EXERCISE 10.1

Show that f (x) = |x – 3| is continuous but not differentiable at x = 3.

Solution:

Given f (x) = |x – 3|

Therefore we can write given function as,

$$f(x) = \begin{cases} -(x-3), x < 3 \\ x-3, x \geq 3 \end{cases}$$

But f (3) = 3 - 3 = 0

LHL = $\lim_{x\to 3} f(x)$

= $\lim_{h\to 0} f(3-h)$

= $\lim_{h\to 0} 3-(3-h)$

= $\lim_{h\to 0} 0$

Now consider,

RHL = $\lim_{x\to 3} f(x)$

= $\lim_{h\to 0} f(3+h)$

= $\lim_{h\to 0} 3+h-3$

= 0

LHL = RHL = f (3)

Since, f(x) is continuous at x = 3

LHD at x = 3 = $\lim_{x-3^-} \frac{f(x)-f(3)}{x-3}$

= $\lim_{h-0^-} \frac{f(3-h)-f(3)}{3-h-3}$

= $\lim_{h-0^-} \frac{3-(3-h)-0}{-h}$

= $\lim_{h-0^-} \frac{h}{-h}$

= - 1

RHD at x = 3 = $\lim_{x-3^+} \frac{f(x)-f(3)}{x-3}$

= $\lim_{h-0^+} \frac{f(3+h)-f(3)}{3+h-3}$

= $\lim_{h-0^+} \frac{3+h-3-0}{h}$

= $\lim_{h-0^+} \frac{h}{h}$

= 1

LHD at x = 3 ≠ RHD at x = 3

Hence, f(x) is continuous but not differentiable at x = 3.

1. Show that f (x) = x $^{1/3}$ is not differentiable at x = 0.

Solution:

For differentiability,

LHD (at x = 0) = RHD (at x = 0)

(LHD at x = 0) = $\lim_{x\to0^-} \frac{f(x)-f(0)}{x-0}$

= $\lim_{h\to0^-} \frac{f(0-h)-f(0)}{0-h-0}$

$= \lim_{h \to 0^-} \frac{(-h)^{\frac{1}{3}} - 0}{-h}$

$= \lim_{h \to 0^-} \frac{(-h)^{\frac{1}{3}}}{-h}$

$= \lim_{h \to 0^-} \frac{(-1)^{\frac{1}{3}}(h)^{\frac{1}{3}}}{(-1)h}$

$= \lim_{h \to 0} (-1)^{\frac{-2}{3}} h^{\frac{-2}{3}}$

= Not defined

(RHD at x = 3) = $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{0 + h - 0}$

$= \lim_{h \to 0^+} \frac{(h)^{\frac{1}{3}} - 0}{+h}$

$= \lim_{h \to 0^+} \frac{(h)^{\frac{1}{3}}}{+h}$

$= \lim_{h \to 0} h^{\frac{-2}{3}}$

= Not defined

Since, LHD and RHD does not exist at x = 0
Hence, f(x) is not differentiable at x = 0

$3.\ Show\ that\ f(x) = \begin{cases} 12x - 13,\ if\ x \le 3 \\ 2x^2 + 5,\ if\ x > 3 \end{cases}\ is\ differentiable\ at\ x = 3.\ Also,\ find\ f'(3)$

Solution:

Now we have to check differentiability of given function at x = 3
That is LHD (at x = 3) = RHD (at x = 3)

(LHD at x = 3) = $\lim_{x \to 3^-} \frac{f(x)-f(3)}{x-3}$

= $\lim_{h \to 0^-} \frac{f(3-h)-f(3)}{3-h-3}$

= $\lim_{h \to 0^-} \frac{[12(3-h)-13]-[12(3)-13]}{-h}$

= $\lim_{h \to 0^-} \frac{36-12h-13-36+13}{-h}$

= $\lim_{h \to 0^-} \frac{-12h}{-h}$

= 12

(RHD at x = 3) = $\lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3}$

= $\lim_{h \to 0^+} \frac{f(3+h)-f(3)}{3+h-3}$

= $\lim_{h \to 0^+} \frac{[2(3+h^2)+5]-[12(3)-13]}{3+h-3}$

= $\lim_{h \to 0^+} \frac{18+12h+2h^2+5-36+13}{h}$

= $\lim_{h \to 0^+} \frac{2h^2+12h}{h}$

= $\lim_{h \to 0^+} \frac{h(2h+12)}{h}$

= 12

Since, (LHD at x = 3) = (RHD at x = 3)

Hence, f(x) is differentiable at x = 3.

4. Show that the function f is defined as follows

$$f(x) = \begin{cases} 3x-2, & 0 < x \le 1 \\ 2x^2-x, & 1 < x \le 2 \\ 5x-4, & x > 2 \end{cases}$$

Is continuous at x = 2, but not differentiable thereat.

Solution:

Given

$$f(x)=\begin{cases}3x-2, & 0<x\le 1\\ 2x^2-x, & 1<x\le 2\\ 5x-4, & x>2\end{cases}$$

Now we have to check continuity at x = 2

For continuity,

LHL (at x = 2) = RHL (at x = 2)

f (2) = $2(2)^2 - 2$

= 8 - 2 = 6

LHL = $\lim_{x\to 2^-} f(x)$

= $\lim_{h\to 0^-} f(2-h)$

= $\lim_{h\to 0^-} [2(2-h)^2-(2-h)]$

= 8 - 2

= 6

RHL = $\lim_{x\to 2^+} f(x)$

= $\lim_{h\to 0^+} f(2+h)$

= $\lim_{h\to 0} 5(2+h)-4$

= 6

Since, LHL = RHL = f (2)

Hence, F(x) is continuous at x = 2

Now we have to differentiability at x = 2

(LHD at x = 2) = $\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$

= $\lim_{h \to 0} \frac{f(2-h) - f(2)}{2 - h - 2}$

= $\lim_{h \to 0} \frac{[2(2-h)^2 - (2-h)] - [8-2]}{-h}$

= $\lim_{h \to 0} \frac{8 - 8h + 2h^2 - h - 6}{-h}$

= $\lim_{h \to 0} \frac{2h^2 - 6h}{-h}$

= $\lim_{h \to 0} \frac{h(2h-6)}{-h}$

= $\lim_{h \to 0} (6 - 2h)$

= 6

Now consider,

(RHD at x = 2) = $\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$

= $\lim_{h \to 0} \frac{f(2+h) - f(2)}{2 + h - 2}$

= $\lim_{h \to 0} \frac{[5(2+h) - 4] - [8-2]}{h}$

= $\lim_{h \to 0} \frac{10 + 5h - 4 - 6}{h}$

= 5

Since, (RHD at x = 2) ≠ (LHD at x = 2)

Hence, f (2) is not differentiable at x = 2.

5. Discuss the continuity and differentiability of the function f (x) = |x| + |x -1| in the interval of (-1, 2).

Solution:

The given function f (x) can be defined as

f (x) = $\begin{cases} x + x + 1, -1 < x < 0 \\ 1, 0 \leq x \leq 1 \\ -x - x + 1, 1 < x < 2 \end{cases}$

f (x) = $\begin{cases} 2x + 1, -1 < x < 0 \\ 1, 0 \leq x \leq 1 \\ -2x + 1, 1 < x < 2 \end{cases}$

We know that a polynomial and a constant function is continuous and differentiable everywhere. So, f(x) is continuous and differentiable for x $\in$
(-1, 0) and x ∈ (0, 1) and (1, 2).
We need to check continuity and differentiability at x = 0 and x = 1.
Continuity at x = 0

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} 2x + 1$ = 1

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} 1$ = 1

F (0) = 1

$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0)$

Since, f(x) is continuous at x = 0

Continuity at x = 1

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} 1$ = 1

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} 1$ = 1

F (1) = 1

$\log_{x\to0^+} f(x) = \log_{x\to0^-} f(x)$ = 1

Since, f(x) is continuous at x = 1
Now we have to check differentiability at x = 0
For differentiability, LHD (at x = 0) = RHD (at x = 0)
Differentiability at x = 0

(LHD at x = 0) = $\lim_{x-0^-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x-0^-} \frac{2x+1-1}{x-0}$

$= \lim_{x-0^-} \frac{2x}{x}$

= 2

(RHD at x = 0) = $\lim_{x-0^+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x-0^+} \frac{1-1}{x}$

$= \lim_{x-0^+} \frac{0}{x}$

= 0

Since, (LHD at x = 0) ≠ (RHD at x = 0)
So, f(x) is differentiable at x = 0.
Now we have to check differentiability at x = 1
For differentiability, LHD (at x = 1) = RHD (at x = 1)
Differentiability at x = 1

(LHD at x = 1) = $\lim_{x-1^-} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x-1^-} \frac{1-1}{x-1}$

= 0

(RHD at x = 1) = $\lim_{x-1^+} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x-1^+} \frac{-2x+1-1}{x-1}$

$= \infty$

Since, f(x) is not differentiable at x = 1.
So, f(x) is continuous on (- 1, 2) but not differentiable at x = 0, 1