

### EXERCISE 3.4

Let  $*$  be a binary operation on  $Z$  defined by  $a * b = a + b - 4$  for all  $a, b \in Z$ .

(i) Show that  $*$  is both commutative and associative.

(ii) Find the identity element in  $Z$

(iii) Find the invertible element in  $Z$ .

**Solution:**

(i) First we have to prove commutativity of  $*$

Let  $a, b \in Z$ . then,

$$a * b = a + b - 4$$

$$= b + a - 4$$

$$= b * a$$

Therefore,

$$a * b = b * a, \forall a, b \in Z$$

Thus,  $*$  is commutative on  $Z$ .

Now we have to prove associativity of  $Z$ .

Let  $a, b, c \in Z$ . then,

$$a * (b * c) = a * (b + c - 4)$$

$$= a + b + c - 4 - 4$$

$$= a + b + c - 8$$

$$(a * b) * c = (a + b - 4) * c$$

$$= a + b - 4 + c - 4$$

$$= a + b + c - 8$$

Therefore,

$$a * (b * c) = (a * b) * c, \text{ for all } a, b, c \in Z$$

Thus,  $*$  is associative on  $Z$ .

(ii) Let  $e$  be the identity element in  $Z$  with respect to  $*$  such that

$$a * e = a = e * a \quad \forall a \in Z$$

$$a * e = a \text{ and } e * a = a, \quad \forall a \in Z$$

$$a + e - 4 = a \text{ and } e + a - 4 = a, \quad \forall a \in Z$$

$$e = 4, \quad \forall a \in Z$$

Thus, 4 is the identity element in  $Z$  with respect to  $*$ .

(iii) Let  $a \in Z$  and  $b \in Z$  be the inverse of  $a$ . Then,

$$a * b = e = b * a$$

$$a * b = e \text{ and } b * a = e$$

$$a + b - 4 = 4 \text{ and } b + a - 4 = 4$$

$$b = 8 - a \in \mathbb{Z}$$

Thus,  $8 - a$  is the inverse of  $a \in \mathbb{Z}$

**2. Let  $*$  be a binary operation on  $\mathbb{Q}_0$  (set of non-zero rational numbers) defined by  $a * b = (3ab/5)$  for all  $a, b \in \mathbb{Q}_0$ . Show that  $*$  is commutative as well as associative. Also, find its identity element, if it exists.**

**Solution:**

First we have to prove commutativity of  $*$

Let  $a, b \in \mathbb{Q}_0$

$$a * b = (3ab/5)$$

$$= (3ba/5)$$

$$= b * a$$

Therefore,  $a * b = b * a$ , for all  $a, b \in \mathbb{Q}_0$

Now we have to prove associativity of  $*$

Let  $a, b, c \in \mathbb{Q}_0$

$$a * (b * c) = a * (3bc/5)$$

$$= [a (3bc/5)] / 5$$

$$= 3abc/25$$

$$(a * b) * c = (3ab/5) * c$$

$$= [(3ab/5)c] / 5$$

$$= 3abc/25$$

Therefore  $a * (b * c) = (a * b) * c$ , for all  $a, b, c \in \mathbb{Q}_0$

Thus  $*$  is associative on  $\mathbb{Q}_0$

Now we have to find the identity element

Let  $e$  be the identity element in  $\mathbb{Z}$  with respect to  $*$  such that

$$a * e = a = e * a \quad \forall a \in \mathbb{Q}_0$$

$$a * e = a \text{ and } e * a = a, \quad \forall a \in \mathbb{Q}_0$$

$$3ae/5 = a \text{ and } 3ea/5 = a, \quad \forall a \in \mathbb{Q}_0$$

$$e = 5/3 \quad \forall a \in \mathbb{Q}_0 \text{ [because } a \text{ is not equal to } 0]$$

Thus,  $5/3$  is the identity element in  $\mathbb{Q}_0$  with respect to  $*$ .

**3. Let  $*$  be a binary operation on  $\mathbb{Q} - \{-1\}$  defined by  $a * b = a + b + ab$  for all  $a, b \in \mathbb{Q} - \{-1\}$ . Then,**

(i) Show that  $*$  is both commutative and associative on  $Q - \{-1\}$

(ii) Find the identity element in  $Q - \{-1\}$

(iii) Show that every element of  $Q - \{-1\}$  is invertible. Also, find inverse of an arbitrary element.

**Solution:**

(i) First we have to check commutativity of  $*$

Let  $a, b \in Q - \{-1\}$

Then  $a * b = a + b + ab$

$= b + a + ba$

$= b * a$

Therefore,

$a * b = b * a, \forall a, b \in Q - \{-1\}$

Now we have to prove associativity of  $*$

Let  $a, b, c \in Q - \{-1\}$ , Then,

$a * (b * c) = a * (b + c + bc)$

$= a + (b + c + bc) + a(b + c + bc)$

$= a + b + c + bc + ab + ac + abc$

$(a * b) * c = (a + b + ab) * c$

$= a + b + ab + c + (a + b + ab)c$

$= a + b + ab + c + ac + bc + abc$

Therefore,

$a * (b * c) = (a * b) * c, \forall a, b, c \in Q - \{-1\}$

Thus,  $*$  is associative on  $Q - \{-1\}$ .

(ii) Let  $e$  be the identity element in  $I^+$  with respect to  $*$  such that

$a * e = a = e * a, \forall a \in Q - \{-1\}$

$a * e = a$  and  $e * a = a, \forall a \in Q - \{-1\}$

$a + e + ae = a$  and  $e + a + ea = a, \forall a \in Q - \{-1\}$

$e + ae = 0$  and  $e + ea = 0, \forall a \in Q - \{-1\}$

$e(1 + a) = 0$  and  $e(1 + a) = 0, \forall a \in Q - \{-1\}$

$e = 0, \forall a \in Q - \{-1\}$  [because  $a$  not equal to  $-1$ ]

Thus,  $0$  is the identity element in  $Q - \{-1\}$  with respect to  $*$ .

(iii) Let  $a \in Q - \{-1\}$  and  $b \in Q - \{-1\}$  be the inverse of  $a$ . Then,

$a * b = e = b * a$

$a * b = e$  and  $b * a = e$

$$a + b + ab = 0 \text{ and } b + a + ba = 0$$

$$b(1 + a) = -a \text{ } Q - \{-1\}$$

$$b = -a/1 + a \text{ } Q - \{-1\} \text{ [because } a \text{ not equal to } -1]$$

Thus,  $-a/1 + a$  is the inverse of  $a \in Q - \{-1\}$

**4. Let  $A = R_0 \times R$ , where  $R_0$  denote the set of all non-zero real numbers. A binary operation 'O' is defined on A as follows:  $(a, b) O (c, d) = (ac, bc + d)$  for all  $(a, b), (c, d) \in R_0 \times R$ .**

**(i) Show that 'O' is commutative and associative on A**

**(ii) Find the identity element in A**

**(iii) Find the invertible element in A.**

**Solution:**

(i) Let  $X = (a, b)$  and  $Y = (c, d) \in A, \forall a, c \in R_0$  and  $b, d \in R$

Then,  $X O Y = (ac, bc + d)$

And  $Y O X = (ca, da + b)$

Therefore,

$$X O Y = Y O X, \forall X, Y \in A$$

Thus, O is not commutative on A.

Now we have to check associativity of O

Let  $X = (a, b), Y = (c, d)$  and  $Z = (e, f), \forall a, c, e \in R_0$  and  $b, d, f \in R$

$$X O (Y O Z) = (a, b) O (ce, de + f)$$

$$= (ace, bce + de + f)$$

$$(X O Y) O Z = (ac, bc + d) O (e, f)$$

$$= (ace, (bc + d)e + f)$$

$$= (ace, bce + de + f)$$

Therefore,  $X O (Y O Z) = (X O Y) O Z, \forall X, Y, Z \in A$

(ii) Let  $E = (x, y)$  be the identity element in A with respect to O,  $\forall x \in R_0$  and  $y \in R$  Such that,

$$X O E = X = E O X, \forall X \in A$$

$$X O E = X \text{ and } E O X = X$$

$$(ax, bx + y) = (a, b) \text{ and } (xa, ya + b) = (a, b)$$

Considering  $(ax, bx + y) = (a, b)$

$$ax = a$$

$$x = 1$$

$$\text{And } bx + y = b$$

$$y = 0 \text{ [since } x = 1\text{]}$$

$$\text{Considering } (xa, ya + b) = (a, b)$$

$$xa = a$$

$$x = 1$$

$$\text{And } ya + b = b$$

$$y = 0 \text{ [since } x = 1\text{]}$$

Therefore  $(1, 0)$  is the identity element in  $A$  with respect to  $O$ .

(iii) Let  $F = (m, n)$  be the inverse in  $A \forall m \in R_0$  and  $n \in R$

$$X O F = E \text{ and } F O X = E$$

$$(am, bm + n) = (1, 0) \text{ and } (ma, na + b) = (1, 0)$$

$$\text{Considering } (am, bm + n) = (1, 0)$$

$$am = 1$$

$$m = 1/a$$

$$\text{And } bm + n = 0$$

$$n = -b/a \text{ [since } m = 1/a\text{]}$$

$$\text{Considering } (ma, na + b) = (1, 0)$$

$$ma = 1$$

$$m = 1/a$$

$$\text{And } na + b = 0$$

$$n = -b/a$$

Therefore the inverse of  $(a, b) \in A$  with respect to  $O$  is  $(1/a, -b/a)$