EXERCISE 19.14

Evaluate the following integrals:

1. $\int \frac{1}{a^2 - b^2x^2} dx$

Solution:

Taking out b^2 as common from the given equation, we get

$$\frac{1}{b^2}\int \frac{1}{\left(\frac{a^2}{b^2}\right)-x^2} dx$$

$$= \frac{1}{b^2}\int \frac{1}{\left(\frac{a^2}{b^2}\right)-x^2} dx = \frac{1}{b^2}\int \frac{1}{\left(\frac{a}{b}\right)^2-x^2} dx$$

On integrating above equation using

$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{x+a}{a-x}\right| + c$, we get

$$= \frac{1}{b^2} \times \frac{1}{2\left(\frac{a}{b}\right)}\log\left|\frac{\frac{a}{b}+x}{\frac{a}{b}-x}\right| + c$$

On simplification we get

$$= \frac{1}{2ab}\log\left|\frac{a+bx}{a-bx}\right| + c$$

2. $\int \frac{1}{a^2x^2 - b^2} dx$

Solution:

Taking out a^2 as common from the given equation, we get

$$= \frac{1}{a^2}\int \frac{1}{x^2-\frac{b^2}{a^2}} dx$$

On integrating above equation using

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

$= \frac{1}{a^2}\int \frac{1}{x^2-(\frac{b}{a})^2}\,dx = \frac{1}{a^2} * \frac{1}{2(\frac{b}{a})}\log\left|\frac{x-(\frac{b}{a})}{x+\frac{b}{a}}\right| + c$

On simplification

$= \frac{1}{2ab}\log\frac{ax-b}{ax+b} + c$

3. $\int \frac{1}{a^2x^2+b^2}\,dx$

Solution:

Taking out a^2 as common from the given equation, we get

$= \frac{1}{a^2}\int \frac{1}{x^2+\frac{b^2}{a^2}}\,dx$

On integrating above equation using

$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$, we get

$= \frac{1}{a^2}\int \frac{1}{x^2+(\frac{b}{a})^2}\,dx = \frac{1}{a^2} * \frac{1}{(\frac{b}{a})}\tan^{-1}[\frac{x}{\frac{b}{a}}] + c$

By simplifying we get

$= \frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right) + c$

4. $\int \frac{x^2-1}{x^2+4}\,dx$

Solution:

Add and subtract 4 in the numerator of given equation, we get

$= \int \frac{x^2+4-4-1}{x^2+4} = \int \frac{(x^2+4)-4-1}{x^2+4}\,dx$

Now separate the numerator terms, we get

$= \int \frac{(x^2+4)-5}{x^2+4}dx = \int \frac{(x^2+4)}{x^2+4}dx - \int \frac{5}{x^2+4}dx$

On computing we get

$= \int dx - \int \frac{5}{x^2+4}dx = \int dx - 5\int \frac{1}{x^2+4}dx$

We know $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$= \int dx - 5\int \frac{1}{x^2+2^2}dx = x - 5 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

On integrating we get

$= x - \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

5. $\int \frac{1}{\sqrt{1+4x^2}}\, dx$

Solution:

Let I = $\int \frac{1}{\sqrt{1+4x^2}}dx$

The above equation can be written as

$= \int \frac{1}{\sqrt{1+(2x)^2}}dx$

Let t = 2x, then dt = 2dx or dx = dt/2

Therefore,

$\int \frac{1}{\sqrt{1+(2x)^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$

We know $\int \frac{1}{\sqrt{(a^2+x^2)}}dx = \log|x + \sqrt{(a^2+x^2)}| + c$

$= \frac{1}{2}\log|t + \sqrt{1+t^2}| + c$

$= \frac{1}{2}\log|2x + \sqrt{1+4x^2}| + c$