Exercise 1.1

1. If a and b are two odd positive integers such that a > b, then prove that one of the two numbers (a+b)/2 and (a-b)/2 is odd and the other is even.

Solution:

We know that any odd positive integer is of the form 4q+1 or, 4q+3 for some whole number q.

Now that it's given a > b

So, we can choose a= 4q+3 and b= 4q+1.

∴ (a+b)/2 = [(4q+3) + (4q+1)]/2

⇒ (a+b)/2 = (8q+4)/2

⇒ (a+b)/2 = 4q+2 = 2(2q+1) which is clearly an even number.

Now, doing (a-b)/2

⇒ (a-b)/2 = [(4q+3)-(4q+1)]/2

⇒ (a-b)/2 = (4q+3-4q-1)/2

⇒ (a-b)/2 = (2)/2

⇒ (a-b)/2 = 1 which is an odd number.

Hence, one of the two numbers (a+b)/2 and (a-b)/2 is odd and the other is even.

2. Prove that the product of two consecutive positive integers is divisible by 2.

Solution:

Let's consider two consecutive positive integers as (n-1) and n.

∴ Their product = (n-1) n

$= n^2 – n$

And then we know that any positive integer is of the form 2q or 2q+1. (From Euclid's division lemma for b= 2)

So, when n= 2q

We have,

⇒ $n^2 – n = (2q)^2 - 2q$

⇒ $n^2 – n = 4q^2 -2q$

⇒ $n^2 – n = 2(2q^2 -q)$

Thus, $n^2 – n$ is divisible by 2.

Now, when n= 2q+1

We have,

⇒ $n^2 – n = (2q+1)^2 – (2q-1)$

⇒ $n^2 – n = (4q^2+4q+1 – 2q+1)$

⇒ $n^2 – n = (4q^2+2q+2)$

⇒ $n^2 – n = 2(2q^2+q+1)$

Thus, $n^2 – n$ is divisible by 2 again.

Hence, the product of two consecutive positive integers is divisible by 2.

3. Prove that the product of three consecutive positive integers is divisible by 6.
Solution:

Let n be any positive integer.
Thus, the three consecutive positive integers are n, n+1 and n+2.
We know that any positive integer can be of the form 6q, or 6q+1, or 6q+2, or 6q+3, or 6q+4, or 6q+5. (From Euclid's division lemma for b= 6)
So,
For n= 6q,
⇒ n(n+1)(n+2)= 6q(6q+1)(6q+2)
⇒ n(n+1)(n+2)= 6[q(6q+1)(6q+2)]
⇒ n(n+1)(n+2)= 6m, which is divisible by 6. [m= q(6q+1)(6q+2)]

For n= 6q+1,
⇒ n(n+1)(n+2)= (6q+1)(6q+2)(6q+3)
⇒ n(n+1)(n+2)= 6[(6q+1)(3q+1)(2q+1)]
⇒ n(n+1)(n+2)= 6m, which is divisible by 6. [m= (6q+1)(3q+1)(2q+1)]

For n= 6q+2,
⇒ n(n+1)(n+2)= (6q+2)(6q+3)(6q+4)
⇒ n(n+1)(n+2)= 6[(3q+1)(2q+1)(6q+4)]
⇒ n(n+1)(n+2)= 6m, which is divisible by 6. [m= (3q+1)(2q+1)(6q+4)]

For n= 6q+3,
⇒ n(n+1)(n+2)= (6q+3)(6q+4)(6q+5)
⇒ n(n+1)(n+2)= 6[(2q+1)(3q+2)(6q+5)]
⇒ n(n+1)(n+2)= 6m, which is divisible by 6. [m= (2q+1)(3q+2)(6q+5)]

For n= 6q+4,
⇒ n(n+1)(n+2)= (6q+4)(6q+5)(6q+6)
⇒ n(n+1)(n+2)= 6[(3q+2)(3q+1)(2q+2)]
⇒ n(n+1)(n+2)= 6m, which is divisible by 6. [m= (3q+2)(3q+1)(2q+2)]

For n= 6q+5,
⇒ n(n+1)(n+2)= (6q+5)(6q+6)(6q+7)
⇒ n(n+1)(n+2)= 6[(6q+5)(q+1)(6q+7)]
⇒ n(n+1)(n+2)= 6m, which is divisible by 6. [m= (6q+5)(q+1)(6q+7)]

Hence, the product of three consecutive positive integers is divisible by 6.

4. For any positive integer n, prove that $n^3 - n$ divisible by 6.
Solution:

Let, n be any positive integer. And since any positive integer can be of the form 6q, or 6q+1, or 6q+2, or 6q+3, or 6q+4, or 6q+5. (From Euclid's division lemma for b= 6)
We have $n^3 - n = n(n^2-1) = (n-1)n(n+1)$,

For n= 6q,
⇒ (n-1)n(n+1)= (6q-1)(6q)(6q+1)
⇒ (n-1)n(n+1)= 6[(6q-1)q(6q+1)]
⇒ (n-1)n(n+1)= 6m, which is divisible by 6. [m= (6q-1)q(6q+1)]
For n= 6q+1,
⇒ (n-1)n(n+1)= (6q)(6q+1)(6q+2)
⇒ (n-1)n(n+1)= 6[q(6q+1)(6q+2)]
⇒ (n-1)n(n+1)= 6m, which is divisible by 6. [m= q(6q+1)(6q+2)]

For n= 6q+2,
⇒ (n-1)n(n+1)= (6q+1)(6q+2)(6q+3)
⇒ (n-1)n(n+1)= 6[(6q+1)(3q+1)(2q+1)]
⇒ (n-1)n(n+1)= 6m, which is divisible by 6. [m= (6q+1)(3q+1)(2q+1)]

For n= 6q+3,
⇒ (n-1)n(n+1)= (6q+2)(6q+3)(6q+4)
⇒ (n-1)n(n+1)= 6[(3q+1)(2q+1)(6q+4)]
⇒ (n-1)n(n+1)= 6m, which is divisible by 6. [m= (3q+1)(2q+1)(6q+4)]

For n= 6q+4,
⇒ (n-1)n(n+1)= (6q+3)(6q+4)(6q+5)
⇒ (n-1)n(n+1)= 6[(2q+1)(3q+2)(6q+5)]
⇒ (n-1)n(n+1)= 6m, which is divisible by 6. [m= (2q+1)(3q+2)(6q+5)]
For n= 6q+5,
⇒ (n-1)n(n+1)= (6q+4)(6q+5)(6q+6)
⇒ (n-1)n(n+1)= 6[(6q+4)(6q+5)(q+1)]
⇒ (n-1)n(n+1)= 6m, which is divisible by 6. [m= (6q+4)(6q+5)(q+1)]

Hence, for any positive integer n, $n^3 – n$ is divisible by 6.

5. Prove that if a positive integer is of form 6q + 5, then it is of the form 3q + 2 for some integer q, but not conversely.

Solution:

Let n= 6q+5 be a positive integer for some integer q.
We know that any positive integer can be of the form 3k, or 3k+1, or 3k+2.
∴ q can be 3k or, 3k+1 or, 3k+2.

If q= 3k, then
⇒ n= 6q+5
⇒ n= 6(3k)+5
⇒ n= 18k+5 = (18k+3)+ 2
⇒ n= 3(6k+1)+2
⇒ n= 3m+2, where m is some integer

If q= 3k+1, then

⇒ $n = 6q+5$

⇒ $n = 6(3k+1)+5$

⇒ $n = 18k+6+5 = (18k+9)+2$

⇒ $n = 3(6k+3)+2$

⇒ $n = 3m+2$, where m is some integer

If q= 3k+2, then

⇒ $n = 6q+5$

⇒ $n = 6(3k+2)+5$

⇒ $n = 18k+12+5 = (18k+15)+2$

⇒ $n = 3(6k+5)+2$

⇒ $n = 3m+2$, where m is some integer

Hence, if a positive integer is of form 6q + 5, then it is of the form 3q + 2 for some integer q.

Conversely,

Let n= 3q+2

And we know that a positive integer can be of the form 6k, or 6k+1, or 6k+2, or 6k+3, or 6k+4, or 6k+5.

So, now if q=6k+1 then

⇒ $n = 3q+2$

⇒ $n = 3(6k+1)+2$

⇒ $n = 18k+5$

⇒ $n = 6m+5$, where m is some integer

So, now if q=6k+2 then

⇒ $n = 3q+2$

⇒ $n = 3(6k+2)+2$

⇒ $n = 18k+6+2 = 18k+8$

⇒ $n = 6(3k+1)+2$

⇒ $n = 6m+2$, where m is some integer

Now, this is not of the form 6q + 5.

Therefore, if n is of the form 3q + 2, then is necessary won't be of the form 6q + 5.

6. Prove that square of any positive integer of the form 5q + 1 is of the same form.

Solution:

Here, the integer 'n' is of the form 5q+1.

⇒ $n = 5q+1$

On squaring it,

⇒ $n^2 = (5q+1)^2$

⇒ $n^2 = (25q^2+10q+1)$

⇒ $n^2 = 5(5q^2+2q)+1$

⇒ $n^2 = 5m+1$, where m is some integer. [For $m = 5q^2+2q$]

Therefore, the square of any positive integer of the form 5q + 1 is of the same form.

7. Prove that the square of any positive integer is of the form 3m or 3m + 1 but not of the form 3m + 2.

Solution:

Let any positive integer ‘n’ be of the form 3q or, 3q+1 or 3q+2. (From Euclid's division lemma for b= 3)

If n= 3q,

Then, on squaring

⇒ $n^2 = (3q)^2 = 9q^2$

⇒ $n^2 = 3(3q^2)$

⇒ $n^2 = 3m$, where m is some integer [$m = 3q^2$]

If n= 3q+1,

Then, on squaring

⇒ $n^2 = (3q+1)^2 = 9q^2 + 6q + 1$

⇒ $n^2 = 3(3q^2 + 2q) + 1$

⇒ $n^2 = 3m + 1$, where m is some integer [$m = 3q^2 + 2q$]

If n= 3q+2,

Then, on squaring

⇒ $n^2 = (3q+2)^2 = 9q^2 + 12q + 4$

⇒ $n^2 = 3(3q^2 + 4q + 1) + 1$

⇒ $n^2 = 3m + 1$, where m is some integer [$m = 3q^2 + 4q + 1$]

Thus, it is observed that the square of any positive integer is of the form 3m or 3m + 1 but not of the form 3m + 2.

8. Prove that the square of any positive integer is of the form 4q or 4q + 1 for some integer q.

Solution:

Let ‘a’ be any positive integer.

Then,

According to Euclid's division lemma,

$a=bq+r$

According to the question, when b = 4.

$a = 4k + r, n \leq r < 4$

When r = 0, we get, a = 4k

$a^2 = 16k^2 = 4(4k^2) = 4q$, where $q = 4k^2$

When r = 1, we get, a = 4k + 1

$a^2 = (4k + 1)^2 = 16k^2 + 1 + 8k = 4(4k + 2) + 1 = 4q + 1$, where $q = k(4k + 2)$

When r = 2, we get, a = 4k + 2

$a^2 = (4k + 2)^2 = 16k^2 + 4 + 16k = 4(4k^2 + 4k + 1) = 4q$, where $q = 4k^2 + 4k + 1$

When r = 3, we get, a = 4k + 3

$a^2 = (4k + 3)^2 = 16k^2 + 9 + 24k = 4(4k^2 + 6k + 2) + 1$

$= 4q + 1$, where $q = 4k^2 + 6k + 2$

Therefore, the square of any positive integer is either of the form 4q or 4q + 1 for some integer q.

9. Prove that the square of any positive integer is of the form 5q or 5q + 1, 5q + 4 for some integer q.

Solution:

Let 'a' be any positive integer.

Then,

According to Euclid's division lemma,

a = bq+r

According to the question, when b = 5.

$a = 5k + r, n \leq r < 5$

When r = 0, we get, a = 5k

$a^2 = 25k^2 = 5(5k^2) = 5q$, where $q = 5k^2$

When r = 1, we get, a = 5k + 1

$a^2 = (5k + 1)^2 = 25k^2 + 1 + 10k = 5k(5k + 2) + 1 = 5q + 1$, where $q = k(5k + 2)$

When r = 2, we get, a = 5k + 2

$a^2 = (5k + 2)^2 = 25k^2 + 4 + 20k = 5(5k^2 + 4k) + 4 = 4q + 4$, where $q = 5k^2 + 4k$

When r = 3, we get, a = 5k + 3

$a^2 = (5k + 3)^2 = 25k^2 + 9 + 30k = 5(5k^2 + 6k + 1) + 4$

$= 5q + 4$, where $q = 5k^2 + 6k + 1$

When r = 4, we get, a = 5k + 4

$a^2 = (5k + 4)^2 = 25k^2 + 16 + 40k = 5(5k^2 + 8k + 3) + 1$

$= 5q + 1$, where $q = 5k^2 + 8k + 3$

Therefore, the square of any positive integer is of the form 5q or, 5q + 1 or 5q + 4 for some integer q.

10. Show that the square of odd integer is of the form 8q + 1, for some integer q.

Solution:

From Euclid's division lemma,

$a = bq+r$; where $0 \leq r < b$

Putting b=4 for the question,

$\Rightarrow \quad a = 4q + r, 0 \leq r < 4$

For r = 0, we get a = 4q, which is an even number.

For r = 1, we get a = 4q + 1, which is an odd number.

On squaring,

$\Rightarrow \quad a^2 = (4q + 1)^2 = 16q^2 + 1 + 8q = 8(2q^2 + q) + 1 = 8m + 1$, where $m = 2q^2 + q$

For r = 2, we get a = 4q + 2 = 2(2q + 1), which is an even number.

For $r = 3$, we get $a = 4q + 3$, which is an odd number.
On squaring,
⇒ $a^2 = (4q + 3)^2 = 16q^2 + 9 + 24q = 8(2q^2 + 3q + 1) + 1$
$= 8m + 1$, where $m = 2q^2 + 3q + 1$

Thus, the square of an odd integer is of the form $8q + 1$, for some integer q.

11. Show that any positive odd integer is of the form 6q +1 or 6q + 3 or 6q + 5, where q is some integer.
Solution:

Let 'a' be any positive integer.
Then from Euclid's division lemma,
$a = bq+r$; where $0 \leq r < b$
Putting b=6 we get,
⇒ $a = 6q + r, 0 \leq r < 6$
For $r = 0$, we get $a = 6q = 2(3q) = 2m$, which is an even number. $[m = 3q]$
For $r = 1$, we get $a = 6q + 1 = 2(3q) + 1 = 2m + 1$, which is an **odd** number. $[m = 3q]$
For $r = 2$, we get $a = 6q + 2 = 2(3q + 1) = 2m$, which is an even number. $[m = 3q + 1]$
For $r = 3$, we get $a = 6q + 3 = 2(3q + 1) + 1 = 2m + 1$, which is an **odd** number. $[m = 3q + 1]$
For $r = 4$, we get $a = 6q + 4 = 2(3q + 2) + 1 = 2m + 1$, which is an even number. $[m = 3q + 2]$
For $r = 5$, we get $a = 6q + 5 = 2(3q + 2) + 1 = 2m + 1$, which is an **odd** number. $[m = 3q + 2]$

Thus, from the above it can be seen that any positive odd integer can be of the form 6q +1 or 6q + 3 or 6q + 5, where q is some integer.

12. Show that the square of any positive integer cannot be of the form 6m + 2 or 6m + 5 for any integer m.
Solution:

Let the positive integer = a

According to Euclid's division algorithm,

$a = 6q + r$, where $0 \leq r < 6$
$a^2 = (6q + r)^2 = 36q^2 + r^2 + 12qr$ $[\because (a+b)^2 = a^2 + 2ab + b^2]$
$a^2 = 6(6q^2 + 2qr) + r^2$ ...(i), where, $0 \leq r < 6$
When $r = 0$, substituting $r = 0$ in Eq.(i), we get

$a^2 = 6 (6q^2) = 6m$, where, $m = 6q^2$ is an integer.

When $r = 1$, substituting $r = 1$ in Eq.(i), we get

$a^2 = 6 (6q^2 + 2q) + 1 = 6m + 1$, where, $m = (6q^2 + 2q)$ is an integer.

When $r = 2$, substituting $r = 2$ in Eq(i), we get

$a^2 = 6(6q^2 + 4q) + 4 = 6m + 4$, where, $m = (6q^2 + 4q)$ is an integer.

When $r = 3$, substituting $r = 3$ in Eq.(i), we get

$a^2 = 6(6q^2 + 6q) + 9 = 6(6q^2 + 6q) + 6 + 3$

$a^2 = 6(6q^2 + 6q + 1) + 3 = 6m + 3$, where, $m = (6q + 6q + 1)$ is integer.

When r = 4, substituting r = 4 in Eq.(i) we get

$a^2 = 6(6q^2 + 8q) + 16$

$= 6(6q^2 + 8q) + 12 + 4$

$\Rightarrow a^2 = 6(6q^2 + 8q + 2) + 4 = 6m + 4$, where, $m = (6q^2 + 8q + 2)$ is integer.

When r = 5, substituting r = 5 in Eq.(i), we get

$a^2 = 6\,(6q^2 + 10q) + 25 = 6(6q^2 + 10q) + 24 + 1$

$a^2 = 6(6q^2 + 10q + 4) + 1 = 6m + 1$, where, $m = (6q^2 + 10q + 4)$ is integer.

Hence, the square of any positive integer cannot be of the form 6m + 2 or 6m + 5 for any integer m.

Hence Proved.

13. Show that the cube of a positive integer of the form 6q + r, q is an integer and r = 0, 1, 2, 3, 4, 5 is also of the form 6m + r.

Solution:

Given, 6q + r is a positive integer, where q is an integer and r = 0, 1, 2, 3, 4, 5

Then, the positive integers are of the form 6q, 6q+1, 6q+2, 6q+3, 6q+4 and 6q+5.

Taking cube on L.H.S and R.H.S,

For 6q,

$(6q)^3 = 216\,q^3 = 6(36q)^3 + 0$

$= 6m + 0$, (where m is an integer $= (36q)^3$)

For 6q+1,

$(6q+1)^3 = 216q^3 + 108q^2 + 18q + 1$

$= 6(36q^3 + 18q^2 + 3q) + 1$

$= 6m + 1$, (where m is an integer $= 36q^3 + 18q^2 + 3q$)

For 6q+2,

$(6q+2)^3 = 216q^3 + 216q^2 + 72q + 8$

$= 6(36q^3 + 36q^2 + 12q + 1) + 2$

$= 6m + 2$, (where m is an integer $= 36q^3 + 36q^2 + 12q + 1$)

For 6q+3,

$(6q+3)^3 = 216q^3 + 324q^2 + 162q + 27$

$= 6(36q^3 + 54q^2 + 27q + 4) + 3$

$= 6m + 3$, (where m is an integer $= 36q^3 + 54q^2 + 27q + 4$)

For 6q+4,

$(6q+4)^3 = 216q^3 + 432q^2 + 288q + 64$

$= 6(36q^3 + 72q^2 + 48q + 10) + 4$

$= 6m + 4$, (where m is an integer $= 36q^3 + 72q^2 + 48q + 10$)

For 6q+5,

$(6q+5)^3 = 216q^3 + 540q^2 + 450q + 125$

$= 6(36q^3 + 90q^2 + 75q + 20) + 5$

$= 6m + 5$, (where m is an integer $= 36q^3 + 90q^2 + 75q + 20$)

Hence, the cube of a positive integer of the form 6q + r, q is an integer and r = 0, 1, 2, 3, 4, 5 is also of the form 6m + r.

14. Show that one and only one out of n, n + 4, n + 8, n + 12 and n + 16 is divisible by 5, where n is any positive integer.

Solution:

According to Euclid's division Lemma,

Let the positive integer = n

And, b=5

n = 5q+r, where q is the quotient and r is the remainder

$0 \leq r < 5$ implies remainders may be 0, 1, 2, 3, 4 and 5

Therefore, n may be in the form of 5q, 5q+1, 5q+2, 5q+3, 5q+4

So, this gives us the following cases:

CASE 1:

When, n = 5q

n+4 = 5q+4

n+8 = 5q+8

n+12 = 5q+12

n+16 = 5q+16

Here, n is only divisible by 5

CASE 2:

When, n = 5q+1

n+4 = 5q+5 = 5(q+1)

n+8 = 5q+9

n+12 = 5q+13

n+16 = 5q+17

Here, n + 4 is only divisible by 5

CASE 3:

When, n = 5q+2

n+4 = 5q+6

n+8 = 5q+10 = 5(q+2)

n+12 = 5q+14

n+16 = 5q+18

Here, n + 8 is only divisible by 5

CASE 4:

When, n = 5q+3

n+4 = 5q+7

n+8 = 5q+11

n+12 = 5q+15 = 5(q+3)

n+16 = 5q+19

Here, n + 12 is only divisible by 5

CASE 5:

When, n = 5q+4

n+4 = 5q+8

n+8 = 5q+12

n+12 = 5q+16

n+16 = 5q+20 = 5(q+4)

Here, n + 16 is only divisible by 5

So, we can conclude that one and only one out of n, n + 4, n + 8, n + 12 and n + 16 is divisible by 5.

Hence Proved

15. Show that the square of an odd integer can be of the form 6q + 1 or 6q + 3, for some integer q.

Solution:

Let 'a' be an odd integer and b = 6.

According to Euclid's algorithm,

a = 6m + r for some integer m ≥ 0

And r = 0, 1, 2, 3, 4, 5 because 0 ≤ r < 6.

So, we have that,

a = 6m or, 6m + 1 or, 6m + 2 or, 6m + 3 or, 6m + 4 or 6m + 5

Thus, we are choosing for a = 6m + 1 or, 6m + 3 or 6m + 5 for it to be an odd integer.

For a = 6m + 1,

$(6m + 1)^2 = 36m^2 + 12m + 1$

$= 6(6m^2 + 2m) + 1$

$= 6q + 1$, where q is some integer and $q = 6m^2 + 2m$.

For $a = 6m + 3$

$(6m + 3)^2 = 36m^2 + 36m + 9$

$= 6(6m^2 + 6m + 1) + 3$

$= 6q + 3$, where q is some integer and $q = 6m^2 + 6m + 1$

For $a = 6m + 5$,

$(6m + 5)^2 = 36m^2 + 60m + 25$

$= 6(6m^2 + 10m + 4) + 1$

$= 6q + 1$, where q is some integer and $q = 6m^2 + 10m + 4$.

Therefore, the square of an odd integer is of the form $6q + 1$ or $6q + 3$, for some integer q.

Hence Proved.

16. A positive integer is of the form 3q + 1, q being a natural number. Can you write its square in any form other than 3m + 1, 3m or 3m + 2 for some integer m? Justify your answer.

Solution:

No.

Justification:

By Euclid's Division Lemma,

$a = bq + r, 0 \le r < b$

Here, a is any positive integer and b = 3,

$\Rightarrow a = 3q + r$

So, a can be of the form 3q, 3q + 1 or 3q + 2.

Now, for a = 3q

$(3q)^2 = 3(3q^2) = 3m$ [where $m = 3q^2$]

for a = 3q + 1

$(3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1 = 3m + 1$ [where $m = 3q^2 + 2q$]

for a = 3q + 2

$(3q + 2)^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1 = 3(3q^2 + 4q + 1) + 1$

$= 3m + 1$ [where $m = 3q^2 + 4q + 1$]

Thus, square of a positive integer of the form 3q + 1 is always of the form 3m + 1 or 3m for some integer m.

17. Show that the square of any positive integer cannot be of the form 3m + 2, where m is a

natural number.
Solution:

Let the positive integer be ‘a’

According to Euclid's division lemma,

a = bm + r

According to the question, we take b = 3

a = 3m + r

So, r = 0, 1, 2.

When r = 0, a = 3m.

When r = 1, a = 3m + 1.

When r = 2, a = 3m + 2.

Now,

When a = 3m

$a^2 = (3m)^2 = 9m^2$

$a^2 = 3(3m^2) = 3q$, where $q = 3m^2$

When a = 3m + 1

$a^2 = (3m + 1)^2 = 9m^2 + 6m + 1$

$a^2 = 3(3m^2 + 2m) + 1 = 3q + 1$, where $q = 3m^2 + 2m$

When a = 3m + 2

$a^2 = (3m + 2)^2$

$a^2 = 9m^2 + 12m + 4$

$a^2 = 3(3m^2 + 4m + 1) + 1$

$a^2 = 3q + 1$ where $q = 3m^2 + 4m + 1$

Therefore, square of any positive integer cannot be of the form 3q + 2, where q is a natural number.

Hence Proved.