Chapter 1
Real numbers

Exercise 1.3

1. **Prove that $\sqrt{5}$ is irrational.**

Solution:

Let $\sqrt{5}$ is a rational number.

we can find two co-prime numbers p, q (q ≠ 0) such that $\sqrt{5} = \frac{p}{q}$

Let 'p' and 'q' have a common factor other than 1.

$p = \sqrt{5q}$

$p^2 = 5\,q^2$

Therefore, p^2 is divisible by 5 and it can be said that 'p' is divisible by 5.

Let p = 5k, where k is an integer

$(5k)^2 = 5q^2$ this mean that q^2 is divisible by 5 and hence, q is divisible by 5.

$q^2 = 5k^2$ this implies that p and q have 5 as a common factor.

And this is a contradiction to the fact that p and q are co-prime.

Hence, $\sqrt{5}$ cannot be expressed as $\frac{p}{q}$ or it can be said that $\sqrt{5}$ is irrational.

2. **Prove that $3 + 2\sqrt{5}$ is irrational.**

Solution:

Let $3 + 2\sqrt{5}$ is rational.

Therefore,

We can find two integers p & q where, (q ≠ 0) such that

$$3 + 2\sqrt{5} = \frac{p}{q}$$

$$2\sqrt{5} = \frac{p}{q} - 3$$

Since p and q are integers, $\frac{1}{2}\left(\frac{p}{q} - 3\right)$ will also be rational and therefore, $\sqrt{5}$ is rational.

This contradicts the fact that $\sqrt{5}$ is irrational.

Therefore, $3 + 2\sqrt{5}$ is irrational.

3. Prove that the following are irrationals:

(i) $\frac{1}{\sqrt{2}}$

Solution:

Let $\frac{1}{\sqrt{2}}$ is rational.

Therefore, we can find two integers p & q where, q $\neq 0$ such that

$$\frac{1}{\sqrt{2}} = \frac{p}{q}$$

$$\sqrt{2} = \frac{q}{p}$$

$\frac{q}{p}$ is rational as p and q are integers.

Therefore, $\sqrt{2}$ is rational which contradicts to the fact that $\sqrt{2}$ is irrational.

Hence, our assumption is false and $\frac{1}{\sqrt{2}}$ is irrational.

(ii) $7\sqrt{5}$

Solution:

Let $7\sqrt{5}$ is rational.

Therefore, we can find two integers p & q where, q$\neq$0 such that

$7\sqrt{5} = \frac{p}{q}$ for some integers p and q

$$\therefore \sqrt{5} = \frac{p}{7q}$$

$\frac{p}{7q}$ is rational as p and q are integers.

Therefore, $\sqrt{5}$ should be rational.

This contradicts the fact that $\sqrt{5}$ is irrational.

Therefore our assumption that $7\sqrt{5}$ is rational is false.

Hence, $7\sqrt{5}$ is irrational.

(iii) $6+\sqrt{2}$

Solution:

Let $6+\sqrt{2}$ be rational.

Therefore, we can find two integers p & q where q≠0, such that

$$6+\sqrt{2}=\frac{p}{q}$$

$$\sqrt{2}=\frac{p}{q}-6$$

Since p and q are integers, $\frac{p}{q}-6$ is also rational

Hence, $\sqrt{2}$ should be rational, this contradicts the fact that $\sqrt{2}$ is irrational.

So, our assumption is false and hence, $6+\sqrt{2}$ is irrational.