EXERCISE 1(B)

1. State whether the following numbers are rational or not:

(i) $(2 + \sqrt{2})^2$
(ii) $(3 - \sqrt{3})^2$
(iii) $(5 + \sqrt{5})(5 - \sqrt{5})$
(iv) $(\sqrt{3} - \sqrt{2})^2$
(v) $(3/2\sqrt{2})^2$
(vi) $(\sqrt{7}/6\sqrt{2})^2$

Solution:

(i) $(2 + \sqrt{2})^2 = 2^2 + 2(2)(\sqrt{2}) + (\sqrt{2})^2$
$= 4 + 4\sqrt{2} + 2$
$= 6 + 4\sqrt{2}$
Therefore, it is irrational

(ii) $(3 - \sqrt{3})^2 = (3)^2 - 2(3)(\sqrt{3}) + (\sqrt{3})^2$
$= 9 - 6\sqrt{3} + 3$
$= 12 - 6\sqrt{3}$
$= 6(2 - \sqrt{3})$
Therefore, it is irrational.

(iii) $(5 + \sqrt{5})(5 - \sqrt{5}) = (5)^2 - (\sqrt{5})^2$
$= 25 - 5$
$= 20$
Therefore, it is rational.

(iv) $(\sqrt{3} - \sqrt{2})^2 = (\sqrt{3})^2 - 2(\sqrt{3})(\sqrt{2}) + (\sqrt{2})^2$
$= 3 - 2\sqrt{6} + 2$
$= 5 - 2\sqrt{6}$
Therefore, it is irrational.

(v) $(3/2\sqrt{2})^2 = 3^2/(2\sqrt{2})^2$
$= 9/(4 \times 2)$
$= 9/8$
Therefore, it is rational.

(vi) $(\sqrt{7}/6\sqrt{2})^2 = (\sqrt{7})^2/(6\sqrt{2})^2$
$= 7/(36 \times 2)$
$= 7/72$
Therefore, it is rational.

2. Find the square of:

(i) $3\sqrt{2}/5$
(ii) $\sqrt{3} + \sqrt{2}$
(iii) $\sqrt{5} - 2$
(iv) $3 + 2\sqrt{5}$

Solution:

(i) $(3\sqrt{2}/5)^2 = (3\sqrt{2})^2/5^2$
$= (9 \times 2)/25$
$= 18/25$
On further implication, we get
$= 1\frac{4}{5}$

(ii) $(\sqrt{3} + \sqrt{2})^2 = (\sqrt{3})^2 + (\sqrt{2})^2 + 2(\sqrt{3})(\sqrt{2})$
$= 3 + 2 + 2\sqrt{6}$
$= 5 + 2\sqrt{6}$

(iii) $(\sqrt{5} - 2)^2 = (\sqrt{5})^2 + (2)^2 - 2(\sqrt{5})(2)$
$= 5 + 4 - 4\sqrt{5}$
$= 9 - 4\sqrt{5}$

(iv) $(3 + 2\sqrt{5})^2 = 3^2 + 2(3)(2\sqrt{5}) + (2\sqrt{5})^2$
$= 9 + 12\sqrt{5} + (4 \times 5)$
$= 9 + 20 + 12\sqrt{5}$
$= 29 + 12\sqrt{5}$

3. State, in each case, whether true or false:
(i) $\sqrt{2} + \sqrt{3} = \sqrt{5}$
(ii) $2\sqrt{4} + 2 = 6$
(iii) $3\sqrt{7} - 2\sqrt{7} = \sqrt{7}$
(iv) $2/7$ is an irrational number.
(v) $5/11$ is a rational number.
(vi) All rational numbers are real numbers.
(vii) All real numbers are rational numbers.
(viii) Some real numbers are rational numbers.
Solution:

(i) False
(ii) True
(iii) True
(iv) False
(v) True
(vi) True
(vii) False
(viii) True

4. Given universal set is {-6, $-5\frac{3}{4}$, $-\sqrt{4}$, -3/5, -3/8, 0, 4/5, 1, $1\frac{2}{3}$, $\sqrt{8}$, 3.01, π, 8.47}
From the given set, find:
(i) Set of Rational numbers
(ii) Set of irrational numbers
(iii) Set of integers
(iv) Set of non-negative integers
Solution:

(i) First find the set of rational numbers
Rational numbers are numbers of the form p/q, where q ≠ 0
U = {-6, -5¾, -√4, -3/5, -3/8, 0, 4/5, 1, 1⅔, √8, 3.01, π, 8.47}
Here, -5¾, -3/5, -3/8, 4/5 and 1⅔ are of the from p/q
Therefore, they are rational numbers
The set of integers is a subset of rational numbers, -6, 0 and 1 are also rational numbers
Here, decimal numbers 3.01 and 8.47 are also rational numbers as they are terminating decimals
Also, -√4 = -2 as square root of 4 is 2
Thus, -2 belongs to the set of integers
From the above set, the set of rational numbers is Q,
Q = {-6, -5¾, -√4, -3/5, -3/8, 0, 4/5, 1, 1⅔, 3.01, 8.47}

(ii) First find the set of irrational numbers
Irrational numbers are numbers which are not rational
From the above subpart, we know that the set of rational numbers is Q,
Q = {-6, -5¾, -√4, -3/5, -3/8, 0, 4/5, 1, 1⅔, 3.01, 8.47}
Here the set of irrational numbers is the set of complement of the rational numbers over real numbers
The set of irrational numbers is U – Q = {√8, π}

(iii) First find the set of integers
Set of integers consists of zero, the natural numbers and their additive inverses
Set of integers is Z
Z = {…, -3, -2, -1, 0, 1, 2, 3, …}
Here, the set of integers is U ⋂ Z = {-6, -√4, 0, 1}

(iv) First find the set of non-negative integers
Set of non-negative integers consists of zero and the natural numbers
Set of non-negative integers is Z^+ and
$Z^+ = \{0, 1, 2, 3, \dots\}$
Set of integers is $U \cap Z^+ = \{0, 1\}$

5. Use method of contradiction to show that √3 and √5 are irrational.
Solution:

Consider √3 and √5 as rational numbers
√3 = a/b and √5 = x/y (where a, b, x, y ∈ Z and b, y ≠ 0)
By squaring on both sides, we have
$3 = a^2/b^2$, $5 = x^2/y^2$
$3b^2 = a^2$, $5y^2 = x^2$ …. (a)
Here,
a^2 and x^2 are odd as $3b^2$ and $5y^2$ are odd.
a and x are odd …. (1)

Take a = 3c, x = 5z
By squaring on both sides
$a^2 = 9c^2$, $x^2 = 25z^2$

Using equation (a)
$3b^2 = 9c^2$, $5y^2 = 25z^2$
By further simplification
$b^2 = 3c^2$, $y^2 = 5z^2$
Here,
B^2 and y^2 are odd as $3c^2$ and $5z^2$ are odd.
b and y are odd …… (2)

Using equation (1) and (2) we know that a, b, x, y are odd integers.
a, b and x, y have common factors 3 and 5 which contradicts our assumption that a/b and x/y are rational
a, b and x, y do not have any common factors
a/b and x/y is not rational
$\sqrt{3}$ and $\sqrt{5}$ are irrational.

6. Prove that each of the following numbers is irrational:
(i) $\sqrt{3} + \sqrt{2}$
(ii) $3 - \sqrt{2}$
(iii) $\sqrt{5} - 2$
Solution:

(i) $\sqrt{3} + \sqrt{2}$
Consider $\sqrt{3} + \sqrt{2}$ be a rational number.
$\sqrt{3} + \sqrt{2} = x$
By squaring on both sides
$(\sqrt{3} + \sqrt{2})^2 = x^2$
$(\sqrt{3})^2 + (\sqrt{2})^2 + 2(\sqrt{3})(\sqrt{2}) = x^2$
$3 + 2 + 2\sqrt{6} = x^2$
$5 + 2\sqrt{6} = x^2$
$2\sqrt{6} = x^2 - 5$
$\sqrt{6} = (x^2 - 5)/2$
Now,
x is a rational number.
x^2 is a rational number.
$x^2 - 5$ is a rational number.
$(x^2 - 5)/2$ is also a rational number.
Considering the equation, $(x^2 - 5)/2 = \sqrt{6}$
$\sqrt{6}$ is an irrational number
But, $(x^2 - 5)/2$ is a rational number
So, $x^2 - 5$ has to be an irrational number.
Then, x^2 should also be an irrational number.
Also, x must be an irrational number.

We assumed that x is a rational number
So, we arrive at a contradiction.
Hence, our assumption that $\sqrt{3} + \sqrt{2}$ is a rational number is wrong.
Therefore, $\sqrt{3} + \sqrt{2}$ is an irrational number.

(ii) $3 - \sqrt{2}$
Consider $3 - \sqrt{2}$ as a rational number.
$3 - \sqrt{2} = x$
By squaring on both sides, we get
$(3 - \sqrt{2})^2 = x^2$
$(3)^2 + (\sqrt{2})^2 - 2(3)(\sqrt{2}) = x^2$
$9 + 2 - 6\sqrt{2} = x^2$
$11 - 6\sqrt{2} = x^2$
$6\sqrt{2} = 11 - x^2$
$\sqrt{2} = (11 - x^2)/6$
Now,
x is a rational number.
x^2 is a rational number.
$11 - x^2$ is a rational number.
$(11 - x^2)/6$ is also a rational number.
Considering the equation, $\sqrt{2} = (11 - x^2)/6$
$\sqrt{2}$ is an irrational number
But, $(11 - x^2)/2$ is a rational number
So, $11 - x^2$ has to be an irrational number.
Then, x^2 should also be an irrational number.
Also, x must be an irrational number.

We assumed that x is a rational number
So, we arrive at a contradiction.
Hence, our assumption that $3 - \sqrt{2}$ is a rational number is wrong.
Therefore, $3 - \sqrt{2}$ is an irrational number.

(iii) $\sqrt{5} - 2$
Consider $\sqrt{5} - 2$ as a rational number.
$\sqrt{5} - 2 = x$
By squaring on both sides
$(\sqrt{5} - 2)^2 = x^2$
$(\sqrt{5})^2 + (2)^2 - 2(\sqrt{5})(2) = x^2$
$5 + 4 - 4\sqrt{5} = x^2$
$9 - 4\sqrt{5} = x^2$
$4\sqrt{5} = 9 - x^2$
$\sqrt{5} = (9 - x^2)/4$
Now,
x is a rational number.
x^2 is a rational number.
$9 - x^2$ is a rational number.
$(9 - x^2)/4$ is also a rational number.
Considering the equation, $\sqrt{5} = (9 - x^2)/4$
$\sqrt{5}$ is an irrational number
But, $(9 - x^2)/4$ is a rational number
So, $9 - x^2$ has to be an irrational number.
Then, x^2 should also be an irrational number.

Also, x must be an irrational number.

We assumed that x is a rational number
So, we arrive at a contradiction.
Hence, our assumption that $\sqrt{5} - 2$ is a rational number is wrong.
Therefore, $\sqrt{5} - 2$ is an irrational number.

7. Write a pair of irrational numbers whose sum is irrational.
Solution:

$\sqrt{3} + 5$ and $\sqrt{5} - 3$ are irrational numbers whose sum is irrational.
Here,
Sum = $(\sqrt{3} + 5) + (\sqrt{5} - 3)$
= $\sqrt{3} + \sqrt{5} + 2$
Hence, the resultant is irrational.

8. Write a pair of irrational numbers whose sum is rational.
Solution:

$\sqrt{3} + 5$ and $4 - \sqrt{3}$ are irrational numbers whose sum is rational.
Here,
Sum = $(\sqrt{3} + 5) + (4 - \sqrt{3})$
= $\sqrt{3} - \sqrt{3} + 9$
= 9
Hence, the resultant is rational.

9. Write a pair of irrational numbers whose difference is irrational.
Solution:

$\sqrt{3} + 2$ and $\sqrt{2} - 3$ are irrational numbers whose sum is irrational.
Here,
Difference = $(\sqrt{3} + 2) - (\sqrt{2} - 3)$
= $\sqrt{3} - \sqrt{2} + 2 + 3$
= $\sqrt{3} - \sqrt{2} + 5$
Hence, the resultant is irrational.

10. Write a pair of irrational numbers whose difference is rational.
Solution:

$\sqrt{5} - 3$ and $\sqrt{5} + 3$ are irrational numbers whose sum is irrational.
Here,
Difference = $(\sqrt{5} - 3) - (\sqrt{5} + 3)$
= $\sqrt{5} - \sqrt{5} - 3 - 3$
= -6
Hence, the resultant is rational.

11. Write a pair of irrational numbers whose product is irrational.

Solution:

Let us take two irrational numbers $(5 + \sqrt{2})$ and $(\sqrt{5} – 2)$
Here the product = $(5 + \sqrt{2}) \times (\sqrt{5} – 2)$
By further calculation
= $5\sqrt{5} – 10 + \sqrt{10} - 2\sqrt{2}$ which is irrational.

12. Write a pair of irrational numbers whose product is rational.
Solution:

Let us consider two irrational numbers $(2\sqrt{3} – 3\sqrt{2})$ and $(2\sqrt{3} + 3\sqrt{2})$

(iii) $6\sqrt{5} = \sqrt{(6^2 \times 5)} = \sqrt{(36 \times 5)} = \sqrt{180}$
$7\sqrt{3} = \sqrt{(7^2 \times 3)} = \sqrt{(49 \times 3)} = \sqrt{147}$
$8\sqrt{2} = \sqrt{(8^2 \times 2)} = \sqrt{(128 \times 2)} = \sqrt{128}$
We know that, $128 < 147 < 180$
So, $\sqrt{128} < \sqrt{147} < \sqrt{180}$
Therefore, $8\sqrt{2} < 7\sqrt{3} < 6\sqrt{5}$

14. Write in descending order:
(i) $2\sqrt[4]{6}$ and $3\sqrt[4]{2}$
(ii) $7\sqrt{3}$ and $3\sqrt{7}$

Solution:

(i) It can be written as
$2\sqrt[4]{6} = \sqrt[4]{(2^4 \times 6)} = \sqrt[4]{96}$
$3\sqrt[4]{2} = \sqrt[4]{(3^4 \times 2)} = \sqrt[4]{162}$
Here, $162 > 96$
So, $\sqrt[4]{162} > \sqrt[4]{96}$
Therefore, $3\sqrt[4]{2} > 2\sqrt[4]{6}$

(ii) It can be written as
$7\sqrt{3} = \sqrt{(7^2 \times 3)} = \sqrt{(49 \times 3)} = \sqrt{141}$

(ii) $\sqrt{24} = (24)^{1/2}$ and $\sqrt[3]{35} = (35)^{1/3}$
In order to make the powers ½ and 1/3 same,
We find L.C.M. of 2 and 3 i.e., 6
½ x 3/3 = 3/6 and 1/3 x 2/2 = 2/6
Now,
(i) $\sqrt[6]{15}$ **and** $\sqrt[4]{12}$
(ii) $\sqrt{24}$ **and** $\sqrt[3]{35}$

$(24)^{1/2} = (24)^{3/6} = (24^3)^{1/6} = (13824)^{1/6}$
$(35)^{1/3} = (35)^{2/6} = (35^2)^{1/6} = (1225)^{1/6}$
On comparing,
$13824 > 1225$
So, $(13824)^{1/6} > (1225)^{1/6}$
Therefore,
$\sqrt{24} > \sqrt[3]{35}$

16. Insert two irrational numbers between 5 and 6.
Solution:

Let's write 5 and 6 as square root
Then, $5 = \sqrt{25}$ and $6 = \sqrt{36}$
Now, take the numbers
$\sqrt{25} < \sqrt{26} < \sqrt{27} < \sqrt{28} < \sqrt{29} < \sqrt{30} < \sqrt{31} < \sqrt{32} < \sqrt{33} < \sqrt{34} < \sqrt{35} < \sqrt{36}$
Hence, any two irrational numbers between 5 and 6 is $\sqrt{29}$ and $\sqrt{30}$

17. Insert five irrational numbers between $2\sqrt{5}$ and $3\sqrt{3}$.
Solution:

Here, $2\sqrt{5} = \sqrt{(2^2 \text{ x } 5)} = \sqrt{(4 \text{ x } 5)} = \sqrt{20}$ and
$3\sqrt{3} = \sqrt{(3^2 \text{ x } 3)} = \sqrt{(9 \text{ x } 3)} = \sqrt{27}$
Now, take the numbers
$\sqrt{20} < \sqrt{21} < \sqrt{22} < \sqrt{23} < \sqrt{24} < \sqrt{25} < \sqrt{26} < \sqrt{27}$
Hence, any five irrational numbers between $2\sqrt{5}$ and $3\sqrt{3}$ are:
$\sqrt{21}, \sqrt{22}, \sqrt{23}, \sqrt{24}$ and $\sqrt{26}$

18. Write two rational numbers between $\sqrt{2}$ and$\sqrt{3}$.
Solution:

Let us take any two rational numbers between 2 and 3 which are perfect squares
For example, let us consider 2.25 and 2.56
Now, we have
$\sqrt{2.25} = 1.5$ and $\sqrt{2.56} = 1.6$
Now,
$\sqrt{2} < \sqrt{2.25} < \sqrt{2.56}\ \sqrt{3}$
$\sqrt{2} < 1.5 < 1.6 < \sqrt{3}$
$\sqrt{2} < 15/10 < 16/10 < \sqrt{3}$
$\sqrt{2} < 3/2 < 8/5 < \sqrt{3}$
Hence, any two rational numbers between $\sqrt{2}$ and $\sqrt{3}$ are: 3/2 and 8/5

19. Write three rational numbers between $\sqrt{3}$ and $\sqrt{5}$.
Solution:

Let us take any two rational numbers between 3 and 5 which are perfect squares
For example, let us consider 3.24, 3.61, 4, 4.41 and 4.84
Now, we have

$\sqrt{3.24} = 1.8$, $\sqrt{3.61} = 1.9$, $\sqrt{4} = 2$, $\sqrt{4.41} = 2.1$ and $\sqrt{4.84} = 2.2$

Now,

$\sqrt{3} < \sqrt{3.24} < \sqrt{3.61} < \sqrt{4} < \sqrt{4.41} < \sqrt{4.84} < \sqrt{5}$

$\sqrt{3} < 1.8 < 1.9 < 2 < 2.1 < 2.2 < \sqrt{5}$

$\sqrt{3} < 18/10 < 19/10 < 2 < 21/10 < 22/10 < \sqrt{5}$

$\sqrt{3} < 9/5 < 19/10 < 2 < 21/10 < 11/5 < \sqrt{5}$

Hence, any three rational numbers between $\sqrt{3}$ and $\sqrt{5}$ are: 9/5, 21/10 and 11/5

20. Simplify each of the following:

$= 3 \times 4 + 3 \times \sqrt{7} + 4 \times \sqrt{2} + \sqrt{2} \times \sqrt{7}$

So, we get

$= 12 + 3\sqrt{7} + 4\sqrt{2} + \sqrt{14}$

(iv) $(\sqrt{3} - \sqrt{2})^2$

It can be written as

$= (\sqrt{3})^2 + (\sqrt{2})^2 - 2 \times \sqrt{3} \times \sqrt{2}$

By further calculation, we get

$= 3 + 2 - 2\sqrt{6}$

$= 5 - 2\sqrt{6}$