

EXERCISE 7.1

1. If $\sin A = 4/5$ and $\cos B = 5/13$, where $0 < A, B < \pi/2$, find the values of the following:

(i) $\sin (A + B)$

(ii) $\cos (A + B)$

(iii) $\sin (A - B)$

(iv) $\cos (A - B)$

Solution:

Given:

$$\sin A = 4/5 \text{ and } \cos B = 5/13$$

We know that $\cos A = \sqrt{1 - \sin^2 A}$ and $\sin B = \sqrt{1 - \cos^2 B}$, where $0 < A, B < \pi/2$

So let us find the value of $\sin A$ and $\cos B$

$$\begin{aligned}\cos A &= \sqrt{1 - \sin^2 A} \\ &= \sqrt{1 - (4/5)^2} \\ &= \sqrt{1 - (16/25)} \\ &= \sqrt{(25 - 16)/25} \\ &= \sqrt{9/25} \\ &= 3/5\end{aligned}$$

$$\begin{aligned}\sin B &= \sqrt{1 - \cos^2 B} \\ &= \sqrt{1 - (5/13)^2} \\ &= \sqrt{1 - (25/169)} \\ &= \sqrt{(169 - 25)/169} \\ &= \sqrt{144/169} \\ &= 12/13\end{aligned}$$

(i) $\sin (A + B)$

We know that $\sin (A + B) = \sin A \cos B + \cos A \sin B$

So,

$$\begin{aligned}\sin (A + B) &= \sin A \cos B + \cos A \sin B \\ &= 4/5 \times 5/13 + 3/5 \times 12/13 \\ &= 20/65 + 36/65 \\ &= (20+36)/65 \\ &= 56/65\end{aligned}$$

(ii) $\cos (A + B)$

We know that $\cos (A + B) = \cos A \cos B - \sin A \sin B$

So,

$$\begin{aligned}\cos (A + B) &= \cos A \cos B - \sin A \sin B \\ &= 3/5 \times 5/13 - 4/5 \times 12/13 \\ &= 15/65 - 48/65 \\ &= -33/65\end{aligned}$$

(iii) $\sin (A - B)$

We know that $\sin (A - B) = \sin A \cos B - \cos A \sin B$

So,

$$\begin{aligned}\sin (A - B) &= \sin A \cos B - \cos A \sin B \\ &= 4/5 \times 5/13 - 3/5 \times 12/13 \\ &= 20/65 - 36/65 \\ &= -16/65\end{aligned}$$

(iv) $\cos (A - B)$

We know that $\cos (A - B) = \cos A \cos B + \sin A \sin B$

So,

$$\begin{aligned}\cos (A - B) &= \cos A \cos B + \sin A \sin B \\ &= 3/5 \times 5/13 + 4/5 \times 12/13 \\ &= 15/65 + 48/65 \\ &= 63/65\end{aligned}$$

2. (a) If $\sin A = 12/13$ and $\sin B = 4/5$, where $\pi/2 < A < \pi$ and $0 < B < \pi/2$, find the following:

(i) $\sin (A + B)$ (ii) $\cos (A + B)$

(b) If $\sin A = 3/5$, $\cos B = -12/13$, where A and B , both lie in second quadrant, find the value of $\sin (A + B)$.

Solution:

(a) Given:

$\sin A = 12/13$ and $\sin B = 4/5$, where $\pi/2 < A < \pi$ and $0 < B < \pi/2$

We know that $\cos A = -\sqrt{1 - \sin^2 A}$ and $\cos B = \sqrt{1 - \sin^2 B}$

So let us find the value of $\cos A$ and $\cos B$

$$\begin{aligned}\cos A &= -\sqrt{1 - \sin^2 A} \\ &= -\sqrt{1 - (12/13)^2} \\ &= -\sqrt{1 - 144/169} \\ &= -\sqrt{(169 - 144)/169} \\ &= -\sqrt{25/169} \\ &= -5/13\end{aligned}$$

$$\begin{aligned}\cos B &= \sqrt{1 - \sin^2 B} \\ &= \sqrt{1 - (4/5)^2} \\ &= \sqrt{1 - 16/25} \\ &= \sqrt{(25 - 16)/25} \\ &= \sqrt{9/25} \\ &= 3/5\end{aligned}$$

(i) $\sin(A + B)$

We know that $\sin(A + B) = \sin A \cos B + \cos A \sin B$

So,

$$\begin{aligned}\sin(A + B) &= \sin A \cos B + \cos A \sin B \\ &= 12/13 \times 3/5 + (-5/13) \times 4/5 \\ &= 36/65 - 20/65 \\ &= 16/65\end{aligned}$$

(ii) $\cos(A + B)$

We know that $\cos(A + B) = \cos A \cos B - \sin A \sin B$

So,

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= -5/13 \times 3/5 - 12/13 \times 4/5 \\ &= -15/65 - 48/65 \\ &= -63/65\end{aligned}$$

(b) Given:

$\sin A = 3/5$, $\cos B = -12/13$, where A and B, both lie in second quadrant.

We know that $\cos A = -\sqrt{1 - \sin^2 A}$ and $\sin B = \sqrt{1 - \cos^2 B}$

So let us find the value of $\cos A$ and $\sin B$

$$\begin{aligned}\cos A &= -\sqrt{1 - \sin^2 A} \\ &= -\sqrt{1 - (3/5)^2} \\ &= -\sqrt{1 - 9/25} \\ &= -\sqrt{(25 - 9)/25} \\ &= -\sqrt{16/25} \\ &= -4/5\end{aligned}$$

$$\begin{aligned}\sin B &= \sqrt{1 - \cos^2 B} \\ &= \sqrt{1 - (-12/13)^2} \\ &= \sqrt{1 - 144/169} \\ &= \sqrt{(169 - 144)/169} \\ &= \sqrt{25/169} \\ &= 5/13\end{aligned}$$

We need to find $\sin (A + B)$

$$\begin{aligned}\text{Since, } \sin (A + B) &= \sin A \cos B + \cos A \sin B \\ &= 3/5 \times (-12/13) + (-4/5) \times 5/13 \\ &= -36/65 - 20/65 \\ &= -56/65\end{aligned}$$

3. If $\cos A = -24/25$ and $\cos B = 3/5$, where $\pi < A < 3\pi/2$ and $3\pi/2 < B < 2\pi$, find the following:

(i) $\sin (A + B)$ (ii) $\cos (A + B)$

Solution:

Given:

$$\cos A = -24/25 \text{ and } \cos B = 3/5, \text{ where } \pi < A < 3\pi/2 \text{ and } 3\pi/2 < B < 2\pi$$

We know that A is in third quadrant, B is in fourth quadrant. So sine function is negative.

By using the formulas,

$$\sin A = -\sqrt{1 - \cos^2 A} \text{ and } \sin B = -\sqrt{1 - \cos^2 B}$$

So let us find the value of $\sin A$ and $\sin B$

$$\begin{aligned}\sin A &= -\sqrt{1 - \cos^2 A} \\ &= -\sqrt{1 - (-24/25)^2} \\ &= -\sqrt{1 - 576/625} \\ &= -\sqrt{(625 - 576)/625} \\ &= -\sqrt{49/625} \\ &= -7/25\end{aligned}$$

$$\begin{aligned}\sin B &= -\sqrt{1 - \cos^2 B} \\ &= -\sqrt{1 - (3/5)^2} \\ &= -\sqrt{1 - 9/25} \\ &= -\sqrt{(25 - 9)/25} \\ &= -\sqrt{16/25} \\ &= -4/5\end{aligned}$$

(i) $\sin (A + B)$

We know that $\sin (A + B) = \sin A \cos B + \cos A \sin B$

So,

$$\begin{aligned}\sin (A + B) &= \sin A \cos B + \cos A \sin B \\ &= -7/25 \times 3/5 + (-24/25) \times (-4/5) \\ &= -21/125 + 96/125 \\ &= 75/125 \\ &= 3/5\end{aligned}$$

(ii) $\cos (A + B)$

We know that $\cos (A + B) = \cos A \cos B - \sin A \sin B$

So,

$$\begin{aligned}\cos (A + B) &= \cos A \cos B - \sin A \sin B \\ &= (-24/25) \times 3/5 - (-7/25) \times (-4/5) \\ &= -72/125 - 28/125 \\ &= -100/125 \\ &= -4/5\end{aligned}$$

4. If $\tan A = 3/4$, $\cos B = 9/41$, where $\pi < A < 3\pi/2$ and $0 < B < \pi/2$, find $\tan (A + B)$.

Solution:

Given:

$\tan A = 3/4$ and $\cos B = 9/41$, where $\pi < A < 3\pi/2$ and $0 < B < \pi/2$

We know that, A is in third quadrant, B is in first quadrant.

So, tan function And sine function are positive.

By using the formula,

$$\sin B = \sqrt{(1 - \cos^2 B)}$$

Let us find the value of $\sin B$.

$$\sin B = \sqrt{(1 - \cos^2 B)}$$

$$= \sqrt{(1 - (9/41)^2)}$$

$$= \sqrt{(1 - 81/1681)}$$

$$= \sqrt{((1681 - 81)/1681)}$$

$$= \sqrt{(1600/1681)}$$

$$= 40/41$$

We know, $\tan B = \sin B / \cos B$

$$= (40/41) / (9/41)$$

$$= 40/9$$

So, $\tan (A + B) = (\tan A + \tan B) / (1 - \tan A \tan B)$

$$= (3/4 + 40/9) / (1 - 3/4 \times 40/9)$$

$$= (187/36) / (1 - 120/36)$$

$$= (187/36) / ((36 - 120)/36)$$

$$= (187/36) / (-84/36)$$

$$= -187/84$$

5. If $\sin A = 1/2$, $\cos B = 12/13$, where $\pi/2 < A < \pi$ and $3\pi/2 < B < 2\pi$, find $\tan(A - B)$.

Solution:

Given:

$\sin A = 1/2$, $\cos B = 12/13$, where $\pi/2 < A < \pi$ and $3\pi/2 < B < 2\pi$

We know that, A is in second quadrant, B is in fourth quadrant.

In the second quadrant, sine function is positive, cosine and tan functions are negative.

In the fourth quadrant, sine and tan functions are negative, cosine function is positive.

By using the formulas,

$$\cos A = -\sqrt{1 - \sin^2 A} \text{ and } \sin B = -\sqrt{1 - \cos^2 B}$$

So let us find the value of $\cos A$ and $\sin B$

$$\begin{aligned} \cos A &= -\sqrt{1 - \sin^2 A} \\ &= -\sqrt{1 - (1/2)^2} \\ &= -\sqrt{1 - 1/4} \\ &= -\sqrt{(4-1)/4} \\ &= -\sqrt{3/4} \\ &= -\sqrt{3}/2 \end{aligned}$$

$$\begin{aligned} \sin B &= -\sqrt{1 - \cos^2 B} \\ &= -\sqrt{1 - (12/13)^2} \\ &= -\sqrt{1 - 144/169} \\ &= -\sqrt{(169-144)/169} \\ &= -\sqrt{25/169} \\ &= -5/13 \end{aligned}$$

We know, $\tan A = \sin A / \cos A$ and $\tan B = \sin B / \cos B$

$$\tan A = (1/2) / (-\sqrt{3}/2) = -1/\sqrt{3} \text{ and}$$

$$\tan B = (-5/13) / (12/13) = -5/12$$

$$\begin{aligned} \text{So, } \tan(A - B) &= (\tan A - \tan B) / (1 + \tan A \tan B) \\ &= ((-1/\sqrt{3}) - (-5/12)) / (1 + (-1/\sqrt{3}) \times (-5/12)) \\ &= ((-12+5\sqrt{3})/12\sqrt{3}) / (1 + 5/12\sqrt{3}) \\ &= ((-12+5\sqrt{3})/12\sqrt{3}) / ((12\sqrt{3} + 5)/12\sqrt{3}) \\ &= (5\sqrt{3} - 12) / (5 + 12\sqrt{3}) \end{aligned}$$

6. If $\sin A = 1/2$, $\cos B = \sqrt{3}/2$, where $\pi/2 < A < \pi$ and $0 < B < \pi/2$, find the following:

(i) $\tan(A + B)$ (ii) $\tan(A - B)$

Solution:

Given:

$$\sin A = 1/2 \text{ and } \cos B = \sqrt{3}/2, \text{ where } \pi/2 < A < \pi \text{ and } 0 < B < \pi/2$$

We know that, A is in second quadrant, B is in first quadrant.

In the second quadrant, sine function is positive. cosine and tan functions are negative.

In first quadrant, all functions are positive.

By using the formulas,

$$\cos A = -\sqrt{1 - \sin^2 A} \text{ and } \sin B = \sqrt{1 - \cos^2 B}$$

So let us find the value of $\cos A$ and $\sin B$

$$\begin{aligned} \cos A &= -\sqrt{1 - \sin^2 A} \\ &= -\sqrt{1 - (1/2)^2} \\ &= -\sqrt{1 - 1/4} \\ &= -\sqrt{(4-1)/4} \\ &= -\sqrt{3/4} \\ &= -\sqrt{3}/2 \end{aligned}$$

$$\begin{aligned} \sin B &= \sqrt{1 - \cos^2 B} \\ &= \sqrt{1 - (\sqrt{3}/2)^2} \\ &= \sqrt{1 - 3/4} \\ &= \sqrt{(4-3)/4} \\ &= \sqrt{1/4} \\ &= 1/2 \end{aligned}$$

We know, $\tan A = \sin A / \cos A$ and $\tan B = \sin B / \cos B$

$$\tan A = (1/2) / (-\sqrt{3}/2) = -1/\sqrt{3} \text{ and}$$

$$\tan B = (1/2) / (\sqrt{3}/2) = 1/\sqrt{3}$$

$$\begin{aligned} \text{(i) } \tan(A + B) &= (\tan A + \tan B) / (1 - \tan A \tan B) \\ &= (-1/\sqrt{3} + 1/\sqrt{3}) / (1 - (-1/\sqrt{3}) \times 1/\sqrt{3}) \\ &= 0 / (1 + 1/3) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{(ii) } \tan(A - B) &= (\tan A - \tan B) / (1 + \tan A \tan B) \\ &= ((-1/\sqrt{3}) - (1/\sqrt{3})) / (1 + (-1/\sqrt{3}) \times (1/\sqrt{3})) \\ &= ((-2/\sqrt{3}) / (1 - 1/3)) \\ &= ((-2/\sqrt{3}) / (3-1)/3) \\ &= ((-2/\sqrt{3}) / 2/3) \\ &= -\sqrt{3} \end{aligned}$$

7. Evaluate the following:

(i) $\sin 78^\circ \cos 18^\circ - \cos 78^\circ \sin 18^\circ$

(ii) $\cos 47^\circ \cos 13^\circ - \sin 47^\circ \sin 13^\circ$

(iii) $\sin 36^\circ \cos 9^\circ + \cos 36^\circ \sin 9^\circ$

(iv) $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$

Solution:

(i) $\sin 78^\circ \cos 18^\circ - \cos 78^\circ \sin 18^\circ$

We know that $\sin (A - B) = \sin A \cos B - \cos A \sin B$
 $\sin 78^\circ \cos 18^\circ - \cos 78^\circ \sin 18^\circ = \sin(78 - 18)^\circ$
 $= \sin 60^\circ$
 $= \sqrt{3}/2$

(ii) $\cos 47^\circ \cos 13^\circ - \sin 47^\circ \sin 13^\circ$
We know that $\cos A \cos B - \sin A \sin B = \cos (A + B)$
 $\cos 47^\circ \cos 13^\circ - \sin 47^\circ \sin 13^\circ = \cos (47 + 13)^\circ$
 $= \cos 60^\circ$
 $= 1/2$

(iii) $\sin 36^\circ \cos 9^\circ + \cos 36^\circ \sin 9^\circ$
We know that $\sin (A + B) = \sin A \cos B + \cos A \sin B$
 $\sin 36^\circ \cos 9^\circ + \cos 36^\circ \sin 9^\circ = \sin (36 + 9)^\circ$
 $= \sin 45^\circ$
 $= 1/\sqrt{2}$

(iv) $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$
We know that $\cos A \cos B + \sin A \sin B = \cos (A - B)$
 $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ = \cos (80 - 20)^\circ$
 $= \cos 60^\circ$
 $= 1/2$

8. If $\cos A = -12/13$ and $\cot B = 24/7$, where A lies in the second quadrant and B in the third quadrant, find the values of the following:

(i) $\sin (A + B)$ (ii) $\cos (A + B)$ (iii) $\tan (A + B)$

Solution:

Given:

$$\cos A = -12/13 \text{ and } \cot B = 24/7$$

We know that, A lies in second quadrant, B in the third quadrant.

In the second quadrant sine function is positive.

In the third quadrant, both sine and cosine functions are negative.

By using the formulas,

$$\sin A = \sqrt{1 - \cos^2 A}, \sin B = -1/\sqrt{1 + \cot^2 B} \text{ and } \cos B = -\sqrt{1 - \sin^2 B},$$

So let us find the value of $\sin A$ and $\sin B$

$$\begin{aligned} \sin A &= \sqrt{1 - \cos^2 A} \\ &= \sqrt{1 - (-12/13)^2} \\ &= \sqrt{1 - 144/169} \\ &= \sqrt{(169-144)/169} \end{aligned}$$

$$\begin{aligned} &= \sqrt{(25/169)} \\ &= 5/13 \end{aligned}$$

$$\begin{aligned} \sin B &= -1/\sqrt{(1 + \cot^2 B)} \\ &= -1/\sqrt{(1 + (24/7)^2)} \\ &= -1/\sqrt{(1 + 576/49)} \\ &= -1/\sqrt{((49+576)/49)} \\ &= -1/\sqrt{(625/49)} \\ &= -1/(25/7) \\ &= -7/25 \end{aligned}$$

$$\begin{aligned} \cos B &= -\sqrt{(1 - \sin^2 B)} \\ &= -\sqrt{(1 - (-7/25)^2)} \\ &= -\sqrt{(1 - (49/625))} \\ &= -\sqrt{((625-49)/625)} \\ &= -\sqrt{(576/625)} \\ &= -24/25 \end{aligned}$$

So, now let us find

(i) $\sin (A + B)$

We know that $\sin (A + B) = \sin A \cos B + \cos A \sin B$

So,

$$\begin{aligned} \sin (A + B) &= \sin A \cos B + \cos A \sin B \\ &= 5/13 \times (-24/25) + (-12/13) \times (-7/25) \\ &= -120/325 + 84/325 \\ &= -36/325 \end{aligned}$$

(ii) $\cos (A + B)$

We know that $\cos (A + B) = \cos A \cos B - \sin A \sin B$

So,

$$\begin{aligned} \cos (A + B) &= \cos A \cos B - \sin A \sin B \\ &= -12/13 \times (-24/25) - (5/13) \times (-7/25) \\ &= 288/325 + 35/325 \\ &= 323/325 \end{aligned}$$

(iii) $\tan (A + B)$

$$\begin{aligned} \text{We know that } \tan (A + B) &= \sin (A+B) / \cos (A+B) \\ &= (-36/325) / (323/325) \\ &= -36/323 \end{aligned}$$

9. Prove that: $\cos 7\pi/12 + \cos \pi/12 = \sin 5\pi/12 - \sin \pi/12$

Solution:

We know that, $7\pi/12 = 105^\circ$, $\pi/12 = 15^\circ$; $5\pi/12 = 75^\circ$

Let us consider LHS: $\cos 105^\circ + \cos 15^\circ$

$$\cos (90^\circ + 15^\circ) + \sin (90^\circ - 75^\circ)$$

$$-\sin 15^\circ + \sin 75^\circ$$

$$\sin 75^\circ - \sin 15^\circ$$

$$= \text{RHS}$$

$$\therefore \text{LHS} = \text{RHS}$$

Hence proved.

10. Prove that: $(\tan A + \tan B) / (\tan A - \tan B) = \sin (A + B) / \sin (A - B)$

Solution:

Let us consider LHS: $(\tan A + \tan B) / (\tan A - \tan B)$

$$\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}}$$

$$= \frac{\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}}{\frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B}}$$

We know that $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$

$$= \frac{\sin (A + B)}{\sin (A - B)}$$

$$= \text{RHS}$$

$$\therefore \text{LHS} = \text{RHS}$$

Hence proved.

11. Prove that:

(i) $(\cos 11^\circ + \sin 11^\circ) / (\cos 11^\circ - \sin 11^\circ) = \tan 56^\circ$

(ii) $(\cos 9^\circ + \sin 9^\circ) / (\cos 9^\circ - \sin 9^\circ) = \tan 54^\circ$

(iii) $(\cos 8^\circ - \sin 8^\circ) / (\cos 8^\circ + \sin 8^\circ) = \tan 37^\circ$

Solution:

(i) $(\cos 11^\circ + \sin 11^\circ) / (\cos 11^\circ - \sin 11^\circ) = \tan 56^\circ$

Let us consider LHS:

$$(\cos 11^\circ + \sin 11^\circ) / (\cos 11^\circ - \sin 11^\circ)$$

Now let us divide the numerator and denominator by $\cos 11^\circ$ we get,

$$\begin{aligned} (\cos 11^\circ + \sin 11^\circ) / (\cos 11^\circ - \sin 11^\circ) &= (1 + \tan 11^\circ) / (1 - \tan 11^\circ) \\ &= (1 + \tan 11^\circ) / (1 - 1 \times \tan 11^\circ) \end{aligned}$$

$$= (\tan 45^\circ + \tan 11^\circ) / (1 - \tan 45^\circ \times \tan 11^\circ)$$

We know that $\tan (A+B) = (\tan A + \tan B) / (1 - \tan A \tan B)$

So,

$$\begin{aligned} (\tan 45^\circ + \tan 11^\circ) / (1 - \tan 45^\circ \times \tan 11^\circ) &= \tan (45^\circ + 11^\circ) \\ &= \tan 56^\circ \end{aligned}$$

= RHS

∴ LHS = RHS

Hence proved.

(ii) $(\cos 9^\circ + \sin 9^\circ) / (\cos 9^\circ - \sin 9^\circ) = \tan 54^\circ$

Let us consider LHS:

$$(\cos 9^\circ + \sin 9^\circ) / (\cos 9^\circ - \sin 9^\circ)$$

Now let us divide the numerator and denominator by $\cos 9^\circ$ we get,

$$\begin{aligned} (\cos 9^\circ + \sin 9^\circ) / (\cos 9^\circ - \sin 9^\circ) &= (1 + \tan 9^\circ) / (1 - \tan 9^\circ) \\ &= (1 + \tan 9^\circ) / (1 - 1 \times \tan 9^\circ) \\ &= (\tan 45^\circ + \tan 9^\circ) / (1 - \tan 45^\circ \times \tan 9^\circ) \end{aligned}$$

We know that $\tan (A+B) = (\tan A + \tan B) / (1 - \tan A \tan B)$

So,

$$\begin{aligned} (\tan 45^\circ + \tan 9^\circ) / (1 - \tan 45^\circ \times \tan 9^\circ) &= \tan (45^\circ + 9^\circ) \\ &= \tan 54^\circ \end{aligned}$$

= RHS

∴ LHS = RHS

Hence proved.

(iii) $(\cos 8^\circ - \sin 8^\circ) / (\cos 8^\circ + \sin 8^\circ) = \tan 37^\circ$

Let us consider LHS:

$$(\cos 8^\circ - \sin 8^\circ) / (\cos 8^\circ + \sin 8^\circ)$$

Now let us divide the numerator and denominator by $\cos 8^\circ$ we get,

$$\begin{aligned} (\cos 8^\circ - \sin 8^\circ) / (\cos 8^\circ + \sin 8^\circ) &= (1 - \tan 8^\circ) / (1 + \tan 8^\circ) \\ &= (1 - \tan 8^\circ) / (1 + 1 \times \tan 8^\circ) \\ &= (\tan 45^\circ - \tan 8^\circ) / (1 + \tan 45^\circ \times \tan 8^\circ) \end{aligned}$$

We know that $\tan (A+B) = (\tan A + \tan B) / (1 - \tan A \tan B)$

So,

$$\begin{aligned} (\tan 45^\circ - \tan 8^\circ) / (1 + \tan 45^\circ \times \tan 8^\circ) &= \tan (45^\circ - 8^\circ) \\ &= \tan 37^\circ \end{aligned}$$

= RHS

∴ LHS = RHS

Hence proved.

12. Prove that:

(i)

$$\sin\left(\frac{\pi}{3} - x\right) \cos\left(\frac{\pi}{6} + x\right) + \cos\left(\frac{\pi}{3} - x\right) \sin\left(\frac{\pi}{6} + x\right) = 1$$

(ii)

$$\sin\left(\frac{4\pi}{9} + 7\right) \cos\left(\frac{\pi}{9} + 7\right) - \cos\left(\frac{4\pi}{9} + 7\right) \sin\left(\frac{\pi}{9} + 7\right) = \frac{\sqrt{3}}{2}$$

(iii)

$$\sin\left(\frac{3\pi}{8} - 5\right) \cos\left(\frac{\pi}{8} + 5\right) + \cos\left(\frac{3\pi}{8} - 5\right) \sin\left(\frac{\pi}{8} + 5\right) = 1$$

Solution:

(i)

$$\sin\left(\frac{\pi}{3} - x\right) \cos\left(\frac{\pi}{6} + x\right) + \cos\left(\frac{\pi}{3} - x\right) \sin\left(\frac{\pi}{6} + x\right) = 1$$

Let us consider LHS:

$$\sin\left(\frac{\pi}{3} - x\right) \cos\left(\frac{\pi}{6} + x\right) + \cos\left(\frac{\pi}{3} - x\right) \sin\left(\frac{\pi}{6} + x\right)$$

We know that $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$\begin{aligned} \sin\left(\frac{\pi}{3} - x\right) \cos\left(\frac{\pi}{6} + x\right) + \cos\left(\frac{\pi}{3} - x\right) \sin\left(\frac{\pi}{6} + x\right) &= \sin\left(\frac{\pi}{3} - x + \frac{\pi}{6} + x\right) \\ &= \sin\left(\frac{2\pi + \pi}{6}\right) \\ &= \sin\left(\frac{\pi}{2}\right) \\ &= \sin 90^\circ \\ &= 1 \end{aligned}$$

= RHS

∴ LHS = RHS

Hence proved.

(ii)

$$\sin\left(\frac{4\pi}{9} + 7\right) \cos\left(\frac{\pi}{9} + 7\right) - \cos\left(\frac{4\pi}{9} + 7\right) \sin\left(\frac{\pi}{9} + 7\right) = \frac{\sqrt{3}}{2}$$

Let us consider LHS:

$$\sin\left(\frac{4\pi}{9} + 7\right) \cos\left(\frac{\pi}{9} + 7\right) - \cos\left(\frac{4\pi}{9} + 7\right) \sin\left(\frac{\pi}{9} + 7\right)$$

We know that $\sin (A - B) = \sin A \cos B - \cos A \sin B$

So,

$$\begin{aligned} \sin \left(\frac{4\pi}{9} + 7 \right) \cos \left(\frac{\pi}{9} + 7 \right) - \cos \left(\frac{4\pi}{9} + 7 \right) \sin \left(\frac{\pi}{9} + 7 \right) &= \sin \left(\frac{4\pi}{9} + 7 - \frac{\pi}{9} - 7 \right) \\ &= \sin \left(\frac{3\pi}{9} \right) \\ &= \sin \left(\frac{\pi}{3} \right) \\ &= \sin 60^\circ \\ &= \sqrt{3}/2 \end{aligned}$$

= RHS

\therefore LHS = RHS

Hence proved.

(iii)

$$\sin \left(\frac{3\pi}{8} - 5 \right) \cos \left(\frac{\pi}{8} + 5 \right) + \cos \left(\frac{3\pi}{8} - 5 \right) \sin \left(\frac{\pi}{8} + 5 \right) = 1$$

Let us consider LHS:

$$\sin \left(\frac{3\pi}{8} - 5 \right) \cos \left(\frac{\pi}{8} + 5 \right) + \cos \left(\frac{3\pi}{8} - 5 \right) \sin \left(\frac{\pi}{8} + 5 \right)$$

We know that $\sin (A + B) = \sin A \cos B + \cos A \sin B$

$$\begin{aligned} \sin \left(\frac{3\pi}{8} - 5 \right) \cos \left(\frac{\pi}{8} + 5 \right) + \cos \left(\frac{3\pi}{8} - 5 \right) \sin \left(\frac{\pi}{8} + 5 \right) &= \sin \left(\frac{3\pi}{8} - 5 + \frac{\pi}{8} + 5 \right) \\ &= \sin \left(\frac{3\pi + \pi}{8} \right) \\ &= \sin \left(\frac{4\pi}{8} \right) \\ &= \sin \left(\frac{\pi}{2} \right) \\ &= \sin 90^\circ \\ &= 1 \end{aligned}$$

= RHS

\therefore LHS = RHS

Hence proved.

13. Prove that: $(\tan 69^\circ + \tan 66^\circ) / (1 - \tan 69^\circ \tan 66^\circ) = -1$

Solution:

Let us consider LHS:

$$(\tan 69^\circ + \tan 66^\circ) / (1 - \tan 69^\circ \tan 66^\circ)$$

We know that, $\tan (A + B) = (\tan A + \tan B) / (1 - \tan A \tan B)$

Here, $A = 69^\circ$ and $B = 66^\circ$

So,

$$\begin{aligned} (\tan 69^\circ + \tan 66^\circ) / (1 - \tan 69^\circ \tan 66^\circ) &= \tan (69 + 66)^\circ \\ &= \tan 135^\circ \\ &= -\tan 45^\circ \\ &= -1 \\ &= \text{RHS} \end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

Hence proved.

14. (i) If $\tan A = 5/6$ and $\tan B = 1/11$, prove that $A + B = \pi/4$

(ii) If $\tan A = m/(m-1)$ and $\tan B = 1/(2m - 1)$, then prove that $A - B = \pi/4$

Solution:

(i) If $\tan A = 5/6$ and $\tan B = 1/11$, prove that $A + B = \pi/4$

Given:

$$\tan A = 5/6 \text{ and } \tan B = 1/11$$

We know that, $\tan (A + B) = (\tan A + \tan B) / (1 - \tan A \tan B)$

$$\begin{aligned} &= [(5/6) + (1/11)] / [1 - (5/6) \times (1/11)] \\ &= (55+6) / (66-5) \\ &= 61/61 \\ &= 1 \\ &= \tan 45^\circ \text{ or } \tan \pi/4 \end{aligned}$$

So, $\tan (A + B) = \tan \pi/4$

$\therefore (A + B) = \pi/4$

Hence proved.

(ii) If $\tan A = m/(m-1)$ and $\tan B = 1/(2m - 1)$, then prove that $A - B = \pi/4$

Given:

$$\tan A = m/(m-1) \text{ and } \tan B = 1/(2m - 1)$$

We know that, $\tan (A - B) = (\tan A - \tan B) / (1 + \tan A \tan B)$

$$\begin{aligned} &= \frac{\frac{m}{m-1} - \frac{1}{2m-1}}{1 + \frac{m}{m-1} \times \frac{1}{2m-1}} \\ &= (2m^2 - m - m + 1) / (2m^2 - m - 2m + 1 + m) \\ &= (2m^2 - 2m + 1) / (2m^2 - 2m + 1) \\ &= 1 \\ &= \tan 45^\circ \text{ or } \tan \pi/4 \end{aligned}$$

So, $\tan (A - B) = \tan \pi/4$

$\therefore (A - B) = \pi/4$
Hence proved.

15. prove that:

(i) $\cos^2 \pi/4 - \sin^2 \pi/12 = \sqrt{3}/4$

(ii) $\sin^2 (n + 1) A - \sin^2 nA = \sin (2n + 1) A \sin A$

Solution:

(i) $\cos^2 \pi/4 - \sin^2 \pi/12 = \sqrt{3}/4$

Let us consider LHS:

$$\cos^2 \pi/4 - \sin^2 \pi/12$$

We know that, $\cos^2 A - \sin^2 B = \cos (A + B) \cos (A - B)$

So,

$$\begin{aligned} \cos^2 \pi/4 - \sin^2 \pi/12 &= \cos (\pi/4 + \pi/12) \cos (\pi/4 - \pi/12) \\ &= \cos 4\pi/12 \cos 2\pi/12 \\ &= \cos \pi/3 \cos \pi/6 \\ &= 1/2 \times \sqrt{3}/2 \\ &= \sqrt{3}/4 \\ &= \text{RHS} \end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

Hence proved.

(ii) $\sin^2 (n + 1) A - \sin^2 nA = \sin (2n + 1) A \sin A$

Let us consider LHS:

$$\sin^2 (n + 1) A - \sin^2 nA$$

We know that, $\sin^2 A - \sin^2 B = \sin (A + B) \sin (A - B)$

Here, $A = (n + 1) A$ and $B = nA$

So,

$$\begin{aligned} \sin^2 (n + 1) A - \sin^2 n A &= \sin ((n + 1) A + nA) \sin ((n + 1) A - nA) \\ &= \sin (nA + A + nA) \sin (nA + A - nA) \\ &= \sin (2nA + A) \sin (A) \\ &= \sin (2n + 1) A \sin A \\ &= \text{RHS} \end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

Hence proved.

16. Prove that:

(i)

$$\frac{\sin (A + B) + \sin (A - B)}{\cos (A + B) + \cos (A - B)} = \tan A$$

$$(ii) \frac{\sin(A - B)}{\cos A \cos B} + \frac{\sin(B - C)}{\cos B \cos C} + \frac{\sin(C - A)}{\cos C \cos A} = 0$$

$$(iii) \frac{\sin(A - B)}{\sin A \sin B} + \frac{\sin(B - C)}{\sin B \sin C} + \frac{\sin(C - A)}{\sin C \sin A} = 0$$

$$(iv) \sin^2 B = \sin^2 A + \sin^2(A - B) - 2\sin A \cos B \sin(A - B)$$

$$(v) \cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A + B) = \sin^2(A + B)$$

$$(vi) \frac{\tan(A + B)}{\cot(A - B)} = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$$

Solution:

$$(i) \frac{\sin(A + B) + \sin(A - B)}{\cos(A + B) + \cos(A - B)} = \tan A$$

Let us consider LHS:

$$\frac{\sin(A + B) + \sin(A - B)}{\cos(A + B) + \cos(A - B)}$$

We know that $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and $\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$

$$\begin{aligned} & \frac{\sin(A + B) + \sin(A - B)}{\cos(A + B) + \cos(A - B)} \\ &= \frac{\sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B}{\cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B} \\ &= \frac{2\sin A \cos B}{2\cos A \cos B} \end{aligned}$$

$$= \tan A$$

$$= \text{RHS}$$

\therefore LHS = RHS

Hence proved.

$$(ii) \frac{\sin(A - B)}{\cos A \cos B} + \frac{\sin(B - C)}{\cos B \cos C} + \frac{\sin(C - A)}{\cos C \cos A} = 0$$

Let us consider LHS:

$$\frac{\sin(A - B)}{\cos A \cos B} + \frac{\sin(B - C)}{\cos B \cos C} + \frac{\sin(C - A)}{\cos C \cos A}$$

We know that, $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$\begin{aligned} & \frac{\sin(A - B)}{\cos A \cos B} + \frac{\sin(B - C)}{\cos B \cos C} + \frac{\sin(C - A)}{\cos C \cos A} \\ &= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B} + \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A - \cos C \sin A}{\cos C \cos A} \end{aligned}$$

$$= \frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B} + \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C} + \frac{\sin C \cos A}{\cos C \cos A} - \frac{\cos C \sin A}{\cos C \cos A}$$

$$= \tan A - \tan B + \tan B - \tan C + \tan C - \tan A$$

$$= 0$$

$$= \text{RHS}$$

$$\therefore \text{LHS} = \text{RHS}$$

Hence proved.

(iii)

$$\frac{\sin(A - B)}{\sin A \sin B} + \frac{\sin(B - C)}{\sin B \sin C} + \frac{\sin(C - A)}{\sin C \sin A} = 0$$

Let us consider LHS:

$$\frac{\sin(A - B)}{\sin A \sin B} + \frac{\sin(B - C)}{\sin B \sin C} + \frac{\sin(C - A)}{\sin C \sin A}$$

We know that, $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$\begin{aligned} & \frac{\sin(A - B)}{\sin A \sin B} + \frac{\sin(B - C)}{\sin B \sin C} + \frac{\sin(C - A)}{\sin C \sin A} \\ &= \frac{\sin A \cos B - \cos A \sin B}{\sin A \sin B} + \frac{\sin B \cos C - \cos B \sin C}{\sin B \sin C} + \frac{\sin C \cos A - \cos C \sin A}{\sin C \sin A} \end{aligned}$$

$$= \frac{\sin A \cos B}{\sin A \sin B} - \frac{\cos A \sin B}{\sin A \sin B} + \frac{\sin B \cos C}{\sin B \sin C} - \frac{\cos B \sin C}{\sin B \sin C} + \frac{\sin C \cos A}{\sin C \sin A} - \frac{\cos C \sin A}{\sin C \sin A}$$

$$= \cot B - \cot A + \cot C - \cot B + \cot A - \cot C$$

$$= 0$$

$$= \text{RHS}$$

$$\therefore \text{LHS} = \text{RHS}$$

Hence proved.

(iv) $\sin^2 B = \sin^2 A + \sin^2(A - B) - 2\sin A \cos B \sin(A - B)$

Let us consider RHS:

$$\sin^2 A + \sin^2 (A - B) - 2 \sin A \cos B \sin (A - B)$$

$$\sin^2 A + \sin (A - B) [\sin (A - B) - 2 \sin A \cos B]$$

$$\text{We know that, } \sin (A - B) = \sin A \cos B - \cos A \sin B$$

So,

$$\sin^2 A + \sin (A - B) [\sin A \cos B - \cos A \sin B - 2 \sin A \cos B]$$

$$\sin^2 A + \sin (A - B) [-\sin A \cos B - \cos A \sin B]$$

$$\sin^2 A - \sin (A - B) [\sin A \cos B + \cos A \sin B]$$

$$\text{We know that, } \sin (A + B) = \sin A \cos B + \cos A \sin B$$

So,

$$\sin^2 A - \sin (A - B) \sin (A + B)$$

$$\sin^2 A - \sin^2 A + \sin^2 B$$

$$\sin^2 B$$

$$= \text{LHS}$$

$$\therefore \text{LHS} = \text{RHS}$$

Hence proved.

$$(v) \cos^2 A + \cos^2 B - 2 \cos A \cos B \cos (A + B) = \sin^2 (A + B)$$

Let us consider LHS:

$$\cos^2 A + \cos^2 B - 2 \cos A \cos B \cos (A + B)$$

$$\cos^2 A + 1 - \sin^2 B - 2 \cos A \cos B \cos (A + B)$$

$$1 + \cos^2 A - \sin^2 B - 2 \cos A \cos B \cos (A + B)$$

$$\text{We know that, } \cos^2 A - \sin^2 B = \cos (A + B) \cos (A - B)$$

So,

$$1 + \cos (A + B) \cos (A - B) - 2 \cos A \cos B \cos (A + B)$$

$$1 + \cos (A + B) [\cos (A - B) - 2 \cos A \cos B]$$

$$\text{We know that, } \cos (A - B) = \cos A \cos B + \sin A \sin B.$$

So,

$$1 + \cos (A + B) [\cos A \cos B + \sin A \sin B - 2 \cos A \cos B]$$

$$1 + \cos (A + B) [-\cos A \cos B + \sin A \sin B]$$

$$1 - \cos (A + B) [\cos A \cos B - \sin A \sin B]$$

$$\text{We know that, } \cos (A + B) = \cos A \cos B - \sin A \sin B.$$

So,

$$1 - \cos^2 (A + B)$$

$$\sin^2 (A + B)$$

$$= \text{RHS}$$

$$\therefore \text{LHS} = \text{RHS}$$

Hence proved.

(vi)

$$\frac{\tan(A+B)}{\cot(A-B)} = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$$

Let us consider LHS:

$$\frac{\tan(A+B)}{\cot(A-B)}$$

We know that,

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \pm \tan A \tan B}$$

$$\frac{\tan(A+B)}{\frac{1}{\tan(A-B)}} = \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B}}{\frac{1}{\frac{\tan A - \tan B}{1 + \tan A \tan B}}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \times \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

We know that, $(x+y)(x-y) = x^2 - y^2$

So,

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} \times \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B} = \text{RHS}$$

\therefore LHS = RHS

Hence proved.

17. Prove that:

(i) $\tan 8x - \tan 6x - \tan 2x = \tan 8x \tan 6x \tan 2x$

(ii) $\tan \pi/12 + \tan \pi/6 + \tan \pi/12 \tan \pi/6 = 1$

(iii) $\tan 36^\circ + \tan 9^\circ + \tan 36^\circ \tan 9^\circ = 1$

(iv) $\tan 13x - \tan 9x - \tan 4x = \tan 13x \tan 9x \tan 4x$

Solution:

(i) $\tan 8x - \tan 6x - \tan 2x = \tan 8x \tan 6x \tan 2x$

Let us consider LHS:

$$\tan 8x - \tan 6x - \tan 2x$$

$$\tan 8x = \tan(6x + 2x)$$

We know that, $\tan(A+B) = (\tan A + \tan B) / (1 - \tan A \tan B)$

So,

$$\tan 8x = (\tan 6x + \tan 2x) / (1 - \tan 6x \tan 2x)$$

By cross-multiplying we get,

$$\tan 8x (1 - \tan 6x \tan 2x) = \tan 6x + \tan 2x$$

$$\tan 8x - \tan 8x \tan 6x \tan 2x = \tan 6x + \tan 2x$$

Upon rearranging we get,

$$\tan 8x - \tan 6x - \tan 2x = \tan 8x \tan 6x \tan 2x$$
$$= \text{RHS}$$

$$\therefore \text{LHS} = \text{RHS}$$

Hence proved.

$$\text{(ii) } \tan \pi/12 + \tan \pi/6 + \tan \pi/12 \tan \pi/6 = 1$$

We know,

$$\pi/12 = 15^\circ \text{ and } \pi/6 = 30^\circ$$

$$\text{So, we have } 15^\circ + 30^\circ = 45^\circ$$

$$\tan (15^\circ + 30^\circ) = \tan 45^\circ$$

$$\text{We know that, } \tan (A + B) = (\tan A + \tan B) / (1 - \tan A \tan B)$$

So,

$$(\tan 15^\circ + \tan 30^\circ) / (1 - \tan 15^\circ \tan 30^\circ) = 1$$

$$\tan 15^\circ + \tan 30^\circ = 1 - \tan 15^\circ \tan 30^\circ$$

Upon rearranging we get,

$$\tan 15^\circ + \tan 30^\circ + \tan 15^\circ \tan 30^\circ = 1$$

Hence proved.

$$\text{(iii) } \tan 36^\circ + \tan 9^\circ + \tan 36^\circ \tan 9^\circ = 1$$

$$\text{We know } 36^\circ + 9^\circ = 45^\circ$$

So we have,

$$\tan (36^\circ + 9^\circ) = \tan 45^\circ$$

$$\text{We know that, } \tan (A + B) = (\tan A + \tan B) / (1 - \tan A \tan B)$$

So,

$$(\tan 36^\circ + \tan 9^\circ) / (1 - \tan 36^\circ \tan 9^\circ) = 1$$

$$\tan 36^\circ + \tan 9^\circ = 1 - \tan 36^\circ \tan 9^\circ$$

Upon rearranging we get,

$$\tan 36^\circ + \tan 9^\circ + \tan 36^\circ \tan 9^\circ = 1$$

Hence proved.

$$\text{(iv) } \tan 13x - \tan 9x - \tan 4x = \tan 13x \tan 9x \tan 4x$$

Let us consider LHS:

$$\tan 13x - \tan 9x - \tan 4x$$

$$\tan 13x = \tan (9x + 4x)$$

$$\text{We know that, } \tan (A + B) = (\tan A + \tan B) / (1 - \tan A \tan B)$$

So,

$$\tan 13x = (\tan 9x + \tan 4x) / (1 - \tan 9x \tan 4x)$$

By cross-multiplying we get,

$$\tan 13x (1 - \tan 9x \tan 4x) = \tan 9x + \tan 4x$$

$$\tan 13x - \tan 13x \tan 9x \tan 4x = \tan 9x + \tan 4x$$

Upon rearranging we get,

$$\tan 13x - \tan 9x - \tan 4x = \tan 13x \tan 9x \tan 4x$$

$$= \text{RHS}$$

$$\therefore \text{LHS} = \text{RHS}$$

Hence proved.



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