EXERCISE

SHORT ANSWER TYPE:

1. The first term of an A.P.is a, and the sum of the first p terms is zero, show that the sum of its next q terms is $\frac{-a(p+q)q}{p-1}$.

[Hint: Required sum = $S_{p+q} - S_p$]

Solution:

Given first term is 'a' and sum of first p terms is $S_p = 0$

Now we have to find the sum of next q terms

Therefore, total terms are p + q

Hence, sum of all terms minus the sum of the first p terms will give the sum of next q terms

But sum of first p terms is 0 hence sum of next q terms will be the same as sum of all terms

So, we have to find sum of p + q terms

Sum of n terms of AP is given by $S_n = \left(\frac{n}{2}\right)(2a + (n-1)d)$

Where a is first term and d is the common difference

Using the given hint we get

⇒ required sum = $S_{p+q} - S_p$

Using S_n formula

$$\Rightarrow \text{required sum} = \frac{p+q}{2}(2a + (p+q-1)d) \quad \text{......1}$$

Now we have to find d

We use the given $S_p = 0$ to find d

$$\Rightarrow S_p = \frac{p}{2}(2a + (p-1)d)$$

⇒ 0 = 2a + (p – 1) d

$$\Rightarrow d = -\frac{2a}{p-1}$$

Put this value of d in 1 we get

$$\Rightarrow \text{required sum} = \frac{p+q}{2}\left(2a + (p+q-1)\left(-\frac{2a}{p-1}\right)\right)$$

Taking LCM and simplifying we get

$$= \frac{p+q}{2}\left(2a - \frac{2ap}{p-1} - \frac{2aq}{p-1} + \frac{2a}{p-1}\right)$$

On computing we get

$$= a(p+q)\left(1 - \frac{p}{p-1} - \frac{q}{p-1} + \frac{1}{p-1}\right)$$

$$= a(p+q)\left(1 + \frac{-(p-1)}{p-1} - \frac{q}{p-1}\right)$$

$$= a(p+q)\left(1 + (-1) - \frac{q}{p-1}\right)$$

$$= -\frac{a(p+q)q}{p-1}$$

Hence proved.

2. A man saved Rs 66000 in 20 years. In each succeeding year after the first year he saved Rs 200 more than what he saved in the previous year. How much did he save in the first year?

Solution:

Given the total amount saved in 20 years is 66000 hence S_{20} = 66000

Let the amount saved in first year be 'a'

Then as he increases 200 Rs every year then the amount in second year will be 'a + 200' then in third year 'a + 400' and so on

The sequence will be a, a + 200, a + 400…

The sequence is AP and the common difference is d = 200

There are 20 terms in the sequence as he saved money for 20 years

Using the sum formula for AP S_n = n/2 (2a + (n – 1) d) we get

Where n is number of terms a is the first term and d is the common difference

Given that S_{20} = 66000

S_{20} = (20/2) (2a + (20 – 1) 200)

⇒ 66000 = 10(2a + 19 × 200)

⇒ 6600 = 2a + 3800

⇒ 6600 – 3800 = 2a

⇒ 2a = 2800

⇒ a = 1400

Hence amount saved in first year is 1400 Rs.

3. A man accepts a position with an initial salary of Rs 5200 per month. It is understood that he will receive an automatic increase of Rs 320 in the very next month and each month thereafter.
(a) Find his salary for the tenth month
(b) What is his total earnings during the first year?

Solution:
Given the man's salary in first month is ₹ 5200 and then it increases every month by ₹ 320
Hence the sequence of his salary per month will be 5200, 5200 + 320, 5200 + 640…
The sequence is in AP with first term as a = 5200,
Common difference as d = 320
a) Now we have to find salary in 10^{th} month that is 10^{th} term of the AP
The n^{th} term of AP is given by $t_n = a + (n – 1) d$
Where a is the first term and d is the common difference
We have to find t_{10}
$\Rightarrow t_{10} = 5200 + (10 – 1) (320)$
$\Rightarrow t_{10} = 5200 + 2880$
$\Rightarrow t_{10} = 8080$
Hence salary in 10^{th} month is ₹ 8080

b) To get the total earnings in first year we have to add first 12 terms of the sequence that is we have to find S_{12}
The sum of first n terms of AP is given by $S_n = n/2 (2a + (n – 1) d)$
Where a is the first term and d is common difference
$S_{12} = (12/2) (2 (5200) + (12 – 1) 320)$
$\Rightarrow S_{12} = 6(10400 + 11(320))$
$\Rightarrow S_{12} = 6(10400 + 3520)$
$\Rightarrow S_{12} = 6(13920)$
$\Rightarrow S_{12} = 83520$
Hence his total earnings in first year is ₹ 83520

4. If the p^{th} and q^{th} terms of a G.P. are q and p respectively, show that its $(p + q)^{th}$ term is
$$\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}.$$

Solution:

The n^{th} term of GP is given by $t_n = ar^{n-1}$ where a is the first term and r is the common difference

p^{th} term is given as q

$\Rightarrow t_p = ar^{p-1}$

The above equation can be written as

$\Rightarrow q = ar^{p-1}$

$$\Rightarrow q = \frac{ar^p}{r}$$

On rearranging the above equation we get

$$\Rightarrow \frac{a}{r} = \frac{q}{r^p} \ldots (a)$$

q^{th} term is given as p

$\Rightarrow t_q = ar^{q-1}$

$\Rightarrow p = ar^{q-1}$

The above equation can be written as

$$\Rightarrow p = \frac{ar^q}{r}$$

On rearranging the above equation we get

$$\Rightarrow \frac{a}{r} = \frac{p}{r^q} \ldots (b)$$

From equation (a) and (b) we have

$$\Rightarrow \frac{q}{r^p} = \frac{p}{r^q}$$

On rearranging we get

$$\Rightarrow \frac{q}{p} = \frac{r^p}{r^q}$$

$$\Rightarrow r^{p-q} = \frac{q}{p}$$

$$\Rightarrow r = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$$

$(p + q)^{th}$ term is given by

$\Rightarrow t_{p+q} = a\, r^{p+q-1}$

$\Rightarrow t_{p+q} = (ar^{p-1})\, r^q$

But $t_p = ar^{p-1}$ and the p^{th} term is q

$\Rightarrow t_{p+q} = q\, r^q$

But

$$r = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$$

$$\Rightarrow t_{p+q} = q\left(\left(\frac{q}{p}\right)^{\frac{1}{p-q}}\right)^{q}$$

Using laws of exponents we get

$$= q\left(\frac{q^{\frac{1}{p-q}}}{p^{\frac{1}{p-q}}}\right)^{q}$$

$$= q\left(\frac{q^{\frac{q}{p-q}}}{p^{\frac{q}{p-q}}}\right)$$

On rearranging

$$= \frac{q^{\frac{q}{p-q}+1}}{p^{q\left(\frac{1}{p-q}\right)}}$$

Taking LCM and simplifying we get

$$= \frac{q^{\frac{q+p-q}{p-q}}}{p^{q\left(\frac{1}{p-q}\right)}}$$

$$= \frac{q^{p\left(\frac{1}{p-q}\right)}}{p^{q\left(\frac{1}{p-q}\right)}}$$

$$\Rightarrow t_{p+q} = \left(\frac{q^{p}}{p^{q}}\right)^{\left(\frac{1}{p-q}\right)}$$

Hence the proof.

5. A carpenter was hired to build 192 window frames. The first day he made five frames and each day, thereafter he made two more frames than he made the day before. How many days did it take him to finish the job?

Solution:

Given first day he made 5 frames then two frames more than the previous that is 7 then 9 and so on

Hence the sequence of making frames each day is 5, 7, 9…
The sequence is AP with first term as a = 5 and common difference d = 2
Total number of frames to be made is 192
Let is requires n days hence $S_n = 192$
The sum of first n terms of AP is given by $S_n = n/2\ (2a + (n – 1)\ d)$
Where a is the first term and d is common difference
$S_n = (n/2)\ (2(5) + (n – 1)\ 2)$
$192 = (n/2)\ (10 + 2n – 2)$
$\Rightarrow 384 = 10n + 2n^2 – 2n$
On computing and simplifying we get
$\Rightarrow 2n^2 + 8n – 384 = 0$
$\Rightarrow n^2 + 4n – 192 = 0$
$\Rightarrow n^2 + 16n – 12n – 192 = 0$
$\Rightarrow n\ (n + 16) – 12\ (n + 16) = 0$
$\Rightarrow (n – 12)\ (n + 16) = 0$
$\Rightarrow n = 12$ and $n = -16$
But n represents number of days which cannot be negative hence n = 12
Hence number of days required to finish the job is 12 days.

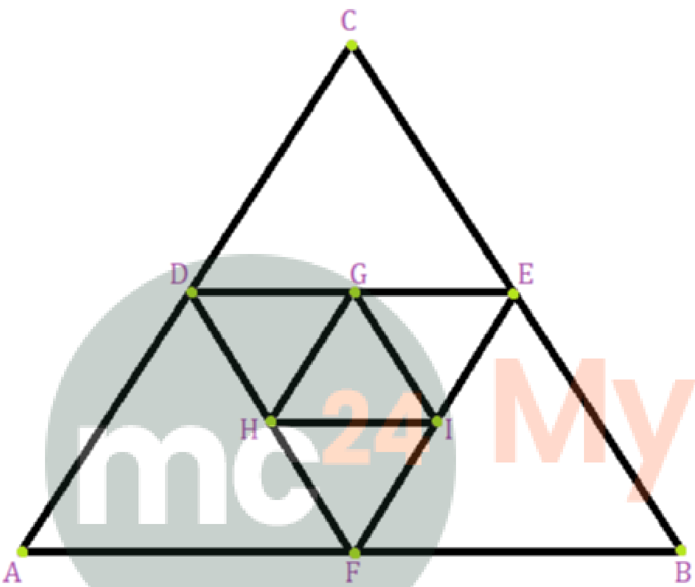
6. We know the sum of the interior angles of a triangle is 180°. Show that the sums of the interior angles of polygons with 3, 4, 5, 6, ... sides form an arithmetic progression. Find the sum of the interior angles for a 21-sided polygon.

Solution:
Given the sum of interior angles of a polygon having 'n' sides is given by (n – 2) × 180°
Sum of angles with three sides that is n = 3 is (3 – 2) × 180° = 180°
Sum of angles with four sides that is n = 4 is (4 – 2) × 180° = 360°
Sum of angles with five sides that is n = 5 is (5 – 2) × 180° = 540°
Sum of angles with six sides that is n = 6 is (6 – 2) × 180° = 720°
As seen as the number of sides increases by 1 the sum of interior angles increases by 180°
Hence the sequence of sum of angles as number of sides' increases is 180°, 360°, 540°, 720° …
The sequence is AP with first term as a = 180° and common difference as d = 180°
We have to find sum of angles of polygon with 21 sides
Using (n – 2) × 180°
⇒ Sum of angles of polygon having 21 sides = (21 – 2) × 180°

⇒ Sum of angles of polygon having 21 sides = $19 \times 180°$
⇒ Sum of angles of polygon having 21 sides = $3420°$

7. A side of an equilateral triangle is 20cm long. A second equilateral triangle is inscribed in it by joining the mid points of the sides of the first triangle. The process is continued as shown in the accompanying diagram. Find the perimeter of the sixth inscribed equilateral triangle.

Solution:



Let ABC be the triangle with AB = BC = AC = 20 cm
Let D, E and F be midpoints of AC, CB and AB respectively which are joined to form an equilateral triangle DEF
Now we have to find the length of side of ΔDEF
Consider ΔCDE
CD = CE = 10 cm … D and E are midpoints of AC and CB
Hence ΔCDE is isosceles
⇒ ∠CDE = ∠CED … base angles of isosceles triangle
But ∠DCE = 60° …∠ABC is equilateral
Hence ∠CDE = ∠CED = 60°
Hence ΔCDE is equilateral
Hence DE = 10 cm
Similarly, we can show that GH = 5 cm
Hence the series of sides of equilateral triangle will be 20, 10, 5 …
The series is GP with first term a = 20 and common ratio r = ½

To find the perimeter of 6th triangle inscribed we first have to find the side of 6th triangle that is the 6th term in the series
nth term in GP is given by $t_n = ar^{n-1}$
⇒ $t_6 = (20)\ (1/2)^{6-1}$
⇒ $t_6 = 20/\ 2^5$
$= 20/\ (4 \times 2^3)$
⇒ $t_6 = 5/8$
Hence the side of 6th equilateral triangle is 5/8 cm and hence its perimeter would be thrice its side length because it's an equilateral triangle
Perimeter of 6th equilateral triangle inscribed is 3 × 5/8 = 15/8 cm

8. In a potato race 20 potatoes are placed in a line at intervals of 4 metres with the first potato 24 metres from the starting point. A contestant is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing back all the potatoes?

Solution:
Given at start he has to run 24m to get the first potato then 28 m as the next potato is 4m away from first and so on
Hence the sequence of his running will be 24, 28, 32 …
There are 20 terms in sequence as there are 20 potatoes
Hence only to get potatoes from starting point he has to run
24 + 28 + 32 + … up to 20 terms
This is only from starting point to potato but he has to get the potato back to starting point hence the total distance will be twice that is
Total distance ran = 2 × (24 + 28 + 32…) … 1
Let us find the sum using he formula to find sum of n terms of AP
That is $S_n = n/2\ (2a + (n - 1)\ d)$
There are 20 terms hence n = 20
⇒ $S_{20} = (20/2)\ (2\ (24) + (20 - 1)\ 4)$
On simplification we get
⇒ $S_{20} = 10(48 + 19(4))$
⇒ $S_{20} = 10(48 + 76)$
On computing we get
⇒ $S_{20} = 10 \times 124$
⇒ $S_{20} = 1240$ m
Using equation 1

⇒ Total distance ran = 2 × 1240
⇒ Total distance ran = 2480 m
Hence total distance he has to run is 2480 m

9. In a cricket tournament 16 school teams participated. A sum of Rs 8000 is to be awarded among themselves as prize money. If the last placed team is awarded Rs 275 in prize money and the award increases by the same amount for successive finishing places, how much amount will the first place team receive?

Solution:
Let the amount received by first place team be a Rs and d be difference in amount
As the difference is same hence the second-place team will receive a – d and the third place a – 2d and so on
The last team receives 275 Rs
As there are 16 teams and all teams are given prizes hence the sequence will have 16 terms because there are 16 teams
Therefore a, a – d, a – 2d… 275
The sequence is in AP with first term as a and common difference is '-d'
As the total prize given is of 8000rs hence
a + a – d + a – 2d … + 275 = 8000
The sum of n terms of AP is given by $S_n = n/2\ (2a + (n – 1)\ d)$ where
A is first term and d is common difference
There are 16 terms n = 16 and the sum S_n = 8000
⇒ 8000 = (16/2) (2a + (16 – 1) (-d))
⇒ 8000 = 8(2a – 15d)
⇒ 1000 = 2a – 15d ……1
The last term of AP is 275 and n^{th} term of AP is $t_n = a + (n – 1)\ d$
The last term is t_n = 275
⇒ 275 = a + (16 – 1) (-d)
⇒ 275 = a – 15d ……2
Subtract 2 from 1 that is 1 – 2
⇒ 1000 – 275 = 2a – 15d – a + 15d
⇒ 725 = a
Hence amount received by first place team is Rs.725

10. If $a_1, a_2, a_3, ..., a_n$ are in A.P., where $a_i > 0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution:

Given $a_1, a_2, a_3, \ldots, a_n$ are in A.P., where $a_i > 0$ for all i

To prove that:

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

$$\Rightarrow \text{LHS}=\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

Multiplying the first term by $\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}-\sqrt{a_2}}$, the second term by $\frac{\sqrt{a_2}-\sqrt{a_3}}{\sqrt{a_2}-\sqrt{a_3}}$ and so on that is rationalizing each term

$$\Rightarrow \text{LHS}=\frac{1}{\sqrt{a_1}+\sqrt{a_2}}\times\frac{\sqrt{a_1}-\sqrt{a_2}}{\sqrt{a_1}-\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}\times\frac{\sqrt{a_2}-\sqrt{a_3}}{\sqrt{a_2}-\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}\times\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{\sqrt{a_{n-1}}-\sqrt{a_n}}$$

Now by using $(a + b)(a - b) = a^2 - b^2$

$$\Rightarrow \text{LHS}=\frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2}+\frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3}+\cdots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$$

As $a_1, a_2, a_3, \ldots, a_n$ are in AP let its common difference be d

$a_2 - a_1 = d$, $a_3 - a_2 = d$ … $a_n - a_{n-1} = d$

Hence multiplying by -1

$a_1 - a_2 = -d$, $a_2 - a_3 = -d$ … $a_n - a_{n-1} = -d$

Put these values in LHS

$$\Rightarrow \text{LHS}=\frac{\sqrt{a_1}-\sqrt{a_2}}{-d}+\frac{\sqrt{a_2}-\sqrt{a_3}}{-d}+\cdots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$$

$$=-\frac{1}{d}\left(\sqrt{a_1}-\sqrt{a_2}+\sqrt{a_2}-\sqrt{a_3}+\cdots+\sqrt{a_{n-1}}-\sqrt{a_n}\right)$$

$$=-\frac{1}{d}\left(\sqrt{a_1}-\sqrt{a_n}\right)$$

To the above equation multiply and divide by $\left(\sqrt{a_1}+\sqrt{a_n}\right)$

$$\Rightarrow \text{LHS} = -\frac{1}{d}\frac{(\sqrt{a_1}-\sqrt{a_n})(\sqrt{a_1}+\sqrt{a_n})}{(\sqrt{a_1}+\sqrt{a_n})}$$

Using $(a + b)(a - b) = a^2 - b^2$

$$\Rightarrow \text{LHS} = -\frac{1}{d}\frac{a_1 - a_n}{(\sqrt{a_1}+\sqrt{a_n})}$$

The n^{th} term of AP is given by $t_n = a + (n - 1)d$

Where the $t_n = a_n$ is the last n^{th} term and $a = a_1$ is the first term

Hence $a_n = a_1 + (n - 1)d$

$\Rightarrow a_1 - a_n = -(n - 1)d$

Substitute $a_1 - a_n$ in LHS

$$\Rightarrow \text{LHS} = -\frac{1}{d}\frac{-(n-1)d}{(\sqrt{a_1}+\sqrt{a_n})}$$

$$= \frac{(n-1)}{(\sqrt{a_1}+\sqrt{a_n})}$$

$\Rightarrow$ LHS = RHS

Hence proved

11. Find the sum of the series $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \ldots$ to
(i) n terms
(ii) 10 terms

Solution:

Given $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \ldots$

Let the series be $S = (3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \ldots$

i) Generalizing the series in terms of i

$$S = \sum_{i=1}^{n}[(2i+1)^3 - (2i)^3]$$

Using the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$$\Rightarrow S = \sum_{i=1}^{n}(2i+1-2i)((2i+1)^2 + (2i+1)(2i) + (2i)^2)$$

$$\Rightarrow S = \sum_{i=1}^{n}(4i^2 + 4i + 1 + 4i^2 + 2i + 4i^2)$$

On simplifying and computing we get

$$\Rightarrow S = \sum_{i=1}^{n}(12i^2 + 6i + 1)$$

Now by splitting the summation we get

$$\Rightarrow S = 12\sum_{i=1}^{n} i^2 + 6\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

We know that $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum n = \frac{n(n+1)}{2}$

Using the above formula we get

$$\Rightarrow S = 12\frac{n(n+1)(2n+1)}{6} + 6\frac{n(n+1)}{2} + n$$

Simplifying we get

$\Rightarrow S = 2n (n+1) (2n+1) + 3n (n+1) + n$

$\Rightarrow S = 2n (2n^2 + 2n + n + 1) + 3n^2 + 3n + n$

$\Rightarrow S = 4n^3 + 6n^2 + 2n + 3n^2 + 4n$

$\Rightarrow S = 4n^3 + 9n^2 + 6n$

Hence sum up to n terms is $4n^3 + 9n^2 + 6n$

ii) Sum of first 10 terms or up to 10 terms

To find sum up to 10 terms put n = 10 in S

$\Rightarrow S = 4 (10)^3 + 9 (10)^2 + 6 (10)$

$\Rightarrow S = 4000 + 900 + 60$

$\Rightarrow S = 4960$

Hence sum of series up to 10 terms is 4960

12. Find the r^{th} term of an A.P. sum of whose first n terms is $2n + 3n^2$.
[Hint: $a_n = S_n – S_{n–1}$]

Solution:

Sum of first n terms be S_n given as $S_n = 2n + 3n^2$

We have to find the r^{th} term that is a_r

Using the given hint n^{th} term is given as $a_n = S_n – S_{n-1}$

$\Rightarrow a_r = S_r – S_{r-1}$

Using $S_n = 2n + 3n^2$

$\Rightarrow a_r = 2r + 3r^2 – (2(r – 1) + 3(r – 1)^2)$

$\Rightarrow a_r = 2r + 3r^2 – (2r – 2 + 3(r^2 – 2r + 1))$

$\Rightarrow a_r = 2r + 3r^2 – (2r – 2 + 3r^2 – 6r + 3) \Rightarrow a_r = 6r – 1$

Hence the r^{th} term is 6r – 1

LONG ANSWER TYPE

13. If A is the arithmetic mean and G_1, G_2 be two geometric means between any two numbers, then prove that

$$2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}.$$

Solution:

Given A is the arithmetic mean and G_1, G_2 be two geometric means between any two numbers

Let the two numbers be ‘a’ and ‘b’

The arithmetic mean is given by $A = \frac{a+b}{2}$ and the geometric mean is given by $G = \sqrt{ab}$

We have to insert two geometric means between a and b

Now that we have the terms a, G_1, G_2, b

G_1 will be the geometric mean of a and G_2 and G_2 will be the geometric mean of G_1 and b

Hence $G_1 = \sqrt{aG_2}$ and $G_2 = \sqrt{G_1 b}$

Square $G_1 = \sqrt{aG_2}$

$\Rightarrow G_1^2 = aG_2$

Put $G_2 = \sqrt{G_1 b}$

$\Rightarrow G_1^2 = a\sqrt{G_1 b}$

Squaring on both sides we get

$\Rightarrow G_1^4 = a^2 (G_1 b)$

$\Rightarrow G_1^3 = a^2 b$

$\Rightarrow G_1 = a^{\frac{2}{3}} b^{\frac{1}{3}}$ 1

Put value of G_1 in $G_2 = \sqrt{G_1 b}$

$\Rightarrow G_2 = \sqrt{a^{\frac{2}{3}} b^{\frac{1}{3}} b}$

$= \left(a^{\frac{2}{3}} b^{\frac{1}{3}+1}\right)^{\frac{1}{2}}$

On simplification we get

$= \left(a^{\frac{2}{3}} b^{\frac{4}{3}}\right)^{\frac{1}{2}}$

$= a^{\frac{1}{3}} b^{\frac{2}{3}}$............2

Now we have to prove that $2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$

Consider RHS

$\Rightarrow \text{RHS} = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$

Substitute values of G_1 and G_2 from 1 and 2

$\Rightarrow \text{RHS} = \frac{\left(a^{\frac{2}{3}} b^{\frac{1}{3}}\right)^2}{a^{\frac{1}{3}} b^{\frac{2}{3}}} + \frac{\left(a^{\frac{1}{3}} b^{\frac{2}{3}}\right)^2}{a^{\frac{2}{3}} b^{\frac{1}{3}}}$

$= \frac{a^{\frac{4}{3}} b^{\frac{2}{3}}}{a^{\frac{1}{3}} b^{\frac{2}{3}}} + \frac{a^{\frac{2}{3}} b^{\frac{4}{3}}}{a^{\frac{2}{3}} b^{\frac{1}{3}}}$

Taking LCM and simplifying we get

$= a^{\frac{4}{3}-\frac{1}{3}} b^{\frac{2}{3}-\frac{2}{3}} + a^{\frac{2}{3}-\frac{2}{3}} b^{\frac{4}{3}-\frac{1}{3}}$

$\Rightarrow$ RHS = a + b

Divide and multiply by 2

$\Rightarrow \text{RHS} = 2\frac{a+b}{2}$

But $A = \frac{a+b}{2}$

Therefore

$\Rightarrow$ RHS = 2A

Hence RHS = LHS

Hence proved

14. If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in A.P., whose common difference is d, show that Sec θ_1 sec θ_2 + sec θ_2 sec θ_3 + ... + sec θ_{n-1} sec θ_n

$$= \frac{\tan\theta_n - \tan\theta_1}{\sin d}.$$

Solution:

Given $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in A.P., and common difference is d,

Now we have to prove that

Sec θ_1 sec θ_2 + sec θ_2 sec θ_3 + ... + sec θ_{n-1} sec θ_n = $\frac{\tan\theta_n - \tan\theta_1}{\sin d}$

On cross multiplication we get

⇒ Sin d (secθ_1 secθ_2 + secθ_2 secθ_3 + ... + sec θ_{n-1} sec θ_n) = tan θ_n – tan θ_1

We know sec x = 1/cos x using this formula we get

$$\Rightarrow \frac{\sin d}{\cos\theta_1\cos\theta_2} + \frac{\sin d}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin d}{\cos\theta_{n-1}\cos\theta_n} = \tan\theta_n - \tan\theta_1$$

Consider LHS

$$\Rightarrow \text{LHS} = \frac{\sin d}{\cos\theta_1\cos\theta_2} + \frac{\sin d}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin d}{\cos\theta_{n-1}\cos\theta_n}$$

Now we have to find value of d in terms of θ so that further simplification can be made

As $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP having common difference as d

Hence

$\theta_2 - \theta_1 = d, \theta_3 - \theta_2 = d, \ldots, \theta_n - \theta_{n-1} = d$

Take sin on both sides

Sin $(\theta_2 - \theta_1)$ = sin d, sin $(\theta_3 - \theta_2)$ = sin d, ..., sin $(\theta_n - \theta_{n-1})$ = sin d

Substitute appropriate value of sin d for each term in LHS

$$\Rightarrow \text{LHS} = \frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1\cos\theta_2} + \frac{\sin(\theta_3 - \theta_2)}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin(\theta_n - \theta_{n-1})}{\cos\theta_{n-1}\cos\theta_n}$$

We know that sin (a – b) = sin a cos b – cos a sin b

Using this formula we get

$$\Rightarrow \text{LHS} = \frac{\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1}{\cos\theta_1\cos\theta_2} + \frac{\sin\theta_3\cos\theta_2 - \cos\theta_3\sin\theta_2}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin\theta_n\cos\theta_{n-1} - \cos\theta_n\sin\theta_{n-1}}{\cos\theta_{n-1}\cos\theta_n}$$

On simplifying we get

$$= \frac{\sin\theta_2\cos\theta_1}{\cos\theta_1\cos\theta_2} - \frac{\cos\theta_2\sin\theta_1}{\cos\theta_1\cos\theta_2} + \frac{\sin\theta_3\cos\theta_2}{\cos\theta_2\cos\theta_3} - \frac{\cos\theta_3\sin\theta_2}{\cos\theta_2\cos\theta_3} + \cdots + \frac{\sin\theta_n\cos\theta_{n-1}}{\cos\theta_{n-1}\cos\theta_n} - \frac{\cos\theta_n\sin\theta_{n-1}}{\cos\theta_{n-1}\cos\theta_n}$$

We know that sin x/cos x = tan x

$= \tan\theta_2 - \tan\theta_1 + \tan\theta_3 - \tan\theta_2 + \ldots + \tan\theta_n - \tan\theta_{n-1}$

$= -\tan\theta_1 + \tan\theta_n$

$= \tan\theta_n - \tan\theta_1$

⇒ LHS = RHS

Hence proved

15. If the sum of p terms of an A.P. is q and the sum of q terms is p, show that the sum of p + q terms is – (p + q). Also, find the sum of first p – q terms (p > q).

Solution:

The sum of n terms of an AP is given by

$S_n = \frac{n}{2}(2a + (n-1)d)$

Where a is the first term and d is the common difference

Given that $S_p = q$ and $S_q = p$

$\Rightarrow S_p = \frac{p}{2}(2a + (p-1)d)$ ….1

We know $S_p = q$

$\Rightarrow q = \frac{p}{2}(2a + (p-1)d)$

On rearranging we get

$\Rightarrow \frac{2q}{p} = 2a + (p-1)d \ldots (i)$

$\Rightarrow S_q = \frac{q}{2}(2a + (q-1)d)$ …….2

Again we have $S_q = p$

$\Rightarrow p = \frac{q}{2}(2a + (q-1)d)$

On rearranging we get

$\Rightarrow \frac{2p}{q} = 2a + (q-1)d \ldots (ii)$

Subtract (i) from (ii) that is (ii) – (i)

$$\Rightarrow \frac{2p}{q} - \frac{2q}{p} = (q-1)d - (p-1)d$$

On simplifying we get

$$\Rightarrow 2\frac{p^2 - q^2}{pq} = (q - 1 - p + 1)d$$

Using $a^2 - b^2 = (a + b)(a - b)$ formula we get

$$\Rightarrow 2\frac{(p+q)(p-q)}{pq} = (q-p)d$$

Computing and simplifying we get

$$\Rightarrow 2\frac{-(p+q)(q-p)}{pq} = (q-p)d$$

$$\Rightarrow -2\frac{(p+q)}{pq} = d \ldots (iii)$$

We have to show that $S_{p+q} = -(p + q)$

$$S_{p+q} = \frac{p+q}{2}(2a + (p+q-1)d)$$

Above equation can be written as

$$= \frac{p}{2}(2a + (p+q-1)d) + \frac{q}{2}(2a + (p+q-1)d)$$

$$= \frac{p}{2}(2a + (p-1)d + qd) + \frac{q}{2}(2a + (q-1)d + pd)$$

$$= \frac{p}{2}(2a + (p-1)d) + \frac{pqd}{2} + \frac{q}{2}(2a + (q-1)d) + \frac{qpd}{2}$$

Using (m) and (n)

$\Rightarrow S_{p+q} = S_p + S_q + p\,q\,d$

$= q + p + p\,q\,d$

Substitute d from (iii)

$$\Rightarrow S_{p+q} = q + p + pq\left(-2\frac{(p+q)}{pq}\right)$$

$= (p + q) - 2(p + q)$

$= -(p + q)$

Now we have to find sum of p – q terms that is S_{p-q}

$$\Rightarrow S_{p-q} = \frac{p-q}{2}(2a + (p-q-1)d)$$

$$= \frac{p}{2}(2a + (p - q - 1)d) - \frac{q}{2}(2a + (p - q - 1)d)$$

The above equation can be written as

$$= \frac{p}{2}(2a + (p - 1)d - qd) - \frac{q}{2}(2a + (p - 1)d - qd)$$

$$= \frac{p}{2}(2a + (p - 1)d) - \frac{pqd}{2} - \frac{q}{2}(2a + (p - 1)d) + \frac{q^2 d}{2}$$

Using (m) and (n)

$$= S_p - \frac{pqd}{2} - \frac{q}{2}\frac{2S_p}{p} + \frac{q^2 d}{2}$$

Substituting the value of S_p = q we get

$$= q - \frac{pqd}{2} - \frac{q^2}{p} + \frac{q^2 d}{2}$$

$$= q - \frac{q^2}{p} + \left(\frac{q^2 - qp}{2}\right)d$$

Substitute d from (iii)

$$= q - \frac{q^2}{p} + \left(\frac{q^2 - qp}{2}\right)\left(-2\frac{(p + q)}{pq}\right)$$

Simplifying and computing we get

$$= \frac{qp - q^2}{p} - (q^2 - qp)\left(\frac{p + q}{pq}\right)$$

$$= \frac{qp - q^2}{p} + (qp - q^2)\left(\frac{p + q}{pq}\right)$$

$$= \frac{qp - q^2}{p} + (qp - q^2)\left(\frac{1}{p} + \frac{1}{q}\right)$$

$$= \frac{qp - q^2}{p} + \frac{(qp - q^2)}{p} + \frac{(qp - q^2)}{q}$$

$$\Rightarrow S_{p-q} = 2\frac{q(p - q)}{p} + p - q$$

Hence sum of p – q terms is $2\frac{q(p-q)}{p} + p - q$

16. If p^{th}, q^{th}, and r^{th} terms of an A.P. and G.P. are both a, b and c respectively, show that

$a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$

Solution:

Let the first term of AP be m and common difference as d

Let the GP first term as l and common ratio as s

The n^{th} term of an AP is given as $t_n = a + (n - 1)d$ where a is the first term and d is the common difference

The n^{th} term of a GP is given by $t_n = ar^{n-1}$ where a is the first term and r is the common ratio

The p^{th} term (t_p) of both AP and GP is a

For AP

$\Rightarrow t_p = m + (p - 1)d$

$\Rightarrow a = m + (p - 1)d$ 1

For GP

$\Rightarrow t_p = ls^{p-1}$

$\Rightarrow a = ls^{p-1}$ 2

The q^{th} term (t_q) of both AP and GP is b

For AP

$\Rightarrow t_q = m + (q - 1)d$

$\Rightarrow b = m + (q - 1)d$ 3

For GP

$\Rightarrow t_q = ls^{q-1}$

$\Rightarrow b = ls^{q-1}$ 4

The r^{th} term (t_r) of both AP and GP is c

For AP

$\Rightarrow t_r = m + (r - 1)d$

$\Rightarrow c = m + (r - 1)d$ 5

For GP

$\Rightarrow t_r = ls^{r-1}$

$\Rightarrow c = ls^{r-1}$ 6

Let us find b – c, c – a and a – b

Using 3 and 5

$\Rightarrow b - c = (q - r)d$ ... (i)

Using 5 and 1

⇒ c – a = (r – p) d … (ii)

Using 1 and 3

⇒ a – b = (p – q) d … (iii)

We have to prove that $a^{b-c}.b^{c-a}.c^{a-b} = 1$

LHS = $a^{b-c}.b^{c-a}.c^{a-b}$)

Using 2, 4 and 6

⇒ LHS = $(ls^{p-1})^{b-c}.(ls^{q-1})^{c-a}.(ls^{r-1})^{a-b}$

$$= \left(\frac{ls^p}{s}\right)^{b-c} . \left(\frac{ls^q}{s}\right)^{c-a} . \left(\frac{ls^r}{s}\right)^{a-b}$$

$$= \frac{l^{b-c}s^{p(b-c)}}{s^{b-c}} . \frac{l^{c-a}s^{q(c-a)}}{s^{c-a}} . \frac{l^{a-b}s^{r(a-b)}}{s^{a-b}}$$

$$= \frac{l^{b-c+c-a+a-b}}{s^{b-c+c-a+a-b}} . s^{p(b-c)} . s^{q(c-a)} . s^{r(a-b)}$$

$= s^{p(b-c)}.s^{q(c-a)}.s^{r(a-b)}$

Substituting values of a – b, c – a and b – c from (iii), (ii) and (i)

$= s^{p(q-r)d}.s^{q(r-p)d}.s^{r(p-q)d}$

$= s^{pqd-prd}.s^{qrd-pqd}.s^{prd-qrd}$

$= s^{pqd-prd+qrd-pqd+prd-qrd}$

$= s^0 = 1$

⇒ LHS = RHS

Hence proved

OBJECTIVE TYPE QUESTIONS

17. If the sum of n terms of an A.P. is given by Sn = 3n + 2n 2, then the common difference of the A.P. is

(A) 3 (B) 2 (C) 6 (D) 4

Solution:

(D) 4

Explanation:

To find: Common Difference of A.P that is d

Consider,

$S_n = 3n + 2n^2$

Putting n = 1, we get

$S_1 = 3(1) + 2(1)^2$
$= 3 + 2$
$S_1 = 5$
Putting n = 2, we get
$S_2 = 3(2) + 2(2)^2$
$= 6 + 2(4)$
$= 6 + 8$
$S_2 = 14$
Now, we know that,
$S_1 = a_1$
$\Rightarrow a_1 = 5$
And $a_2 = S_2 - S_1$
$= 14 - 5$
$= 9$
$\therefore$ Common Difference, $d = a_2 - a_1$
$= 9 - 5$
$= 4$
Hence, the correct option is (D)

18. The third term of G.P. is 4. The product of its first 5 terms is
(A) 4^3 (B) 4^4 (C) 4^5 (D) None of these

Solution:
(C) 45

Explanation:
Given third term of G.P, $T_3 = 4$
To find Product of first five terms
We know that,
$T_n = a\, r^{n-1}$
It is given that, $T_3 = 4$
$\Rightarrow ar^{3-1} = 4$
$\Rightarrow ar^2 = 4$ … (i)
Product of first 5 terms = $a \times ar \times ar^2 \times ar^3 \times ar^4$
$= a^5 r^{1+2+3+4}$
$= a^5 r^{10}$
$= (ar^2)^5$

$= (4)^5$ [from (i)]

Hence, the correct option is (C)