tan 2) = 2 – π

(vii) Given $\tan^{-1}$(tan 4)
As $\tan^{-1}$(tan x) = x if x ϵ [-π/2, π/2]
But here x = 4 which does not belongs to above range
We also have tan (π – θ) = –tan (θ)
Therefore tan (θ – π) = tan (θ)
Tan (4 – π) = tan (4)
Now 4 – π is in the given range
Hence $\tan^{-1}$ (tan 2) = 4 – π

(viii) Given $\tan^{-1}$(tan 12)
As $\tan^{-1}$(tan x) = x if x ϵ [-π/2, π/2]
But here x = 12 which does not belongs to above range
We know that tan (2nπ – θ) = –tan (θ)
Tan (θ – 2nπ) = tan (θ)
Here n = 2
Tan (12 – 4π) = tan (12)
Now 12 – 4π is in the given range
∴ $\tan^{-1}$ (tan 12) = 12 – 4π.