

EXERCISE 9.3

Prove that:

1. $\sin^2 2\pi/5 - \sin^2 \pi/3 = (\sqrt{5} - 1)/8$

Solution:

Let us consider LHS:

$$\sin^2 2\pi/5 - \sin^2 \pi/3 = \sin^2 (\pi/2 - \pi/10) - \sin^2 \pi/3$$

we know, $\sin (90^\circ - A) = \cos A$

So, $\sin^2 (\pi/2 - \pi/10) = \cos^2 \pi/10$

$\sin \pi/3 = \sqrt{3}/2$

Then the above equation becomes,

$$= \cos^2 \pi/10 - (\sqrt{3}/2)^2$$

We know, $\cos \pi/10 = \sqrt{(10+2\sqrt{5})}/4$

the above equation becomes,

$$= [\sqrt{(10+2\sqrt{5})}/4]^2 - 3/4$$

$$= [10 + 2\sqrt{5}]/16 - 3/4$$

$$= [10 + 2\sqrt{5} - 12]/16$$

$$= [2\sqrt{5} - 2]/16$$

$$= [\sqrt{5} - 1]/8$$

$$= \text{RHS}$$

Hence proved.

2. $\sin^2 24^\circ - \sin^2 6^\circ = (\sqrt{5} - 1)/8$

Solution:

Let us consider LHS:

$$\sin^2 24^\circ - \sin^2 6^\circ$$

we know, $\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B$

Then the above equation becomes,

$$\sin^2 24^\circ - \sin^2 6^\circ = \sin (24^\circ + 6^\circ) \times \sin (24^\circ - 6^\circ)$$

$$= \sin 30^\circ \times \sin 18^\circ$$

$$= \sin 30^\circ \times (\sqrt{5} - 1)/4 \text{ [since, } \sin 18^\circ = (\sqrt{5} - 1)/4]$$

$$= 1/2 \times (\sqrt{5} - 1)/4$$

$$= (\sqrt{5} - 1)/8$$

$$= \text{RHS}$$

Hence proved.

3. $\sin^2 42^\circ - \cos^2 78^\circ = (\sqrt{5} + 1)/8$

Solution:

Let us consider LHS:

$$\begin{aligned} \sin^2 42^\circ - \cos^2 78^\circ &= \sin^2 (90^\circ - 48^\circ) - \cos^2 (90^\circ - 12^\circ) \\ &= \cos^2 48^\circ - \sin^2 12^\circ \text{ [since, } \sin (90 - A) = \cos A \text{ and } \cos (90 - A) = \sin A \text{]} \end{aligned}$$

We know, $\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B$

Then the above equation becomes,

$$\begin{aligned} &= \cos^2 (48^\circ + 12^\circ) \cos (48^\circ - 12^\circ) \\ &= \cos 60^\circ \cos 36^\circ \text{ [since, } \cos 36^\circ = (\sqrt{5} + 1)/4 \text{]} \\ &= 1/2 \times (\sqrt{5} + 1)/4 \\ &= (\sqrt{5} + 1)/8 \\ &= \text{RHS} \end{aligned}$$

Hence proved.

4. $\cos 78^\circ \cos 42^\circ \cos 36^\circ = 1/8$

Solution:

Let us consider LHS:

$$\cos 78^\circ \cos 42^\circ \cos 36^\circ$$

Let us multiply and divide by 2 we get,

$$\cos 78^\circ \cos 42^\circ \cos 36^\circ = 1/2 (2 \cos 78^\circ \cos 42^\circ \cos 36^\circ)$$

We know, $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

Then the above equation becomes,

$$\begin{aligned} &= 1/2 (\cos (78^\circ + 42^\circ) + \cos (78^\circ - 42^\circ)) \times \cos 36^\circ \\ &= 1/2 (\cos 120^\circ + \cos 36^\circ) \times \cos 36^\circ \\ &= 1/2 (\cos (180^\circ - 60^\circ) + \cos 36^\circ) \times \cos 36^\circ \\ &= 1/2 (-\cos (60^\circ) + \cos 36^\circ) \times \cos 36^\circ \text{ [since, } \cos(180^\circ - A) = -\cos A \text{]} \\ &= 1/2 (-1/2 + (\sqrt{5} + 1)/4) ((\sqrt{5} + 1)/4) \text{ [since, } \cos 36^\circ = (\sqrt{5} + 1)/4 \text{]} \\ &= 1/2 (\sqrt{5} + 1 - 2)/4 ((\sqrt{5} + 1)/4) \\ &= 1/2 (\sqrt{5} - 1)/4 ((\sqrt{5} + 1)/4) \\ &= 1/2 ((\sqrt{5})^2 - 1^2)/16 \\ &= 1/2 (5-1)/16 \\ &= 1/2 (4/16) \\ &= 1/8 \\ &= \text{RHS} \end{aligned}$$

Hence proved.

5. $\cos \pi/15 \cos 2\pi/15 \cos 4\pi/15 \cos 7\pi/15 = 1/16$

Solution:

Let us consider LHS:

$$\cos \pi/15 \cos 2\pi/15 \cos 4\pi/15 \cos 7\pi/15$$

Let us multiply and divide by $2 \sin \pi/15$, we get,

$$= [2 \sin \pi/15 \cos \pi/15] \cos 2\pi/15 \cos 4\pi/15 \cos 7\pi/15 / 2 \sin \pi/15$$

We know, $2\sin A \cos A = \sin 2A$

Then the above equation becomes,

$$= [(\sin 2\pi/15) \cos 2\pi/15 \cos 4\pi/15 \cos 7\pi/15] / 2 \sin \pi/15$$

Now, multiply and divide by 2 we get,

$$= [(2 \sin 2\pi/15 \cos 2\pi/15) \cos 4\pi/15 \cos 7\pi/15] / 2 \times 2 \sin \pi/15$$

We know, $2\sin A \cos A = \sin 2A$

Then the above equation becomes,

$$= [(\sin 4\pi/15) \cos 4\pi/15 \cos 7\pi/15] / 4 \sin \pi/15$$

Now, multiply and divide by 2 we get,

$$= [(2 \sin 4\pi/15 \cos 4\pi/15) \cos 7\pi/15] / 2 \times 4 \sin \pi/15$$

We know, $2\sin A \cos A = \sin 2A$

Then the above equation becomes,

$$= [(\sin 8\pi/15) \cos 7\pi/15] / 8 \sin \pi/15$$

Now, multiply and divide by 2 we get,

$$= [2 \sin 8\pi/15 \cos 7\pi/15] / 2 \times 8 \sin \pi/15$$

We know, $2\sin A \cos B = \sin (A+B) + \sin (A-B)$

Then the above equation becomes,

$$= [\sin (8\pi/15 + 7\pi/15) + \sin (8\pi/15 - 7\pi/15)] / 16 \sin \pi/15$$

$$= [\sin (\pi) + \sin (\pi/15)] / 16 \sin \pi/15$$

$$= [0 + \sin (\pi/15)] / 16 \sin \pi/15$$

$$= \sin (\pi/15) / 16 \sin \pi/15$$

$$= 1/16$$

$$= \text{RHS}$$

Hence proved.