

EXERCISE 16.2

1. Find the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = a$, at the point $(a^2/4, a^2/4)$.

Solution:

$$\text{Given } \sqrt{x} + \sqrt{y} = a$$

To find the slope of the tangent of the given curve we have to differentiate the given equation

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \left(\frac{dy}{dx} \right) = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{x}}{\sqrt{y}}$$

At $(\frac{a^2}{4}, \frac{a^2}{4})$ slope m , is -1

The equation of the tangent is given by $y - y_1 = m(x - x_1)$

$$y - \frac{a^2}{4} = -1 \left(x - \frac{a^2}{4} \right)$$

$$x + y = \frac{a^2}{2}$$

2. Find the equation of the normal to $y = 2x^3 - x^2 + 3$ at $(1, 4)$.

Solution:

$$\text{Given } y = 2x^3 - x^2 + 3$$

By differentiating the given curve, we get the slope of the tangent

$$m = \frac{dy}{dx} = 6x^2 - 2x$$

$$m = 4 \text{ at } (1, 4)$$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

$$m(\text{normal}) = -\frac{1}{4}$$

Equation of normal is given by $y - y_1 = m(\text{normal})(x - x_1)$

$$y - 4 = \left(-\frac{1}{4}\right)(x - 1)$$

$$x + 4y = 17$$

3. Find the equation of the tangent and the normal to the following curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(0, 5)$

Solution:

Given $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(0, 5)$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$m(\text{tangent}) \text{ at } (0, 5) = -10$$

$$m(\text{normal}) \text{ at } (0, 5) = \frac{1}{10}$$

Equation of tangent is given by $y - y_1 = m(\text{tangent})(x - x_1)$

$$y - 5 = -10x$$

$$y + 10x = 5$$

Equation of normal is given by $y - y_1 = m(\text{normal})(x - x_1)$

$$y - 5 = \frac{1}{10}x$$

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $x = 1, y = 3$

Solution:

Given $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $x = 1$ $y = 3$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

m (tangent) at $(x = 1) = 2$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

$$m(\text{normal}) \text{ at } (x = 1) = -\frac{1}{2}$$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - 3 = 2(x - 1)$$

$$y = 2x + 1$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$2y = 7 - x$$

(iii) $y = x^2$ at $(0, 0)$

Solution:

Given $y = x^2$ at $(0, 0)$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 2x$$

m (tangent) at $(x = 0) = 0$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

$$m(\text{normal}) \text{ at } (x = 0) = \frac{1}{0}$$

We can see that the slope of normal is not defined

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y = 0$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$x = 0$$

(iv) $y = 2x^2 - 3x - 1$ at $(1, -2)$

Solution:

Given $y = 2x^2 - 3x - 1$ at $(1, -2)$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 4x - 3$$

m (tangent) at $(1, -2) = 1$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

m (normal) at $(1, -2) = -1$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y + 2 = 1(x - 1)$$

$$y = x - 3$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y + 2 = -1(x - 1)$$

$$y + x + 1 = 0$$

(v) $y^2 = \frac{x^3}{4 - x}$

Solution:

By differentiating the given curve, we get the slope of the tangent

$$2y \frac{dy}{dx} = \frac{(4 - x)3x^2 + x^4}{(4 - x)^2}$$

$$\frac{dy}{dx} = \frac{(4 - x)3x^2 + x^4}{2y(4 - x)^2}$$

m (tangent) at $(2, -2) = -2$

m (normal) at $(2, -2) = \frac{1}{2}$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y + 2 = -2(x - 2)$$

$$y + 2x = 2$$

Equation of normal is given by $y - y_1 = m \text{ (normal)} (x - x_1)$

$$y + 2 = \frac{1}{2}(x - 2)$$

$$2y + 4 = x - 2$$

$$2y - x + 6 = 0$$

4. Find the equation of the tangent to the curve $x = \theta + \sin \theta$, $y = 1 + \cos \theta$ at $\theta = \pi/4$.

Solution:

Given $x = \theta + \sin \theta$, $y = 1 + \cos \theta$ at $\theta = \pi/4$

By differentiating the given equation with respect to θ , we get the slope of the tangent

$$\frac{dx}{d\theta} = 1 + \cos \theta$$

$$\frac{dy}{d\theta} = -\sin \theta$$

Dividing both the above equations

$$\frac{dy}{dx} = \frac{\sin \theta}{1 + \cos \theta}$$

$$m \text{ at } \theta = (\pi/4) = -1 + \frac{1}{\sqrt{2}}$$

Equation of tangent is given by $y - y_1 = m \text{ (tangent)} (x - x_1)$

$$y - 1 - \frac{1}{\sqrt{2}} = \left(-1 + \frac{1}{\sqrt{2}}\right) \left(x - \frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$$

5. Find the equation of the tangent and the normal to the following curves at the indicated points:

(i) $x = \theta + \sin \theta$, $y = 1 + \cos \theta$ at $\theta = \pi/2$

Solution:

Given $x = \theta + \sin \theta$, $y = 1 + \cos \theta$ at $\theta = \pi/2$

By differentiating the given equation with respect to θ , we get the slope of the tangent

$$\frac{dx}{d\theta} = 1 + \cos\theta$$

$$\frac{dy}{d\theta} = -\sin\theta$$

Dividing both the above equations

$$\frac{dy}{dx} = -\frac{\sin\theta}{1 + \cos\theta}$$

$$m \text{ (tangent) at } \theta = (\pi/2) = -1$$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

$$m \text{ (normal) at } \theta = (\pi/2) = 1$$

Equation of tangent is given by $y - y_1 = m \text{ (tangent)} (x - x_1)$

$$y - 1 = -1 \left(x - \frac{\pi}{2} - 1 \right)$$

Equation of normal is given by $y - y_1 = m \text{ (normal)} (x - x_1)$

$$y - 1 = 1 \left(x - \frac{\pi}{2} - 1 \right)$$

$$(ii) \ x = \frac{2at^2}{1+t^2}, \ y = \frac{2at^3}{1+t^2} \text{ at } t = \frac{1}{2}$$

Solution:

By differentiating the given equation with respect to t , we get the slope of the tangent

$$\frac{dx}{dt} = \frac{(1+t^2)4at - 2at^2(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{4at}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1+t^2)6at^2 - 2at^3(2t)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at^2 + 2at^4}{(1 + t^2)^2}$$

Now dividing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{6at^2 + 2at^4}{4at}$$

m (tangent) at $t = \frac{1}{2}$ is $\frac{13}{16}$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

m (normal) at $t = \frac{1}{2}$ is $-\frac{16}{13}$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - \frac{a}{5} = \frac{13}{16} \left(x - \frac{2a}{5} \right)$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - \frac{a}{5} = -\frac{16}{13} \left(x - \frac{2a}{5} \right)$$

(iii) $x = at^2$, $y = 2at$ at $t = 1$.

Solution:

Given $x = at^2$, $y = 2at$ at $t = 1$.

By differentiating the given equation with respect to t , we get the slope of the tangent

$$\frac{dx}{dt} = 2at$$

$$\frac{dy}{dt} = 2a$$

Now dividing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{1}{t}$$

m (tangent) at $t = 1$ is 1

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

m (normal) at $t = 1$ is -1

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - 2a = 1(x - a)$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - 2a = -1(x - a)$$

(iv) $x = a \sec t$, $y = b \tan t$ at t .

Solution:

Given $x = a \sec t$, $y = b \tan t$ at t .

By differentiating the given equation with respect to t , we get the slope of the tangent

$$\frac{dx}{dt} = a \sec t \tan t$$

$$\frac{dy}{dt} = b \sec^2 t$$

Now dividing $\frac{dy}{dt}$ and $\frac{dx}{dt}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{b \operatorname{cosec} t}{a}$$

$$m \text{ (tangent) at } t = \frac{b \operatorname{cosec} t}{a}$$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

$$m \text{ (normal) at } t = -\frac{a}{b} \sin t$$

Equation of tangent is given by $y - y_1 = m \text{ (tangent)} (x - x_1)$

$$y - b \tan t = \frac{b \operatorname{cosec} t}{a} (x - a \operatorname{sect})$$

Equation of normal is given by $y - y_1 = m \text{ (normal)} (x - x_1)$

$$y - b \tan t = -\frac{a \sin t}{b} (x - a \operatorname{sect})$$

(v) $x = a (\theta + \sin \theta)$, $y = a (1 - \cos \theta)$ at θ

Solution:

Given $x = a (\theta + \sin \theta)$, $y = a (1 - \cos \theta)$ at θ

By differentiating the given equation with respect to θ , we get the slope of the tangent

$$\frac{dx}{d\theta} = a(1 + \cos \theta)$$

$$\frac{dy}{d\theta} = a(\sin \theta)$$

Now dividing $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{\sin \theta}{1 + \cos \theta}$$

$m \text{ (tangent) at } \theta \text{ is } \frac{\sin \theta}{1 + \cos \theta}$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

$m \text{ (normal) at } \theta \text{ is } -\frac{\sin \theta}{1 + \cos \theta}$

Equation of tangent is given by $y - y_1 = m \text{ (tangent)} (x - x_1)$

$$y - a(1 - \cos \theta) = \frac{\sin \theta}{1 + \cos \theta} (x - a(\theta + \sin \theta))$$

Equation of normal is given by $y - y_1 = m \text{ (normal)} (x - x_1)$

$$y - a(1 - \cos \theta) = \frac{1 + \cos \theta}{-\sin \theta} (x - a(\theta + \sin \theta))$$

(vi) $x = 3 \cos \theta - \cos^3 \theta$, $y = 3 \sin \theta - \sin^3 \theta$

Solution:

Given $x = 3 \cos \theta - \cos^3 \theta$, $y = 3 \sin \theta - \sin^3 \theta$

By differentiating the given equation with respect to θ , we get the slope of the tangent

$$\frac{dx}{d\theta} = -3 \sin \theta + 3 \cos^2 \theta \sin \theta$$

$$\frac{dy}{d\theta} = 3 \cos \theta - 3 \sin^2 \theta \cos \theta$$

Now dividing $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ to obtain the slope of tangent

$$\frac{dy}{dx} = \frac{3 \cos \theta - 3 \sin^2 \theta \cos \theta}{-3 \sin \theta + 3 \cos^2 \theta \sin \theta} = -\tan^3 \theta$$

m (tangent) at theta is $-\tan^3 \theta$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

m (normal) at theta is $\cot^3 \theta$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

$$y - 3 \sin \theta + \sin^3 \theta = -\tan^3 \theta (x - 3 \cos \theta + 3 \cos^3 \theta)$$

Equation of normal is given by $y - y_1 = m$ (normal) $(x - x_1)$

$$y - 3 \sin \theta + \sin^3 \theta = \cot^3 \theta (x - 3 \cos \theta + 3 \cos^3 \theta)$$

6. Find the equation of the normal to the curve $x^2 + 2y^2 - 4x - 6y + 8 = 0$ at the point whose abscissa is 2.

Solution:

$$\text{Given } x^2 + 2y^2 - 4x - 6y + 8 = 0$$

By differentiating the given curve, we get the slope of the tangent

$$2x + 4y \frac{dy}{dx} - 4 - 6 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{4 - 2x}{4y - 6}$$

Finding y co-ordinate by substituting x in the given curve

$$2y^2 - 6y + 4 = 0$$

$$y^2 - 3y + 2 = 0$$

$$y = 2 \text{ or } y = 1$$

m (tangent) at $x = 2$ is 0

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

m (normal) at $x = 2$ is $1/0$, which is undefined

Equation of normal is given by $y - y_1 = m \text{ (normal)} (x - x_1)$

$$x = 2$$

7. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3) .

Solution:

$$\text{Given } ay^2 = x^3$$

By differentiating the given curve, we get the slope of the tangent

$$2ay \frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$

m (tangent) at (am^2, am^3) is $\frac{3m}{2}$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

m (normal) at (am^2, am^3) is $-\frac{2}{3m}$

Equation of normal is given by $y - y_1 = m \text{ (normal)} (x - x_1)$

$$y - am^3 = -\frac{2}{3m}(x - am^2)$$

8. The equation of the tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of a and b .

Solution:

Given $y^2 = ax^3 + b$ is $y = 4x - 5$

By differentiating the given curve, we get the slope of the tangent

$$2y \frac{dy}{dx} = 3ax^2$$

$$\frac{dy}{dx} = \frac{3ax^2}{2y}$$

m (tangent) at (2, 3) = $2a$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

Now comparing the slope of a tangent with the given equation

$$2a = 4$$

$$a = 2$$

Now (2, 3) lies on the curve, these points must satisfy

$$3^2 = 2 \times 2^3 + b$$

$$b = -7$$

9. Find the equation of the tangent line to the curve $y = x^2 + 4x - 16$ which is parallel to the line $3x - y + 1 = 0$.

Solution:

Given $y = x^2 + 4x - 16$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 2x + 4$$

m (tangent) = $2x + 4$

Equation of tangent is given by $y - y_1 = m$ (tangent) $(x - x_1)$

Now comparing the slope of a tangent with the given equation

$$2x + 4 = 3$$

$$x = -\frac{1}{2}$$

Now substituting the value of x in the curve to find y

$$y = \frac{1}{4} - 2 - 16 = -\frac{71}{4}$$

Therefore, the equation of tangent parallel to the given line is

$$y + \frac{71}{4} = 3\left(x + \frac{1}{2}\right)$$

10. Find the equation of normal line to the curve $y = x^3 + 2x + 6$ which is parallel to the line $x + 14y + 4 = 0$.

Solution:

Given $y = x^3 + 2x + 6$

By differentiating the given curve, we get the slope of the tangent

$$\frac{dy}{dx} = 3x^2 + 2$$

$$m(\text{tangent}) = 3x^2 + 2$$

Normal is perpendicular to tangent so, $m_1 m_2 = -1$

$$m(\text{normal}) = \frac{-1}{3x^2 + 2}$$

Equation of normal is given by $y - y_1 = m(\text{normal})(x - x_1)$

Now comparing the slope of normal with the given equation

$$m(\text{normal}) = -\frac{1}{14}$$

$$-\frac{1}{14} = -\frac{1}{3x^2 + 2}$$

$$x = 2 \text{ or } -2$$

Hence the corresponding value of y is 18 or -6

So, equations of normal are

$$y - 18 = -\frac{1}{14}(x - 2)$$

Or

$$y + 6 = -\frac{1}{14}(x + 2)$$



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