EXERCISE 28.3

The vertices of the triangle are A(5, 4, 6), B(1, -1, 3) and C(4, 3, 2). The internal bisector of angle A meets BC at D. Find the coordinates of D and the length AD. Solution:

Given:

The vertices of the triangle are A (5, 4, 6), B (1, -1, 3) and C (4, 3, 2).

By using the formulas let us find the coordinates of D and the length of AD

The distance between any two points (a, b, c) and (m, n, o) is given by,

$$\sqrt{(a-m)^2 + (b-n)^2 + (c-o)^2}$$

The section formula is given as

$$\left(\frac{nx+ma}{m+n}, \frac{ny+mb}{m+n}, \frac{nz+mc}{m+n}\right)$$

The distance between the points A (5, 4, 6) and B (1, -1, 3) is AB,

$= \sqrt{(5-1)^2 + (4-(-1))^2 + (6-3)^2}$

$= \sqrt{4^2 + 5^2 + 3^2}$

$= \sqrt{16 + 25 + 9}$

$= \sqrt{50}$

$= 5\sqrt{2}$

The distance between the points A (5, 4, 6) and C (4, 3, 2) is AC,

$= \sqrt{(5-4)^2 + (4-3)^2 + (6-2)^2}$

$= \sqrt{1^2 + 1^2 + 4^2}$

$= \sqrt{1 + 1 + 16}$

$= \sqrt{18}$

$= 3\sqrt{2}$

So,

$$\frac{AB}{AC} = \frac{5\sqrt{2}}{3\sqrt{2}} = \frac{5}{3}$$

AB : AC = 5:3

BD: DC = 5:3
So, m = 5 and n = 3

B(1, -1, 3) and C(4, 3, 2)

Coordinates of D using section formula:

$$= \left(\frac{3(1)+5(4)}{5+3}, \frac{3(-1)+5(3)}{5+3}, \frac{3(3)+5(2)}{5+3}\right)$$

$$= \left(\frac{3+20}{8}, \frac{-3+15}{8}, \frac{9+10}{8}\right)$$

$$= \left(\frac{23}{8}, \frac{12}{8}, \frac{19}{8}\right)$$

$$= \left(\frac{23}{8}, \frac{3}{2}, \frac{19}{8}\right)$$

The distance between the points A (5, 4, 6) and D (23/8, 3/2, 19/8) is AD,

$$= \sqrt{\left(5-\frac{23}{8}\right)^2 + \left(4-\frac{3}{2}\right)^2 + \left(6-\frac{19}{8}\right)^2}$$

$$= \sqrt{\left(\frac{40-23}{8}\right)^2 + \left(\frac{8-3}{2}\right)^2 + \left(\frac{48-19}{8}\right)^2}$$

$$= \sqrt{\left(\frac{17}{8}\right)^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{29}{8}\right)^2}$$

$$= \sqrt{\frac{289}{64} + \frac{25}{4} + \frac{361}{64}}$$

$$= \sqrt{\frac{289+400+841}{64}}$$

$$= \sqrt{\frac{1530}{64}}$$

$$= \frac{\sqrt{1530}}{8}$$ Units

∴ The Coordinates of D are $\left(\frac{23}{8}, \frac{3}{2}, \frac{19}{8}\right)$ and the length of AD is $\frac{\sqrt{1530}}{8}$ units.

1. A point C with z-coordinate 8 lies on the line segment joining the points A(2, -3, 4) and B(8, 0, 10). Find the coordinates.

Solution:

Given:

The points A (2, -3, 4) and B (8, 0, 10)

By using the section formula,

$$\left(\frac{nx + ma}{m + n}, \frac{ny + mb}{m + n}, \frac{nz + mc}{m + n}\right)$$

Let Point C(x, y, 8), and C divides AB in ratio k: 1

So, m = k and n = 1

A(2, -3, 4) and B(8, 0, 10)

Coordinates of C are:

$$(x,y,8) = \left(\frac{k(8) + 1(2)}{k + 1}, \frac{k(0) + 1(-3)}{k + 1}, \frac{k(10) + 1(4)}{k + 1}\right)$$
$$= \left(\frac{8k + 2}{k + 1}, \frac{-3}{k + 1}, \frac{10k + 4}{k + 1}\right)$$

On comparing we get,

[10k + 4] / [k + 1] = 8

10k + 4 = 8(k + 1)

10k + 4 = 8k + 8

10k – 8k = 8 – 4

2k = 4

k = 4/2

= 2

Here C divides AB in ratio 2:1

x = [8k + 2] / [k + 1]

= [8(2) + 2] / [2 + 1]

= [16 + 2] / [3]

= 18/3

= 6

y = -3 / [k + 1]

= -3 / [2 + 1]

= -3 / 3
= -1
∴ The Coordinates of C are (6, -1, 8).

2. Show that the three points A(2, 3, 4), B(-1, 2, -3) and C(-4, 1, -10) are collinear and find the ratio in which C divides AB.
Solution:
Given:
The points A (2, 3, 4), B (-1, 2, -3) and C (-4, 1, -10)
By using the section formula,

$$\left(\frac{nx + ma}{m + n}, \frac{ny + mb}{m + n}, \frac{nz + mc}{m + n}\right)$$

Let C divides AB in ratio k: 1
Three points are collinear if the value of k is the same for x, y and z coordinates.
So, m = k and n = 1
A(2, 3, 4), B(-1, 2, -3) and C(-4, 1, -10)

Coordinates of C are:

$$(-4, 1, -10) = \left(\frac{k(-1) + 1(2)}{k + 1}, \frac{k(2) + 1(3)}{k + 1}, \frac{k(-3) + 1(4)}{k + 1}\right)$$

$$= \left(\frac{-k + 2}{k + 1}, \frac{2k + 3}{k + 1}, \frac{-3k + 4}{k + 1}\right)$$

On comparing we get,
[-k + 2] / [k + 1] = -4
-k + 2 = -4(k + 1)
-k + 2 = -4k – 4
4k – k = - 2 – 4
3k = -6
k = -6/3
= -2

[2k + 3] / [k + 1] = 1
2k + 3 = k + 1
2k – k = 1 – 3
k = – 2

[-3k + 4] / [k + 1] = -10
-3k + 4 = -10(k + 1)
-3k + 4 = -10k – 10

-3k + 10k = -10 – 4
7k = -14
k = -14/7
= -2

The value of k is the same in all three cases.
So, A, B and C are collinear [as k = -2]
∴We can say that, C divides AB externally in ratio 2:1

3. Find the ratio in which the line joining (2, 4, 5) and (3, 5, 4) is divided by the yz-plane.
Solution:
Given:
The points (2, 4, 5) and (3, 5, 4)
By using the section formula,
$$\left(\frac{nx+ma}{m+n}, \frac{ny+mb}{m+n}, \frac{nz+mc}{m+n}\right)$$
We know X coordinate is always 0 on yz-plane
So, let Point C(0, y, z), and let C divide AB in ratio k: 1
Then, m = k and n = 1
A(2, 4, 5) and B(3, 5, 4)

The coordinates of C are:
$$(0,y,z) = \left(\frac{k(3)+1(2)}{k+1}, \frac{k(5)+1(4)}{k+1}, \frac{k(4)+1(5)}{k+1}\right)$$
$$= \left(\frac{3k+2}{k+1}, \frac{5k+4}{k+1}, \frac{4k+5}{k+1}\right)$$
On comparing we get,
[3k + 2] / [k + 1] = 0
3k + 2 = 0(k + 1)
3k + 2 = 0
3k = – 2
k = -2/3
∴We can say that, C divides AB externally in ratio 2: 3

4. Find the ratio in which the line segment joining the points (2, -1, 3) and (-1, 2, 1) is divided by the plane x + y + z = 5.
Solution:
Given:

The points(2, -1, 3) and (-1, 2, 1)
By using the section formula,
$$\left(\frac{nx + ma}{m + n}, \frac{ny + mb}{m + n}, \frac{nz + mc}{m + n}\right)$$
Let C(x, y, z) be any point on the given plane and C divides AB in ratio k: 1
Then, m = k and n = 1
A(2, -1, 3) and B(-1, 2, 1)

Coordinates of C are:
$$(x,y,z) = \left(\frac{k(-1) + 1(2)}{k + 1}, \frac{k(2) + 1(-1)}{k + 1}, \frac{k(-1) + 1(3)}{k + 1}\right)$$
$$= \left(\frac{-k + 2}{k + 1}, \frac{2k - 1}{k + 1}, \frac{-k + 3}{k + 1}\right)$$
On comparing we get,
[-k + 2] / [k + 1] = x
[2k - 1] / [k + 1] = y
[-k + 3] / [k + 1] = z

We know that x + y + z = 5
$$\frac{-k + 2}{k + 1} + \frac{2k - 1}{k + 1} + \frac{-k + 3}{k + 1} = 5$$

$$\frac{-k + 2 + 2k - 1 - k + 3}{k + 1} = 5$$
$$\frac{4}{k + 1} = 5$$
5(k + 1) = 4
5k + 5 = 4
5k = 4 – 5
5k = – 1
k = -1/5
∴We can say that, the plane divides AB externally in the ratio 1:5

5. If the points A(3, 2, -4), B(9, 8, -10) and C(5, 4, -6) are collinear, find the ratio in which C divided AB.
Solution:
Given:
The points A (3, 2, -4), B (9, 8, -10) and C (5, 4, -6)

By using the section formula,

$$\left(\frac{nx+ma}{m+n}, \frac{ny+mb}{m+n}, \frac{nz+mc}{m+n}\right)$$

Let C divides AB in ratio k: 1

Three points are collinear if the value of k is the same for x, y and z coordinates.

Then, m = k and n = 1

A(3, 2, -4), B(9, 8, -10) and C(5, 4, -6)

Coordinates of C are:

$$(5,4,-6) = \left(\frac{k(9)+1(3)}{k+1}, \frac{k(8)+1(2)}{k+1}, \frac{k(-10)+1(-4)}{k+1}\right)$$
$$= \left(\frac{9k+3}{k+1}, \frac{8k+2}{k+1}, \frac{-10k-4}{k+1}\right)$$

On comparing we get,

[9k + 3] / [k + 1] = 5

9k + 3 = 5(k + 1)

9k + 3 = 5k + 5

9k – 5k = 5 – 3

4k = 2

k = 2/4

= ½

[8k + 2] / [k + 1] = 4

8k + 2 = 4(k + 1)

8k + 2 = 4k + 4

8k – 4k = 4 – 2

4k = 2

k = 2/4

= ½

[-10k - 4] / [k + 1] = -6

-10k – 4 = -6(k + 1)

-10k – 4 = -6k – 6

-10k + 6k = 4 – 6

-4k = -2

k = -2/-4

= ½

The value of k is the same in all three cases.

So, A, B and C are collinear [as, k = ½]

∴We can say that, C divides AB externally in ratio 1:2

6. The mid-points of the sides of a triangle ABC are given by (-2, 3, 5), (4, -1, 7) and (6, 5, 3). Find the coordinates of A, B and C.

Solution:

Given:

The mid-points of the sides of a triangle ABC is given as (-2, 3, 5), (4, -1, 7) and (6, 5, 3).

By using the section formula,

$$\left(\frac{nx+ma}{m+n}, \frac{ny+mb}{m+n}, \frac{nz+mc}{m+n}\right)$$

We know the mid-point divides side in the ratio of 1:1.

The coordinates of C is given by,

$$\left(\frac{x+a}{2}, \frac{y+b}{2}, \frac{z+c}{2}\right)$$

P(-2, 3, 5) is mid-point of A(x_1, y_1, z_1) and B(x_2, y_2, z_2)

Then,

$$(-2, 3, 5) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

$(-4, 6, 10) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ …. (1)

Q(4, -1, 7) is mid-point of B(x_2, y_2, z_2) and C(x_3, y_3, z_3)

Then,

$$(4, -1, 7) = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$$

$(8, -2, 14) = (x_2 + x_3, y_3 + y_3, z_3 + z_3)$ …. (2)

R(6, 5, 3) is mid-point of A(x_1, y_1, z_1) and C(x_3, y_3, z_3)

Then,

$$(6, 5, 3) = \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, \frac{z_1+z_3}{2}\right)$$

$(12, 10, 6) = (x_1 + x_3, y_1 + y_3, z_1 + z_3)$ …. (3)

Now solving for 'x' terms

$x_1 + x_2 = -4$…...........................(4)

$x_2 + x_3 = 8$.................................. (5)

$x_1 + x_3 = 12$.............................. (6)

By adding equation (4), (5), (6)

$x_1 + x_2 + x_2 + x_3 + x_1 + x_3 = 8 + 12 – 4$

$2x_1 + 2x_2 + 2x_3 = 16$
$2(x_1 + x_2 + x_3) = 16$
$x_1 + x_2 + x_3 = 8$……………………………(7)

Now, subtract equation (4), (5) and (6) from equation (7) separately:
$x_1 + x_2 + x_3 – x_1 – x_2 = 8 – (-4)$
$x_3 = 12$

$x_1 + x_2 + x_3 – x_2 – x_3 = 8 – 8$
$x_1 = 0$

$x_1 + x_2 + x_3 – x_1 – x_3 = 8 – 12$
$x_2 = -4$

Now solving for 'y' terms
$y_1 + y_2 = 6$………………………… (8)
$y_2 + y_3 = -2$…………………………(9)
$y_1 + y_3 = 10$………………………… (10)

By adding equation (8), (9) and (10) we get,
$y_1 + y_2 + y_2 + y_3 + y_1 + y_3 = 10 + 6 – 2$
$2y_1 + 2y_2 + 2y_3 = 14$
$2(y_1 + y_2 + y_3) = 14$
$y_1 + y_2 + y_3 = 7$……………………………(11)

Now, subtract equation (8), (9) and (10) from equation (11) separately:
$y_1 + y_2 + y_3 – y_1 – y_2 = 7 – 6$
$y_3 = 1$

$y_1 + y_2 + y_3 – y_2 – y_3 = 7 – (-2)$
$y_1 = 9$

$y_1 + y_2 + y_3 – y_1 – y_3 = 7 – 10$
$y_2 = -3$

Now solving for 'z' terms
$z_1 + z_2 = 10$…………………………(12)
$z_2 + z_3 = 14$…………………………(13)
$z_1 + z_3 = 6$…………………………(14)

By adding equation (12), (13) and (14) we get,
$z_1 + z_2 + z_2 + z_3 + z_1 + z_3 = 6 + 14 + 10$
$2z_1 + 2z_2 + 2z_3 = 30$
$2(z_1 + z_2 + z_3) = 30$
$z_1 + z_2 + z_3 = 15$… .... (15)

Now, subtract equation (8), (9) and (10) from equation (11) separately:
$z_1 + z_2 + z_3 - z_1 - z_2 = 15 - 10$
$z_3 = 5$

$z_1 + z_2 + z_3 - z_2 - z_3 = 15 - 14$
$z_1 = 1$

$z_1 + z_2 + z_3 - z_1 - z_3 = 15 - 6$
$z_2 = 9$

∴Thevertices of sides of a triangle ABC are A(0, 9, 1) B(-4,-3, 9) and C(12, 1, 5).

7. A(1, 2, 3), B(0, 4, 1), C(-1, -1, -3) are the vertices of a triangle ABC. Find the point in which the bisector of the angle ∠BAC meets BC.
Solution:
Given:
The vertices of a triangle are A (1, 2, 3), B (0, 4, 1), C (-1, -1, -3)
By using the distance formula,
$\sqrt{(a-m)^2 + (b-n)^2 + (c-o)^2}$
So,
The distance between the points A (1, 2, 3) and B (0, 4, 1) is AB,
$= \sqrt{(1-0)^2 + (2-4)^2 + (3-1)^2}$
$= \sqrt{1^2 + (-2)^2 + 2^2}$
$= \sqrt{1 + 4 + 4}$
$= \sqrt{9}$
$= 3$

The distance between the points A (1, 2, 3) and C (-1, -1, -3) is AC,

$= \sqrt{(1-(-1))^2 + (2-(-1))^2 + (3-(-3))^2}$

$= \sqrt{2^2 + 3^2 + 6^2}$

$= \sqrt{4 + 9 + 36}$

$= \sqrt{49}$

$= 7$

So, AB/AC = 3/7

AB: AC = 3:7

BD: DC = 3:7

Then, m = 3 and n = 7

B(0, 4, 1) and C(-1, -1, -3)

Coordinates of D by using section formula is given as

$= \left(\frac{7(0)+3(-1)}{7+3}, \frac{7(4)+3(-1)}{7+3}, \frac{7(1)+3(-3)}{7+3}\right)$

$= \left(\frac{0-3}{10}, \frac{28-3}{10}, \frac{7-9}{10}\right)$

$= \left(\frac{-3}{10}, \frac{25}{10}, \frac{-2}{10}\right)$

$= \left(\frac{-3}{10}, \frac{5}{2}, \frac{-1}{5}\right)$

∴The coordinates of D are (-3/10, 5/2, -1/5).