

EXERCISE 10.2

In any ΔABC , prove the following:

1. In a ΔABC , if $a = 5$, $b = 6$ and $C = 60^\circ$, show that its area is $(15\sqrt{3})/2$ sq. units.

Solution:

Given:

In a ΔABC , $a = 5$, $b = 6$ and $C = 60^\circ$

By using the formula,

Area of $\Delta ABC = 1/2 ab \sin \theta$ where, a and b are the lengths of the sides of a triangle and θ is the angle between sides.

So,

$$\begin{aligned}\text{Area of } \Delta ABC &= 1/2 ab \sin \theta \\ &= 1/2 \times 5 \times 6 \times \sin 60^\circ \\ &= 30/2 \times \sqrt{3}/2 \\ &= (15\sqrt{3})/2 \text{ sq. units}\end{aligned}$$

Hence proved.

2. In a ΔABC , if $a = \sqrt{2}$, $b = \sqrt{3}$ and $c = \sqrt{5}$ show that its area is $1/2 \sqrt{6}$ sq. units.

Solution:

Given:

In a ΔABC , $a = \sqrt{2}$, $b = \sqrt{3}$ and $c = \sqrt{5}$

By using the formulas,

We know, $\cos A = (b^2 + c^2 - a^2)/2bc$

By substituting the values we get,

$$\begin{aligned}&= [(\sqrt{3})^2 + (\sqrt{5})^2 - (\sqrt{2})^2] / [2 \times \sqrt{3} \times \sqrt{5}] \\ &= 3/\sqrt{15}\end{aligned}$$

We know, Area of $\Delta ABC = 1/2 bc \sin A$

To find $\sin A$:

$$\begin{aligned}\sin A &= \sqrt{1 - \cos^2 A} \text{ [by using trigonometric identity]} \\ &= \sqrt{1 - (3/\sqrt{15})^2} \\ &= \sqrt{1 - (9/15)} \\ &= \sqrt{6/15}\end{aligned}$$

Now,

$$\begin{aligned}\text{Area of } \Delta ABC &= 1/2 bc \sin A \\ &= 1/2 \times \sqrt{3} \times \sqrt{5} \times \sqrt{6/15} \\ &= 1/2 \sqrt{6} \text{ sq. units}\end{aligned}$$

Hence proved.

3. The sides of a triangle are $a = 4$, $b = 6$ and $c = 8$, show that: $8 \cos A + 16 \cos B + 4 \cos C = 17$.

Solution:

Given:

Sides of a triangle are $a = 4$, $b = 6$ and $c = 8$

By using the formulas,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

So now let us substitute the values of a , b and c we get,

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{6^2 + 8^2 - 4^2}{2 \times 6 \times 8} \\ &= \frac{36 + 64 - 16}{96} \\ &= \frac{84}{96} \\ &= \frac{7}{8}\end{aligned}$$

$$\begin{aligned}\cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{4^2 + 8^2 - 6^2}{2 \times 4 \times 8} \\ &= \frac{16 + 64 - 36}{64} \\ &= \frac{44}{64}\end{aligned}$$

$$\begin{aligned}\cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{4^2 + 6^2 - 8^2}{2 \times 4 \times 6} \\ &= \frac{16 + 36 - 64}{48} \\ &= \frac{-12}{48} \\ &= \frac{-1}{4}\end{aligned}$$

Now considering LHS:

$$\begin{aligned}8 \cos A + 16 \cos B + 4 \cos C &= 8 \times \frac{7}{8} + 16 \times \frac{44}{64} + 4 \times \left(\frac{-1}{4}\right) \\ &= 7 + 11 - 1 \\ &= 17\end{aligned}$$

Hence proved.

4. In a $\triangle ABC$, if $a = 18$, $b = 24$, $c = 30$, find $\cos A$, $\cos B$ and $\cos C$

Solution:

Given:

Sides of a triangle are $a = 18$, $b = 24$ and $c = 30$

By using the formulas,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = (a^2 + c^2 - b^2)/2ac$$

$$\cos C = (a^2 + b^2 - c^2)/2ab$$

So now let us substitute the values of a, b and c we get,

$$\cos A = (b^2 + c^2 - a^2)/2bc$$

$$= (24^2 + 30^2 - 18^2)/2 \times 24 \times 30$$

$$= 1152/1440$$

$$= 4/5$$

$$\cos B = (a^2 + c^2 - b^2)/2ac$$

$$= (18^2 + 30^2 - 24^2)/2 \times 18 \times 30$$

$$= 648/1080$$

$$= 3/5$$

$$\cos C = (a^2 + b^2 - c^2)/2ab$$

$$= (18^2 + 24^2 - 30^2)/2 \times 18 \times 24$$

$$= 0/864$$

$$= 0$$

$$\therefore \cos A = 4/5, \cos B = 3/5, \cos C = 0$$

5. For any ΔABC , show that $b(c \cos A - a \cos C) = c^2 - a^2$

Solution:

Let us consider LHS:

$$b(c \cos A - a \cos C)$$

As LHS contain $bc \cos A$ and $ab \cos C$ which can be obtained from cosine formulae.

From cosine formula we have:

$$\cos A = (b^2 + c^2 - a^2)/2bc$$

$$bc \cos A = (b^2 + c^2 - a^2)/2 \dots (i)$$

$$\cos C = (a^2 + b^2 - c^2)/2ab$$

$$ab \cos C = (a^2 + b^2 - c^2)/2 \dots (ii)$$

Now let us subtract equation (i) and (ii) we get,

$$\begin{aligned} bc \cos A - ab \cos C &= (b^2 + c^2 - a^2)/2 - (a^2 + b^2 - c^2)/2 \\ &= c^2 - a^2 \end{aligned}$$

$$\therefore b(c \cos A - a \cos C) = c^2 - a^2$$

Hence proved.

6. For any ΔABC show that $c(a \cos B - b \cos A) = a^2 - b^2$

Solution:

Let us consider LHS:

$$c(a \cos B - b \cos A)$$

As LHS contain $ca \cos B$ and $cb \cos A$ which can be obtained from cosine formulae.

From cosine formula we have:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$bc \cos A = \frac{b^2 + c^2 - a^2}{2} \dots (i)$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$ac \cos B = \frac{a^2 + c^2 - b^2}{2} \dots (ii)$$

Now let us subtract equation (ii) from (i) we get,

$$\begin{aligned} ac \cos B - bc \cos A &= \frac{a^2 + c^2 - b^2}{2} - \frac{b^2 + c^2 - a^2}{2} \\ &= a^2 - b^2 \end{aligned}$$

$$\therefore c(a \cos B - b \cos A) = a^2 - b^2$$

Hence proved.

7. For any ΔABC show that

$$2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$$

Solution:

Let us consider LHS:

$$2(bc \cos A + ca \cos B + ab \cos C)$$

As LHS contain $2ca \cos B$, $2ab \cos C$ and $2cb \cos A$, which can be obtained from cosine formulae.

From cosine formula we have:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$2bc \cos A = \frac{b^2 + c^2 - a^2}{1} \dots (i)$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$2ac \cos B = \frac{a^2 + c^2 - b^2}{1} \dots (ii)$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$2ab \cos C = \frac{a^2 + b^2 - c^2}{1} \dots (iii)$$

Now let us add equation (i), (ii) and (iii) we get,

$$2bc \cos A + 2ac \cos B + 2ab \cos C = (b^2 + c^2 - a^2) + (a^2 + c^2 - b^2) + (a^2 + b^2 - c^2)$$

Upon simplification we get,

$$= c^2 + b^2 + a^2$$

$$2(bc \cos A + ac \cos B + ab \cos C) = a^2 + b^2 + c^2$$

Hence proved.

8. For any ΔABC show that

$$(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$$

Solution:

Let us consider LHS:

$$(c^2 - a^2 + b^2), (a^2 - b^2 + c^2), (b^2 - c^2 + a^2)$$

We know sine rule in ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

As LHS contain $(c^2 - a^2 + b^2)$, $(a^2 - b^2 + c^2)$ and $(b^2 - c^2 + a^2)$, which can be obtained from cosine formulae.

From cosine formula we have:

$$\cos A = (b^2 + c^2 - a^2)/2bc$$

$$2bc \cos A = (b^2 + c^2 - a^2)$$

Let us multiply both the sides by $\tan A$ we get,

$$2bc \cos A \tan A = (b^2 + c^2 - a^2) \tan A$$

$$2bc \cos A (\sin A/\cos A) = (b^2 + c^2 - a^2) \tan A$$

$$2bc \sin A = (b^2 + c^2 - a^2) \tan A \dots (i)$$

$$\cos B = (a^2 + c^2 - b^2)/2ac$$

$$2ac \cos B = (a^2 + c^2 - b^2)$$

Let us multiply both the sides by $\tan B$ we get,

$$2ac \cos B \tan B = (a^2 + c^2 - b^2) \tan B$$

$$2ac \cos B (\sin B/\cos B) = (a^2 + c^2 - b^2) \tan B$$

$$2ac \sin B = (a^2 + c^2 - b^2) \tan B \dots (ii)$$

$$\cos C = (a^2 + b^2 - c^2)/2ab$$

$$2ab \cos C = (a^2 + b^2 - c^2)$$

Let us multiply both the sides by $\tan C$ we get,

$$2ab \cos C \tan C = (a^2 + b^2 - c^2) \tan C$$

$$2ab \cos C (\sin C/\cos C) = (a^2 + b^2 - c^2) \tan C$$

$$2ab \sin C = (a^2 + b^2 - c^2) \tan C \dots (iii)$$

As we are observing that sin terms are being involved so let's use sine formula.

From sine formula we have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Let us multiply abc to each of the expression we get,

$$\frac{abc \sin A}{a} = \frac{abc \sin B}{b} = \frac{abc \sin C}{c}$$

$$bc \sin A = ac \sin B = ab \sin C$$

$$2bc \sin A = 2ac \sin B = 2ab \sin C$$

∴ From equation (i), (ii) and (iii) we have,
 $(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$
Hence proved.

9. For any ΔABC show that:

$$\frac{c - b \cos A}{b - c \cos A} = \frac{\cos B}{\cos C}$$

Solution:

Let us consider LHS:

$$\frac{c - b \cos A}{b - c \cos A}$$

We can observe that we can get terms $c - b \cos A$ and $b - c \cos A$ from projection formula

From projection formula we get,

$$c = a \cos B + b \cos A$$

$$c - b \cos A = a \cos B \dots (i)$$

And,

$$b = c \cos A + a \cos C$$

$$b - c \cos A = a \cos C \dots (ii)$$

Dividing equation (i) by (ii), we get,

$$\begin{aligned} \frac{c - b \cos A}{b - c \cos A} &= \frac{a \cos B}{a \cos C} \\ &= \frac{\cos B}{\cos C} \\ &= \text{RHS} \end{aligned}$$

Hence proved.