

EXERCISE 19.5

If $1/a, 1/b, 1/c$ are in A.P., prove that:

(i) $(b+c)/a, (c+a)/b, (a+b)/c$ are in A.P.

(ii) $a(b+c), b(c+a), c(a+b)$ are in A.P.

Solution:

(i) $(b+c)/a, (c+a)/b, (a+b)/c$ are in A.P.

We know that, if a, b, c are in AP, then $b - a = c - b$

If $1/a, 1/b, 1/c$ are in AP

Then, $1/b - 1/a = 1/c - 1/b$

If $(b+c)/a, (c+a)/b, (a+b)/c$ are in AP

Then, $(c+a)/b - (b+c)/a = (a+b)/c - (c+a)/b$

Let us take LCM

$$\frac{ca + a^2 - b^2 - cb}{ab} = \frac{ab + b^2 - c^2 - ac}{bc}$$

Now let us consider LHS:

$$\frac{ca + a^2 - b^2 - cb}{ab}$$

Multiply both numerator and denominator by 'c', we get,

$$\begin{aligned} \frac{ca + a^2 - b^2 - cb}{ab} &= \frac{c^2a + ca^2 - cb^2 - c^2b}{abc} \\ &= \frac{C(b-a)(a+b+c)}{abc} \end{aligned}$$

Now let us consider RHS:

$$\frac{ab + b^2 - c^2 - ac}{bc}$$

Multiply both numerator and denominator by 'a', we get,

$$\begin{aligned} \frac{ab + b^2 - c^2 - ac}{bc} &= \frac{a^2b + ab^2 - ac^2 - a^2c}{abc} \\ &= \frac{a(b-c)(a+b+c)}{abc} \end{aligned}$$

LHS = RHS

$$\frac{C(b-a)(a+b+c)}{abc} = \frac{a(b-c)(a+b+c)}{abc}$$

Since, $1/a, 1/b, 1/c$ are in AP

$$1/b - 1/a = 1/c - 1/b$$

$$C(b-a) = a(b-c)$$

Hence, the given terms are in AP.

(ii) $a(b + c)$, $b(c + a)$, $c(a + b)$ are in A.P.

We know that if, $b(c + a) - a(b+c) = c(a+b) - b(c+a)$

Consider LHS:

$$b(c + a) - a(b+c)$$

Upon simplification we get,

$$\begin{aligned} b(c + a) - a(b+c) &= bc + ba - ab - ac \\ &= c(b-a) \end{aligned}$$

Now, taking RHS

$$\begin{aligned} c(a+b) - b(c+a) &= ca + cb - bc - ba \\ &= a(c-b) \end{aligned}$$

We know,

$1/a$, $1/b$, $1/c$ are in AP

So, $1/a - 1/b = 1/b - 1/c$

Or $c(b-a) = a(c-b)$

Hence, given terms are in AP.

2. If a^2 , b^2 , c^2 are in AP., prove that $a/(b+c)$, $b/(c+a)$, $c/(a+b)$ are in AP.

Solution:

If a^2 , b^2 , c^2 are in AP then, $b^2 - a^2 = c^2 - b^2$

If $a/(b+c)$, $b/(c+a)$, $c/(a+b)$ are in AP then,

$$b/(c+a) - a/(b+c) = c/(a+b) - b/(c+a)$$

Let us take LCM on both the sides we get,

$$\frac{b^2 + bc - a^2 - ac}{(a + c)(b + c)} = \frac{ca + c^2 - b^2 - ab}{(a + b)(b + c)}$$

$$\frac{(b - a)(a + b + c)}{(a + c)(b + c)} = \frac{(c - b)(a + b + c)}{(a + b)(b + c)}$$

Since, $b^2 - a^2 = c^2 - b^2$

Substituting $b^2 - a^2 = c^2 - b^2$ in above, we get

LHS = RHS

Hence, given terms are in AP

3. If a , b , c are in A.P., then show that:

(i) $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are also in A.P.

(ii) $b + c - a$, $c + a - b$, $a + b - c$ are in A.P.

(iii) $bc - a^2$, $ca - b^2$, $ab - c^2$ are in A.P.

Solution:

(i) $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are also in A.P.

$$\text{If } b^2(c + a) - a^2(b + c) = c^2(a + b) - b^2(c + a)$$

$$b^2c + b^2a - a^2b - a^2c = c^2a + c^2b - b^2a - b^2c$$

$$\text{Given, } b - a = c - b$$

And since a, b, c are in AP,

$$c(b^2 - a^2) + ab(b - a) = a(c^2 - b^2) + bc(c - b)$$

$$(b - a)(ab + bc + ca) = (c - b)(ab + bc + ca)$$

Upon cancelling, $ab + bc + ca$ from both sides

$$b - a = c - b$$

$$2b = c + a \text{ [which is true]}$$

Hence, given terms are in AP

(ii) $b + c - a$, $c + a - b$, $a + b - c$ are in A.P.

$$\text{If } (c + a - b) - (b + c - a) = (a + b - c) - (c + a - b)$$

Then, $b + c - a$, $c + a - b$, $a + b - c$ are in A.P.

Let us consider LHS and RHS

$$(c + a - b) - (b + c - a) = (a + b - c) - (c + a - b)$$

$$2a - 2b = 2b - 2c$$

$$b - a = c - b$$

And since a, b, c are in AP,

$$b - a = c - b$$

Hence, given terms are in AP.

(iii) $bc - a^2$, $ca - b^2$, $ab - c^2$ are in A.P.

$$\text{If } (ca - b^2) - (bc - a^2) = (ab - c^2) - (ca - b^2)$$

Then, $bc - a^2$, $ca - b^2$, $ab - c^2$ are in A.P.

Let us consider LHS and RHS

$$(ca - b^2) - (bc - a^2) = (ab - c^2) - (ca - b^2)$$

$$(a - b^2 - bc + a^2) = (ab - c^2 - ca + b^2)$$

$$(a - b)(a + b + c) = (b - c)(a + b + c)$$

$$a - b = b - c$$

And since a, b, c are in AP,

$$b - c = a - b$$

Hence, given terms are in AP

4. If $(b+c)/a$, $(c+a)/b$, $(a+b)/c$ are in AP., prove that:

(i) $1/a$, $1/b$, $1/c$ are in AP

(ii) bc , ca , ab are in AP

Solution:

(i) $1/a, 1/b, 1/c$ are in AP

If $1/a, 1/b, 1/c$ are in AP then,

$$1/b - 1/a = 1/c - 1/b$$

Let us consider LHS:

$$1/b - 1/a = (a-b)/ab$$

$$= c(a-b)/abc \text{ [by multiplying with 'c' on both the numerator and denominator]}$$

Let us consider RHS:

$$1/c - 1/b = (b-c)/bc$$

$$= a(b-c)/bc \text{ [by multiplying with 'a' on both the numerator and denominator]}$$

Since, $(b+c)/a, (c+a)/b, (a+b)/c$ are in AP

$$\frac{c+a}{b} - \frac{b+c}{a} = \frac{a+b}{c} - \frac{c+a}{b}$$

$$\frac{b^2 + bc - a^2 - ac}{ab} = \frac{ca + c^2 - b^2 - ab}{bc}$$

$$\frac{(b-a)(a+b+c)}{ab} = \frac{(c-b)(a+b+c)}{bc}$$

$$\frac{a(b-c)}{abc} = \frac{c(a-b)}{abc}$$

$$a(b-c) = c(a-b)$$

LHS = RHS

Hence, the given terms are in AP

(ii) bc, ca, ab are in AP

If bc, ca, ab are in AP then,

$$ca - bc = ab - ca$$

$$c(a-b) = a(b-c)$$

If $1/a, 1/b, 1/c$ are in AP then,

$$1/b - 1/a = 1/c - 1/b$$

$$c(a-b) = a(b-c)$$

Hence, the given terms are in AP

5. If a, b, c are in A.P., prove that:

(i) $(a - c)^2 = 4(a - b)(b - c)$

(ii) $a^2 + c^2 + 4ac = 2(ab + bc + ca)$

(iii) $a^3 + c^3 + 6abc = 8b^3$

Solution:

(i) $(a - c)^2 = 4(a - b)(b - c)$

Let us expand the above expression

$$a^2 + c^2 - 2ac = 4(ab - ac - b^2 + bc)$$

$$a^2 + 4c^2b^2 + 2ac - 4ab - 4bc = 0$$

$$(a + c - 2b)^2 = 0$$

$$a + c - 2b = 0$$

Since a, b, c are in AP

$$b - a = c - b$$

$$a + c - 2b = 0$$

$$a + c = 2b$$

Hence, $(a - c)^2 = 4(a - b)(b - c)$

(ii) $a^2 + c^2 + 4ac = 2(ab + bc + ca)$

Let us expand the above expression

$$a^2 + c^2 + 4ac = 2(ab + bc + ca)$$

$$a^2 + c^2 + 2ac - 2ab - 2bc = 0$$

$$(a + c - b)^2 - b^2 = 0$$

$$a + c - b = b$$

$$a + c - 2b = 0$$

$$2b = a + c$$

$$b = (a + c)/2$$

Since a, b, c are in AP

$$b - a = c - b$$

$$b = (a + c)/2$$

Hence, $a^2 + c^2 + 4ac = 2(ab + bc + ca)$

(iii) $a^3 + c^3 + 6abc = 8b^3$

Let us expand the above expression

$$a^3 + c^3 + 6abc = 8b^3$$

$$a^3 + c^3 - (2b)^3 + 6abc = 0$$

$$a^3 + (-2b)^3 + c^3 + 3a(-2b)c = 0$$

Since, if $a + b + c = 0$, $a^3 + b^3 + c^3 = 3abc$

$$(a - 2b + c)^3 = 0$$

$$a - 2b + c = 0$$

$$a + c = 2b$$

$$b = (a + c)/2$$

Since a, b, c are in AP

$$a - b = c - b$$

$$b = (a+c)/2$$

$$\text{Hence, } a^3 + c^3 + 6abc = 8b^3$$

6. If $a(1/b + 1/c)$, $b(1/c + 1/a)$, $c(1/a + 1/b)$ are in AP., prove that a , b , c are in AP.

Solution:

Here, we know $a(1/b + 1/c)$, $b(1/c + 1/a)$, $c(1/a + 1/b)$ are in AP

Also, $a(1/b + 1/c) + 1$, $b(1/c + 1/a) + 1$, $c(1/a + 1/b) + 1$ are in AP

Let us take LCM for each expression then we get,

$(ac+ab+bc)/bc$, $(ab+bc+ac)/ac$, $(cb+ac+ab)/ab$ are in AP

$1/bc$, $1/ac$, $1/ab$ are in AP

Let us multiply numerator with 'abc', we get

abc/bc , abc/ac , abc/ab are in AP

$\therefore a$, b , c are in AP.

Hence proved.

7. Show that $x^2 + xy + y^2$, $z^2 + zx + x^2$ and $y^2 + yz + z^2$ are in consecutive terms of an A.P., if x , y and z are in A.P.

Solution:

x , y , z are in AP

Given, $x^2 + xy + y^2$, $z^2 + zx + x^2$ and $y^2 + yz + z^2$ are in AP

$$(z^2 + zx + x^2) - (x^2 + xy + y^2) = (y^2 + yz + z^2) - (z^2 + zx + x^2)$$

Let d = common difference,

So, $Y = x + d$ and $z = x + 2d$

Let us consider the LHS:

$$= (z^2 + zx + x^2) - (x^2 + xy + y^2)$$

$$= z^2 + zx - xy - y^2$$

$$= (x + 2d)^2 + (x + 2d)x - x(x + d) - (x + d)^2$$

$$= x^2 + 4xd + 4d^2 + x^2 + 2xd - x^2 - xd - x^2 - 2xd - d^2$$

$$= 3xd + 3d^2$$

Now, let us consider RHS:

$$= (y^2 + yz + z^2) - (z^2 + zx + x^2)$$

$$= y^2 + yz - zx - x^2$$

$$= (x + d)^2 + (x + d)(x + 2d) - (x + 2d)x - x^2$$

$$= x^2 + 2dx + d^2 + x^2 + 2dx + xd + 2d^2 - x^2 - 2dx - x^2$$

$$= 3xd + 3d^2$$

LHS = RHS

$\therefore x^2 + xy + y^2$, $z^2 + zx + x^2$ and $y^2 + yz + z^2$ are in consecutive terms of A.P

Hence proved.