

EXERCISE 1.1

1. Let A be the set of all human beings in a town at a particular time. Determine whether of the following relation is reflexive, symmetric and transitive:

(i) $R = \{(x, y): x \text{ and } y \text{ work at the same place}\}$

(ii) $R = \{(x, y): x \text{ and } y \text{ live in the same locality}\}$

(iii) $R = \{(x, y): x \text{ is wife of } y\}$

(iv) $R = \{(x, y): x \text{ is father of } y\}$

Solution:

(i) Given $R = \{(x, y): x \text{ and } y \text{ work at the same place}\}$

Now we have to check whether the relation is reflexive:

Let x be an arbitrary element of R .

Then, $x \in R$

$\Rightarrow x$ and x work at the same place is true since they are the same.

$\Rightarrow (x, x) \in R$ [condition for reflexive relation]

So, R is a reflexive relation.

Now let us check Symmetric relation:

Let $(x, y) \in R$

$\Rightarrow x$ and y work at the same place [given]

$\Rightarrow y$ and x work at the same place

$\Rightarrow (y, x) \in R$

So, R is a symmetric relation.

Transitive relation:

Let $(x, y) \in R$ and $(y, z) \in R$.

Then, x and y work at the same place. [Given]

y and z also work at the same place. [$(y, z) \in R$]

$\Rightarrow x, y$ and z all work at the same place.

$\Rightarrow x$ and z work at the same place.

$\Rightarrow (x, z) \in R$

So, R is a transitive relation.

Hence R is reflexive, symmetric and transitive.

(ii) Given $R = \{(x, y): x \text{ and } y \text{ live in the same locality}\}$

Now we have to check whether the relation R is reflexive, symmetric and transitive.

Let x be an arbitrary element of R .

Then, $x \in R$

It is given that x and x live in the same locality is true since they are the same.

So, R is a reflexive relation.

Symmetry:

Let $(x, y) \in R$

$\Rightarrow x$ and y live in the same locality [given]

$\Rightarrow y$ and x live in the same locality

$\Rightarrow (y, x) \in R$

So, R is a symmetric relation.

Transitivity:

Let $(x, y) \in R$ and $(y, z) \in R$.

Then,

x and y live in the same locality and y and z live in the same locality

$\Rightarrow x, y$ and z all live in the same locality

$\Rightarrow x$ and z live in the same locality

$\Rightarrow (x, z) \in R$

So, R is a transitive relation.

Hence R is reflexive, symmetric and transitive.

(iii) Given $R = \{(x, y) : x \text{ is wife of } y\}$

Now we have to check whether the relation R is reflexive, symmetric and transitive.

First let us check whether the relation is reflexive:

Let x be an element of R .

Then, x is wife of x cannot be true.

$\Rightarrow (x, x) \notin R$

So, R is not a reflexive relation.

Symmetric relation:

Let $(x, y) \in R$

$\Rightarrow x$ is wife of y

$\Rightarrow x$ is female and y is male

$\Rightarrow y$ cannot be wife of x as y is husband of x

$\Rightarrow (y, x) \notin R$

So, R is not a symmetric relation.

Transitive relation:

Let $(x, y) \in R$, but $(y, z) \notin R$

Since x is wife of y , but y cannot be the wife of z , y is husband of x .

$\Rightarrow x$ is not the wife of z

$\Rightarrow (x, z) \in R$

So, R is a transitive relation.

(iv) Given $R = \{(x, y) : x \text{ is father of } y\}$

Now we have to check whether the relation R is reflexive, symmetric and transitive.

Reflexivity:

Let x be an arbitrary element of R.

Then, x is father of x cannot be true since no one can be father of himself.

So, R is not a reflexive relation.

Symmetry:

Let $(x, y) \in R$

$\Rightarrow x$ is father of y

$\Rightarrow y$ is son/daughter of x

$\Rightarrow (y, x) \notin R$

So, R is not a symmetric relation.

Transitivity:

Let $(x, y) \in R$ and $(y, z) \in R$.

Then, x is father of y and y is father of z

$\Rightarrow x$ is grandfather of z

$\Rightarrow (x, z) \notin R$

So, R is not a transitive relation.

2. Three relations R_1 , R_2 and R_3 are defined on a set $A = \{a, b, c\}$ as follows:

$R_1 = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

$R_2 = \{(a, a)\}$

$R_3 = \{(b, c)\}$

$R_4 = \{(a, b), (b, c), (c, a)\}$.

Find whether or not each of the relations R_1 , R_2 , R_3 , R_4 on A is (i) reflexive (ii) symmetric and (iii) transitive.

Solution:

(i) Consider R_1

Given $R_1 = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

Now we have check R_1 is reflexive, symmetric and transitive

Reflexive:

Given (a, a) , (b, b) and $(c, c) \in R_1$

So, R_1 is reflexive.

Symmetric:

We see that the ordered pairs obtained by interchanging the components of R_1 are also in R_1 .

So, R_1 is symmetric.

Transitive:

Here, $(a, b) \in R_1$, $(b, c) \in R_1$ and also $(a, c) \in R_1$

So, R_1 is transitive.

(ii) Consider R_2

Given $R_2 = \{(a, a)\}$

Reflexive:

Clearly $(a, a) \in R_2$.

So, R_2 is reflexive.

Symmetric:

Clearly $(a, a) \in R$

$\Rightarrow (a, a) \in R$.

So, R_2 is symmetric.

Transitive:

R_2 is clearly a transitive relation, since there is only one element in it.

(iii) Consider R_3

Given $R_3 = \{(b, c)\}$

Reflexive:

Here, $(b, b) \notin R_3$ neither $(c, c) \in R_3$

So, R_3 is not reflexive.

Symmetric:

Here, $(b, c) \in R_3$, but $(c, b) \notin R_3$

So, R_3 is not symmetric.

Transitive:

Here, R_3 has only two elements.

Hence, R_3 is transitive.

(iv) Consider R_4

Given $R_4 = \{(a, b), (b, c), (c, a)\}$.

Reflexive:

Here, $(a, a) \notin R_4$, $(b, b) \notin R_4$, $(c, c) \notin R_4$

So, R_4 is not reflexive.

Symmetric:

Here, $(a, b) \in R_4$, but $(b, a) \notin R_4$.

So, R_4 is not symmetric

Transitive:

Here, $(a, b) \in R_4$, $(b, c) \in R_4$, but $(a, c) \notin R_4$

So, R_4 is not transitive.

3. Test whether the following relation R_1 , R_2 , and R_3 are (i) reflexive (ii) symmetric and (iii) transitive:

(i) R_1 on Q_0 defined by $(a, b) \in R_1 \Leftrightarrow a = 1/b$.

(ii) R_2 on Z defined by $(a, b) \in R_2 \Leftrightarrow |a - b| \leq 5$

(iii) R_3 on R defined by $(a, b) \in R_3 \Leftrightarrow a^2 - 4ab + 3b^2 = 0$.

Solution:

(i) Given R_1 on Q_0 defined by $(a, b) \in R_1 \Leftrightarrow a = 1/b$.

Reflexivity:

Let a be an arbitrary element of R_1 .

Then, $a \in R_1$

$\Rightarrow a \neq 1/a$ for all $a \in Q_0$

So, R_1 is not reflexive.

Symmetry:

Let $(a, b) \in R_1$

Then, $(a, b) \in R_1$

Therefore we can write 'a' as $a = 1/b$

$\Rightarrow b = 1/a$

$\Rightarrow (b, a) \in R_1$

So, R_1 is symmetric.

Transitivity:

Here, $(a, b) \in R_1$ and $(b, c) \in R_2$

$\Rightarrow a = 1/b$ and $b = 1/c$

$\Rightarrow a = 1/(1/c) = c$

$\Rightarrow a \neq 1/c$

$\Rightarrow (a, c) \notin R_1$

So, R_1 is not transitive.

(ii) Given R_2 on Z defined by $(a, b) \in R_2 \Leftrightarrow |a - b| \leq 5$

Now we have check whether R_2 is reflexive, symmetric and transitive.

Reflexivity:

Let a be an arbitrary element of R_2 .

Then, $a \in R_2$

On applying the given condition we get,

$$\Rightarrow |a-a| = 0 \leq 5$$

So, R_1 is reflexive.

Symmetry:

Let $(a, b) \in R_2$

$$\Rightarrow |a-b| \leq 5 \quad [\text{Since, } |a-b| = |b-a|]$$

$$\Rightarrow |b-a| \leq 5$$

$$\Rightarrow (b, a) \in R_2$$

So, R_2 is symmetric.

Transitivity:

Let $(1, 3) \in R_2$ and $(3, 7) \in R_2$

$$\Rightarrow |1-3| \leq 5 \text{ and } |3-7| \leq 5$$

But $|1-7| \not\leq 5$

$$\Rightarrow (1, 7) \notin R_2$$

So, R_2 is not transitive.

(iii) Given R_3 on R defined by $(a, b) \in R_3 \Leftrightarrow a^2 - 4ab + 3b^2 = 0$.

Now we have check whether R_2 is reflexive, symmetric and transitive.

Reflexivity:

Let a be an arbitrary element of R_3 .

Then, $a \in R_3$

$$\Rightarrow a^2 - 4a \times a + 3a^2 = 0$$

So, R_3 is reflexive

Symmetry:

Let $(a, b) \in R_3$

$$\Rightarrow a^2 - 4ab + 3b^2 = 0$$

But $b^2 - 4ba + 3a^2 \neq 0$ for all $a, b \in R$

So, R_3 is not symmetric.

Transitivity:

Let $(1, 2) \in R_3$ and $(2, 3) \in R_3$

$$\Rightarrow 1 - 8 + 6 = 0 \text{ and } 4 - 24 + 27 = 0$$

But $1 - 12 + 9 \neq 0$

So, R_3 is not transitive.

4. Let $A = \{1, 2, 3\}$, and let $R_1 = \{(1, 1), (1, 3), (3, 1), (2, 2), (2, 1), (3, 3)\}$, $R_2 = \{(2, 2), (3, 1), (1, 3)\}$, $R_3 = \{(1, 3), (3, 3)\}$. Find whether or not each of the relations R_1, R_2, R_3 on A is (i) reflexive (ii) symmetric (iii) transitive.

Solution:

Consider R_1

Given $R_1 = \{(1, 1), (1, 3), (3, 1), (2, 2), (2, 1), (3, 3)\}$

Reflexivity:

Here, $(1, 1), (2, 2), (3, 3) \in R_1$

So, R_1 is reflexive.

Symmetry:

Here, $(2, 1) \in R_1$,

But $(1, 2) \notin R_1$

So, R_1 is not symmetric.

Transitivity:

Here, $(2, 1) \in R_1$ and $(1, 3) \in R_1$,

But $(2, 3) \notin R_1$

So, R_1 is not transitive.

Now consider R_2

Given $R_2 = \{(2, 2), (3, 1), (1, 3)\}$

Reflexivity:

Clearly, $(1, 1)$ and $(3, 3) \notin R_2$

So, R_2 is not reflexive.

Symmetry:

Here, $(1, 3) \in R_2$ and $(3, 1) \in R_2$

So, R_2 is symmetric.

Transitivity:

Here, $(1, 3) \in R_2$ and $(3, 1) \in R_2$

But $(3, 3) \notin R_2$

So, R_2 is not transitive.

Consider R_3

Given $R_3 = \{(1, 3), (3, 3)\}$

Reflexivity:

Clearly, $(1, 1) \notin R_3$

So, R_3 is not reflexive.

Symmetry:

Here, $(1, 3) \in R_3$, but $(3, 1) \notin R_3$

So, R_3 is not symmetric.

Transitivity:

Here, $(1, 3) \in R_3$ and $(3, 3) \in R_3$

Also, $(1, 3) \in R_3$

So, R_3 is transitive.

5. The following relation is defined on the set of real numbers.

(i) aRb if $a - b > 0$

(ii) aRb iff $1 + a b > 0$

(iii) aRb if $|a| \leq b$.

Find whether relation is reflexive, symmetric or transitive.

Solution:

(i) Consider aRb if $a - b > 0$

Now for this relation we have to check whether it is reflexive, transitive and symmetric.

Reflexivity:

Let a be an arbitrary element of R .

Then, $a \in R$

But $a - a = 0 \not> 0$

So, this relation is not reflexive.

Symmetry:

Let $(a, b) \in R$

$\Rightarrow a - b > 0$

$\Rightarrow -(b - a) > 0$

$\Rightarrow b - a < 0$

So, the given relation is not symmetric.

Transitivity:

Let $(a, b) \in R$ and $(b, c) \in R$.

Then, $a - b > 0$ and $b - c > 0$

Adding the two, we get

$a - b + b - c > 0$

$\Rightarrow a - c > 0$

$\Rightarrow (a, c) \in R$.

So, the given relation is transitive.

(ii) Consider aRb iff $1 + a b > 0$

Now for this relation we have to check whether it is reflexive, transitive and symmetric.

Reflexivity:

Let a be an arbitrary element of R .

Then, $a \in R$

$$\Rightarrow 1 + a \times a > 0$$

i.e. $1 + a^2 > 0$ [Since, square of any number is positive]

So, the given relation is reflexive.

Symmetry:

Let $(a, b) \in R$

$$\Rightarrow 1 + a b > 0$$

$$\Rightarrow 1 + b a > 0$$

$$\Rightarrow (b, a) \in R$$

So, the given relation is symmetric.

Transitivity:

Let $(a, b) \in R$ and $(b, c) \in R$

$$\Rightarrow 1 + a b > 0 \text{ and } 1 + b c > 0$$

But $1 + ac \not> 0$

$$\Rightarrow (a, c) \notin R$$

So, the given relation is not transitive.

(iii) Consider aRb if $|a| \leq b$.

Now for this relation we have to check whether it is reflexive, transitive and symmetric.

Reflexivity:

Let a be an arbitrary element of R .

Then, $a \in R$ [Since, $|a| = a$]

$$\Rightarrow |a| \leq a$$

So, R is not reflexive.

Symmetry:

Let $(a, b) \in R$

$$\Rightarrow |a| \leq b$$

$$\Rightarrow |b| \not\leq a \text{ for all } a, b \in R$$

$$\Rightarrow (b, a) \notin R$$

So, R is not symmetric.

Transitivity:

Let $(a, b) \in R$ and $(b, c) \in R$

$$\Rightarrow |a| \leq b \text{ and } |b| \leq c$$

Multiplying the corresponding sides, we get

$$|a| \times |b| \leq b c$$

$$\Rightarrow |a| \leq c$$

$$\Rightarrow (a, c) \in R$$

Thus, R is transitive.

6. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b): b = a + 1\}$ is reflexive, symmetric or transitive.

Solution:

$$\text{Given } R = \{(a, b): b = a + 1\}$$

Now for this relation we have to check whether it is reflexive, transitive and symmetric

Reflexivity:

Let a be an arbitrary element of R.

Then, $a = a + 1$ cannot be true for all $a \in A$.

$$\Rightarrow (a, a) \notin R$$

So, R is not reflexive on A.

Symmetry:

$$\text{Let } (a, b) \in R$$

$$\Rightarrow b = a + 1$$

$$\Rightarrow -a = -b + 1$$

$$\Rightarrow a = b - 1$$

Thus, $(b, a) \notin R$

So, R is not symmetric on A.

Transitivity:

$$\text{Let } (1, 2) \text{ and } (2, 3) \in R$$

$$\Rightarrow 2 = 1 + 1 \text{ and } 3$$

$2 + 1$ is true.

But $3 \neq 1 + 1$

$$\Rightarrow (1, 3) \notin R$$

So, R is not transitive on A.

7. Check whether the relation R on R defined as $R = \{(a, b): a \leq b^3\}$ is reflexive, symmetric or transitive.

Solution:

$$\text{Given } R = \{(a, b): a \leq b^3\}$$

It is observed that $(1/2, 1/2)$ in R as $1/2 > (1/2)^3 = 1/8$

\therefore R is not reflexive.

Now,

$(1, 2) \in R$ (as $1 < 2^3 = 8$)

But,

$(2, 1) \notin R$ (as $2 > 1^3 = 1$)

$\therefore R$ is not symmetric.

We have $(3, 3/2), (3/2, 6/5)$ in " R " as " $3 < (3/2)^3$ " and $3/2 < (6/5)^3$

But $(3, 6/5) \notin R$ as $3 > (6/5)^3$

$\therefore R$ is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

8. Prove that every identity relation on a set is reflexive, but the converse is not necessarily true.

Solution:

Let A be a set.

Then, Identity relation $I_A = I_A$ is reflexive, since $(a, a) \in A \forall a$

The converse of it need not be necessarily true.

Consider the set $A = \{1, 2, 3\}$

Here,

Relation $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 3)\}$ is reflexive on A .

However, R is not an identity relation.

9. If $A = \{1, 2, 3, 4\}$ define relations on A which have properties of being

(i) Reflexive, transitive but not symmetric

(ii) Symmetric but neither reflexive nor transitive.

(iii) Reflexive, symmetric and transitive.

Solution:

(i) The relation on A having properties of being reflexive, transitive, but not symmetric is

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 1)\}$

Relation R satisfies reflexivity and transitivity.

$\Rightarrow (1, 1), (2, 2), (3, 3) \in R$

And $(1, 1), (2, 1) \in R \Rightarrow (1, 1) \in R$

However, $(2, 1) \in R$, but $(1, 2) \notin R$

(ii) The relation on A having properties of being reflexive, transitive, but not symmetric is

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 1)\}$$

Relation R satisfies reflexivity and transitivity.

$$\Rightarrow (1, 1), (2, 2), (3, 3) \in R$$

$$\text{And } (1, 1), (2, 1) \in R \Rightarrow (1, 1) \in R$$

However, $(2, 1) \in R$, but $(1, 2) \notin R$

(iii) The relation on A having properties of being symmetric, reflexive and transitive is

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$$

The relation R is an equivalence relation on A.



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