EXERCISE 19.22

Evaluate the following integrals:

1. $$\int \frac{1}{4\cos^2 x + 9\sin^2 x}\,dx$$

Solution:

Given $I = \int \frac{1}{4\cos^2 x + 9\sin^2 x} dx$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4\cos^2 x + 9\sin^2 x} dx = \int \frac{\sec^2 x}{4 + 9\tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$\Rightarrow I = \int \frac{dt}{4 + 9t^2} = \frac{1}{9}\int \frac{dt}{\frac{4}{9} + t^2}$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{1}{9}\int \frac{dt}{\frac{4}{9} + t^2} = \frac{1}{9} \times \frac{1}{\frac{2}{3}} \tan^{-1}\left(\frac{t}{\frac{2}{3}}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3t}{2}\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

$$\therefore I = \int \frac{1}{4\cos^2 x + 9\sin^2 x} dx = \frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

2. $$\int \frac{1}{4\sin^2 x + 5\cos^2 x}\,dx$$

Solution:

Given $I = \int \frac{1}{4\sin^2 x + 5\cos^2 x} dx$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4\sin^2 x + 5\cos^2 x} dx = \int \frac{\sec^2 x}{4\tan^2 x + 5} dx$$

Putting tan x = t and $\sec^2 x$ dx = dt, we get

$$\Rightarrow I = \int \frac{dt}{4t^2 + 5} = \frac{1}{4}\int \frac{dt}{t^2 + \left(\frac{5}{4}\right)}$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{1}{4}\int \frac{dt}{t^2 + \left(\frac{5}{4}\right)} = \frac{1}{4} \times \frac{1}{\frac{\sqrt{5}}{2}} \tan^{-1}\left(\frac{t}{\frac{\sqrt{5}}{2}}\right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2t}{\sqrt{5}}\right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + c$$

$$\therefore I = \int \frac{1}{4\sin^2 x + 5\cos^2 x} dx = \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + c$$

3. $\int \frac{2}{2 + \sin 2x} dx$

Solution:

Given $I = \int \frac{2}{2 + \sin 2x} dx$

We know that sin 2x = 2 sin x cos x

$$\Rightarrow \int \frac{2}{2 + \sin 2x} dx = \int \frac{2}{2 + 2\sin x \cos x} dx$$

$$= \int \frac{1}{1 + \sin x \cos x} dx$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{1 + \sin x \cos x} dx = \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\Rightarrow \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx = \int \frac{\sec^2 x}{1 + \tan^2 x + \tan x} dx$$

Putting $\tan x = t$ so that $\sec^2 x\, dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan^2 x + \tan x + 1} dx = \int \frac{dt}{t^2 + t + 1}$$

$$= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t + 1}{\sqrt{3}}\right) + c$$

$$\therefore I = \int \frac{2}{2 + \sin 2x} dx = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t + 1}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2 \tan x + 1}{\sqrt{3}}\right) + C$$

4. $\int \frac{\cos x}{\cos 3x} dx$

Solution:

Given $I = \int \frac{\cos x}{\cos 3x} dx$

$$\Rightarrow \int \frac{\cos x}{\cos 3x} dx = \int \frac{\cos x}{4\cos^3 x - 3\cos x} dx$$

$$= \int \frac{1}{4\cos^2 x - 3} dx$$

Dividing numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{4\cos^2 x - 3} dx = \int \frac{\sec^2 x}{4 - 3\sec^2 x} dx$$

Replacing $\sec^2 x$ by $1 + \tan^2 x$ in denominator,

$$\Rightarrow \int \frac{\sec^2 x}{4 - 3\sec^2 x} dx = \int \frac{\sec^2 x}{4 - 3 - 3\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 - 3\tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$I = \int \frac{dt}{1 - 3t^2} = \frac{1}{3}\int \frac{1}{\frac{1}{3} - t^2} dt$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

$$\Rightarrow \frac{1}{3}\int \frac{1}{\frac{1}{3} - t^2} dt = \frac{1}{3} \times \frac{1}{2\left(\frac{1}{\sqrt{3}}\right)} \log\left|\frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t}\right| + c$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{1 + \sqrt{3}t}{1 - \sqrt{3}t}\right| + c$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{1 + \sqrt{3}\tan x}{1 - \sqrt{3}\tan x}\right| + c$$

$$\therefore I = \int \frac{\cos x}{\cos 3x} dx = \frac{1}{2\sqrt{3}} \log\left|\frac{1 + \sqrt{3}\tan x}{1 - \sqrt{3}\tan x}\right| + c$$