$$\int \frac{x}{\sqrt{9-x^4}}dx$$

Answer

Tip – $d(x^2) = 2xdx$ i.e. $xdx = (1/2)\times d(x^2)$

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{xdx}{\sqrt{9-x^4}}$$

$$= \frac{1}{2}\int \frac{d(x^2)}{\sqrt{3^2-(x^2)^2}}$$

$= \frac{1}{2}\sin^{-1}\frac{x^2}{3} + c$, c being the integrating constant

11. Question

Evaluate:

$$\int \frac{3x^2}{\sqrt{9-16x^6}}dx$$

Answer

Tip – $d(x^3) = 3x^2dx$ so, $d(4x^3) = 4\times 3x^2dx$ i.e $3x^2dx = (1/4)d(2x^3)$

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{3x^2dx}{\sqrt{9-16x^6}}$$

$$= \frac{1}{4}\int \frac{d(2x^3)}{\sqrt{3^2-(4x^3)^2}}$$

$= \frac{1}{4}\sin^{-1}\frac{4x^3}{3} + c$, c being the integrating constant

12. Question

Evaluate:

$$\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}dx$$

Answer

Tip – $d(\tan x) = \sec^2 x dx$

Formula to be used - $\int \frac{dx}{\sqrt{x^2\pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{\sec^2 x dx}{\sqrt{16 + \tan^2 x}}$$

$$= \int \frac{d(\tan x)}{\sqrt{4^2 + (\tan x)^2}}$$

$= \log|\tan x + \sqrt{16 + \tan^2 x}| + c$, c being the integrating constant

13. Question

Evaluate:

$$\int \frac{\sin x}{\sqrt{4 + \cos^2 x}} dx$$

Answer

Tip – d(cosx) = - sinxdx i.e. sinxdx = - d(cosx)

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{\sin x dx}{\sqrt{4 + \cos^2 x}}$$

$$= \int \frac{-d(\cos x)}{\sqrt{(\cos x)^2 + 2^2}}$$

$= -\log|\cos x + \sqrt{4 + \cos^2 x}| + c$, c being the integrating constant

14. Question

Evaluate:

$$\int \frac{\cos x}{\sqrt{9\sin^2 x - 1}} dx$$

Answer

Tip – d(sinx) = cosxdx so, d(3sinx) = 3cosxdx i.e. cosxdx = (1/3)d(3sinx)

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{\cos x dx}{\sqrt{9\sin^2 x - 1}}$$

$$= \frac{1}{3} \int \frac{d(3\sin x)}{\sqrt{(3\sin x)^2 - 1^2}}$$

$= \frac{1}{3}\log|\cos x + \sqrt{4 + \cos^2 x}| + c$, c being the integrating constant

15. Question

Evaluate:

$$\int \frac{e^x}{\sqrt{4 + e^{2x}}} dx$$

Answer

Tip – $d(e^x) = e^x dx$

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{e^x dx}{\sqrt{4 + e^{2x}}}$$

$= \int \frac{d(e^x)}{\sqrt{2^2 + (e^x)^2}}$

$= \log|e^x + \sqrt{4 + e^{2x}}| + c$, c being the integrating constant

16. Question

Evaluate:

$$\int \frac{2e^x}{\sqrt{4 - e^{2x}}} dx$$

Answer

Tip – $d(e^x) = e^x dx$

Formula to be used - $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$\therefore \int \frac{2e^x dx}{\sqrt{4 - e^{2x}}}$

$= 2\int \frac{d(e^x)}{\sqrt{2^2 - (e^x)^2}}$

$= 2\sin^{-1}\left(\frac{e^x}{2}\right) + c$, c being the integrating constant

17. Question

Evaluate:

$$\int \frac{dx}{\sqrt{1 - e^x}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$\therefore \int \frac{dx}{\sqrt{1 - e^x}}$

$= \int \frac{dx}{\sqrt{e^x(e^{-x} - 1)}}$

$= \int \frac{e^{-\frac{x}{2}} dx}{\sqrt{e^{-x} - 1}}$

$= \int \frac{e^{-\frac{x}{2}} dx}{\sqrt{\left(e^{-\frac{x}{2}}\right)^2 - 1^2}}$

Tip – Assuming $e^{-(x/2)} = a$, $-(1/2)\, e^{-(x/2)} dx = da$ i.e. $e^{-(x/2)} dx = -2da$

$\therefore \int \frac{e^{-\frac{x}{2}} dx}{\sqrt{\left(e^{-\frac{x}{2}}\right)^2 - 1^2}}$

$= \int \frac{-2da}{\sqrt{a^2 - 1^2}}$

$= -2\log|a + \sqrt{a^2 - 1}| + c$

$= -2\log|e^{-\frac{x}{2}} + \sqrt{e^{-x} - 1}| + c$, c being the integrating constant

18. Question

Evaluate:

$$\int \sqrt{\frac{a - x}{a + x}} dx$$

Answer

Tip – Taking $x = a\cos 2\theta$,

$dx = -2a\sin 2\theta\, d\theta$ and $\theta = \frac{1}{2}\cos^{-1}\frac{x}{a}$

$x = a\cos 2\theta$ i.e $\cos 2\theta = \frac{x}{a}$

$$\therefore \sin 2\theta = \sqrt{1 - \frac{x^2}{a^2}}$$

$$\therefore \int \sqrt{\frac{a - x}{a + x}} dx$$

$$= \int \sqrt{\frac{a - a\cos 2\theta}{a + a\cos 2\theta}} \times (-2a\sin 2\theta\, d\theta)$$

$$= \int \sqrt{\frac{a(1 - \cos 2\theta)}{a(1 + \cos 2\theta)}} \times (-2a\sin 2\theta\, d\theta)$$

Formula to be used - $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\therefore \int \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}} \times (-2a\sin 2\theta\, d\theta)$$

$$= \int \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \times (-2a\sin 2\theta\, d\theta)$$

$$= \int \frac{\sin\theta}{\cos\theta} \times (-2a \times 2\sin\theta\cos\theta\, d\theta)$$

$$= -2a \int 2\sin^2\theta\, d\theta$$

$$= -2a \int 1 - \cos 2\theta\, d\theta$$

$$= -2a\left[\theta - \frac{\sin 2\theta}{2}\right]$$

$$= -2a\left[\theta - \frac{\sin 2\theta}{2}\right] + c$$

$= -2a\left[\frac{1}{2}\cos^{-1}\frac{x}{a} - \frac{\sqrt{1-\frac{x^2}{a^2}}}{2}\right] + c$

$= -a\cos^{-1}\frac{x}{a} + a\sqrt{1-\frac{x^2}{a^2}} + c$

$= a\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2} + c$, c being the integrating constant

19. Question

Evaluate:

$$\int \frac{dx}{\sqrt{x^2+6x+5}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$\therefore \int \frac{dx}{\sqrt{x^2 + 6x + 5}}$

$= \int \frac{dx}{\sqrt{(x^2 + 2 \times x \times 3 + 3^2) + 5 - 3^2}}$

$= \int \frac{dx}{\sqrt{(x + 3)^2 - 2^2}}$

$= \log|(x + 3) + \sqrt{x^2 + 6x + 5}| + c$, c being the integrating constant

20. Question

Evaluate:

$$\int \frac{dx}{\sqrt{(2-x)^2+1}}$$

Answer

Tip – d(2 - x) = - dx i.e. dx = - d(2 - x)

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$\therefore \int \frac{dx}{\sqrt{(2-x)^2 + 1}}$

$= \int \frac{-d(2-x)}{\sqrt{(2-x)^2 + 1}}$

$= -\log|(2-x) + \sqrt{(2-x)^2 + 1}| + c$

$= -\log|(2-x) + \sqrt{x^2 - 4x + 5}| + c$, c being the integrating constant

21. Question

Evaluate:

$$\int \frac{dx}{\sqrt{(x-3)^2+1}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{(x-3)^2 + 1}}$$

$$= \log|(x-3) + \sqrt{(x-3)^2 + 1}| + c$$

$= \log|(x-3) + \sqrt{x^2 - 6x + 10}| + c$, c being the integrating constant

22. Question

Evaluate:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{x^2 - 6x + 10}}$$

$$= \int \frac{dx}{\sqrt{(x-3)^2 + 1}}$$

$$= \log|(x-3) + \sqrt{(x-3)^2 + 1}| + c$$

$= \log|(x-3) + \sqrt{x^2 - 6x + 10}| + c$, c being the integrating constant

23. Question

Evaluate:

$$\int \frac{dx}{\sqrt{2 + 2x - x^2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{2 + 2x - x^2}}$$

$$= \int \frac{dx}{\sqrt{3 - (x^2 - 2x + 1)}}$$

$$= \int \frac{dx}{\sqrt{\left(\sqrt{3}\right)^2 - (x-1)^2}}$$

$= \sin^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$, c being the integrating constant

24. Question

Evaluate:

$$\int \frac{dx}{\sqrt{8-4x-2x^2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{8-4x-2x^2}}$$

$$= \int \frac{dx}{\sqrt{10-2(x^2+2x+1)}}$$

$$= \int \frac{dx}{\sqrt{\left(\sqrt{10}\right)^2-2(x+1)^2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\sqrt{5}\right)^2-(x+1)^2}}$$

$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c$, c being the integrating constant

25. Question

Evaluate:

$$\int \frac{dx}{\sqrt{16-6x-x^2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{16-6x-x^2}}$$

$$= \int \frac{dx}{\sqrt{25-(x^2+6x+9)}}$$

$$= \int \frac{dx}{\sqrt{(5)^2-(x+3)^2}}$$

$= \sin^{-1}\left(\frac{x+3}{5}\right) + c$, c being the integrating constant

26. Question

Evaluate:

$$\int \frac{dx}{\sqrt{7-6x-x^2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{7-6x-x^2}}$$

$$= \int \frac{dx}{\sqrt{16-(x^2+6x+9)}}$$

$$= \int \frac{dx}{\sqrt{(4)^2-(x+3)^2}}$$

$= \sin^{-1}\left(\frac{x+3}{4}\right) + c$, c being the integrating constant

27. Question

Evaluate:

$$\int \frac{dx}{\sqrt{x-x^2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{x-x^2}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-(x^2-2\times x\times\frac{1}{2}+\left(\frac{1}{2}\right)^2)}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-(x-\frac{1}{2})^2}}$$

$$= \sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + c$$

$= \sin^{-1}(2x-1) + c$, c being the integrating constant

28. Question

Evaluate:

$$\int \frac{dx}{\sqrt{8+2x-x^2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{8+2x-x^2}}$$

$$= \int \frac{dx}{\sqrt{9-(x^2-2x+1)}}$$

$$= \int \frac{dx}{\sqrt{(3)^2 - (x-1)^2}}$$

$= \sin^{-1}\left(\frac{x-1}{3}\right) + c$, c being the integrating constant

29. Question

Evaluate:

$$\int \frac{dx}{\sqrt{x^2 - 3x + 2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{x^2 - 3x + 2}}$$

$$= \int \frac{dx}{\sqrt{x^2 - 2 \times x \times \frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2}}$$

$$= \int \frac{dx}{\sqrt{(x - \frac{3}{2})^2 - \frac{1}{4}}}$$

$= \log|(x - \frac{3}{2}) + \sqrt{x^2 - 3x + 2}| + c$, c being the integrating constant

30. Question

Evaluate:

$$\int \frac{dx}{\sqrt{2x^2 + 3x - 2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{2x^2 + 3x - 2}}$$

$$= \int \frac{dx}{\sqrt{2\left(x^2 + 2 \times x \times \frac{3}{4} + \left(\frac{3}{4}\right)^2\right) - \frac{7}{8}}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x + \frac{3}{4})^2 - \left(\frac{\sqrt{7}}{4}\right)^2}}$$

$= \frac{1}{\sqrt{2}} \log|(x + \frac{3}{4}) + \sqrt{2x^2 + 3x - 2}| + c$, c being the integrating constant

31. Question

Evaluate:

$$\int \frac{dx}{\sqrt{2x^2+4x+6}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{2x^2 + 4x + 6}}$$

$$= \int \frac{dx}{\sqrt{2(x^2 + 2x + 1) + 4}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x + 1)^2 + (\sqrt{2})^2}}$$

$= \frac{1}{\sqrt{2}} \log|(x + 1) + \sqrt{2x^2 + 4x + 6}| + c$, c being the integrating constant

32. Question

Evaluate:

$$\int \frac{dx}{\sqrt{1+2x-3x^2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{1 + 2x - 3x^2}}$$

$$= \int \frac{dx}{\sqrt{\left(1 - \frac{1}{3}\right) - 3\left(x^2 - 2 \times x \times \frac{1}{3} + \left(\frac{1}{3}\right)^2\right)}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 - 3\left(x - \frac{1}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{\left(\frac{\sqrt{2}}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{x - \frac{1}{3}}{\frac{\sqrt{2}}{3}}\right) + c$$

$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3x-1}{\sqrt{2}}\right) + c$, c being the integrating constant

33. Question

Evaluate:

$$\int \frac{dx}{\sqrt{x}\sqrt{5-x}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{5x-x^2}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x^2 - 2\times x\times\frac{5}{2} + \left(\frac{5}{2}\right)^2\right)}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - \left(x-\frac{5}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{x-\frac{5}{2}}{\frac{5}{2}}\right) + c$$

$= \sin^{-1}\left(\frac{2x-5}{5}\right) + c$, c being the integrating constant

34. Question

Evaluate:

$$\int \frac{dx}{\sqrt{3+4x-2x^2}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{dx}{\sqrt{3 + 4x - 2x^2}}$$

$$= \int \frac{dx}{\sqrt{5 - 2(x^2 - 2x + 1)}}$$

$$= \int \frac{dx}{\sqrt{\left(\sqrt{5}\right)^2 - 2(x-1)^2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\sqrt{\frac{5}{2}}\right)^2 - (x-1)^2}}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x-1}{\sqrt{\frac{5}{2}}}\right) + c$$

$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}(x-5)}{\sqrt{5}}\right) + c$, c being the integrating constant

35. Question

Evaluate:

$$\int \frac{x^2}{\sqrt{x^6 + 2x^3 + 3}} dx$$

Answer

Tip – $d(x^3) = 3x^2 dx$ i.e. $x^2 dx = (1/3)d(x^3)$

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{x^2 dx}{\sqrt{x^6 + 2x^3 + 3}}$$

$$= \int \frac{\frac{1}{3} d(x^3)}{\sqrt{(x^3)^2 + 2x^3 + 3}}$$

$$= \frac{1}{3} \int \frac{d(x^3)}{\sqrt{(x^3 + 1)^2 + (\sqrt{2})^2}}$$

$= \frac{1}{3} \log|(x^3 + 1) + \sqrt{x^6 + 2x^3 + 3}| + c$, c being the integrating constant

36. Question

Evaluate:

$$\int \frac{(2x + 3)}{\sqrt{x^2 + x + 1}} dx$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{(2x + 3)}{\sqrt{x^2 + x + 1}} dx$$

$$= \int \frac{(2x + 1) + 2}{\sqrt{x^2 + x + 1}} dx$$

$$= \int \frac{(2x + 1)}{\sqrt{x^2 + x + 1}} dx + \int \frac{2}{\sqrt{x^2 + x + 1}} dx$$

Tip – Assuming $x^2 + x + 1 = a^2$, $(2x + 1)dx = 2ada$

$$\therefore \int \frac{(2x + 1)}{\sqrt{x^2 + x + 1}} dx$$

$$= \int \frac{2ada}{a}$$

$$= \int 2da$$

$$= 2a + c_1$$

$$= 2\sqrt{x^2 + x + 1} + c_1$$

$\therefore \int \frac{2}{\sqrt{x^2 + x + 1}} dx$

$= 2\int \frac{dx}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$

$= 2\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1}\right| + c_2$

$\therefore \int \frac{(2x + 1)}{\sqrt{x^2 + x + 1}} dx + \int \frac{2}{\sqrt{x^2 + x + 1}} dx$

$= 2\sqrt{x^2 + x + 1} + 2\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1}\right| + c$, c is the integrating constant

37. Question

Evaluate:

$$\int \frac{(5x + 3)}{\sqrt{x^2 + 4x + 10}} dx$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$\therefore \int \frac{(5x + 3)}{\sqrt{x^2 + 4x + 10}} dx$

$= \int \frac{\frac{5}{2} \times (2x + 4) - 7}{\sqrt{x^2 + 4x + 10}} dx$

$= \frac{5}{2}\int \frac{(2x + 4)}{\sqrt{x^2 + 4x + 10}} dx - \int \frac{7}{\sqrt{x^2 + 4x + 10}} dx$

Tip – Assuming $x^2 + 4x + 10 = a^2$, $(2x + 4)dx = 2ada$

$\therefore \frac{5}{2}\int \frac{(2x + 4)}{\sqrt{x^2 + 4x + 10}} dx$

$= \frac{5}{2}\int \frac{2ada}{a}$

$= \frac{5}{2}\int 2da$

$= 5a + c_1$

$= 5\sqrt{x^2 + 4x + 10} + c_1$

$\therefore \int \frac{7}{\sqrt{x^2 + 4x + 10}} dx$

$= 7\int \frac{dx}{\sqrt{(x + 2)^2 + (\sqrt{6})^2}}$

$= 7\log\left|(x + 2) + \sqrt{x^2 + 4x + 10}\right| + c_2$

$$\therefore \frac{5}{2}\int \frac{(2x + 4)}{\sqrt{x^2 + 4x + 10}}dx - \int \frac{7}{\sqrt{x^2 + 4x + 10}}dx$$

$= 5\sqrt{x^2 + 4x + 10} - 7\log\left|(x + 2) + \sqrt{x^2 + 4x + 10}\right| + c$, c is the integrating constant

38. Question

Evaluate:

$$\int \frac{(4x + 3)}{\sqrt{2x^2 + 2x - 3}}$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{(4x + 3)}{\sqrt{2x^2 + 2x - 3}}dx$$

$$= \int \frac{(4x + 2) + 1}{\sqrt{2x^2 + 2x - 3}}dx$$

$$= \int \frac{(4x + 2)}{\sqrt{2x^2 + 2x - 3}}dx + \int \frac{1}{\sqrt{2x^2 + 2x - 3}}dx$$

Tip – Assuming $2x^2 + 2x - 3 = a^2$, $(4x + 2)dx = 2ada$

$$\therefore \int \frac{(4x + 2)}{\sqrt{2x^2 + 2x - 3}}dx$$

$$= \int \frac{2ada}{a}$$

$$= \int 2da$$

$$= 2a + c_1$$

$$= 2\sqrt{2x^2 + 2x - 3} + c_1$$

$$\therefore \int \frac{1}{\sqrt{2x^2 + 2x - 3}}dx$$

$$= \int \frac{dx}{\sqrt{2\left(x + \frac{1}{2}\right)^2 - \left(\sqrt{\frac{7}{2}}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{7}}{2}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x - \frac{3}{2}}\right| + c_2$$

$$\therefore \int \frac{(4x + 2)}{\sqrt{2x^2 + 2x - 3}}dx + \int \frac{1}{\sqrt{2x^2 + 2x - 3}}dx$$

$= 2\sqrt{2x^2 + 2x - 3} + \frac{1}{\sqrt{2}}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x - \frac{3}{2}}\right| + c$, c is the integrating constant

39. Question

Evaluate:

$$\int \frac{(3-2x)}{\sqrt{2+x-x^2}}dx$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{(3-2x)}{\sqrt{2 + x - x^2}}dx$$

$$= \int \frac{(1-2x) + 2}{\sqrt{2 + x - x^2}}dx$$

$$= \int \frac{(1-2x)}{\sqrt{2 + x - x^2}}dx + \int \frac{2}{\sqrt{2 + x - x^2}}dx$$

Tip – Assuming $2 + x - x^2 = a^2$, $(1 - 2x)dx = 2ada$

$$\therefore \int \frac{(1-2x)}{\sqrt{2 + x - x^2}}dx$$

$$= \int \frac{2ada}{a}$$

$$= 2a + c_1$$

$$= 2\sqrt{2 + x - x^2} + c_1$$

$$\therefore \int \frac{2}{\sqrt{2 + x - x^2}}dx$$

$$= 2\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}}$$

$$= 2\sin^{-1}\frac{\left(x - \frac{1}{2}\right)}{\left(\frac{3}{2}\right)} + c_2$$

$$= 2\sin^{-1}\left(\frac{2x-1}{3}\right) + c_2$$

$$\therefore \int \frac{(1-2x)}{\sqrt{2 + x - x^2}}dx + \int \frac{2}{\sqrt{2 + x - x^2}}dx$$

$= 2\sqrt{2 + x - x^2} + 2\sin^{-1}\left(\frac{2x-1}{3}\right) + c$, c is the integrating constant

40. Question

Evaluate:

$$\int \frac{(x+2)}{\sqrt{2x^2+2x-3}}dx$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\therefore \int \frac{(x + 2)}{\sqrt{2x^2 + 2x - 3}} dx$$

$$= \int \frac{\frac{1}{4} \times (4x + 2) + \frac{3}{2}}{\sqrt{2x^2 + 2x - 3}} dx$$

$$= \frac{1}{4} \int \frac{(4x + 2)}{\sqrt{2x^2 + 2x - 3}} dx + \frac{3}{2} \int \frac{1}{\sqrt{2x^2 + 2x - 3}} dx$$

Tip – Assuming $2x^2 + 2x - 3 = a^2$, $(4x + 2)dx = 2ada$

$$\therefore \frac{1}{4} \int \frac{(4x + 2)}{\sqrt{2x^2 + 2x - 3}} dx$$

$$= \frac{1}{4} \int \frac{2ada}{a}$$

$$= \frac{1}{2} \int da$$

$$= \frac{a}{2} + c_1$$

$$= \frac{\sqrt{2x^2 + 2x - 3}}{2} + c_1$$

$$\therefore \frac{3}{2} \int \frac{1}{\sqrt{2x^2 + 2x - 3}} dx$$

$$= \frac{3}{2} \int \frac{dx}{\sqrt{2\left(x + \frac{1}{2}\right)^2 - \left(\sqrt{\frac{7}{2}}\right)^2}}$$

$$= \frac{3}{2\sqrt{2}} \int \frac{dx}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{7}}{2}\right)^2}}$$

$$= \frac{3}{2\sqrt{2}} \log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x - \frac{3}{2}}\right| + c_2$$

$$\therefore \frac{1}{4} \int \frac{(4x + 2)}{\sqrt{2x^2 + 2x - 3}} dx + \frac{3}{2} \int \frac{1}{\sqrt{2x^2 + 2x - 3}} dx$$

$= \frac{\sqrt{2x^2 + 2x - 3}}{2} + \frac{3}{2\sqrt{2}} \log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x - \frac{3}{2}}\right| + c$, c is the integrating constant

41. Question

Evaluate:

$$\int \frac{(3x + 1)}{\sqrt{5 - 2x - x^2}} dx$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{(3x+1)}{\sqrt{5-2x-x^2}}dx$$

$$= \int \frac{3(x+1)-2}{\sqrt{5-2x-x^2}}dx$$

$$= \int \frac{3(x+1)}{\sqrt{5-2x-x^2}}dx - \int \frac{2}{\sqrt{5-2x-x^2}}dx$$

Tip – Assuming 5 - 2x - x^2 = a^2, (- 2 - 2x)dx = 2ada i.e. (x + 1)dx = - ada

$$\therefore \int \frac{3(x+1)}{\sqrt{5-2x-x^2}}dx$$

$$= -3\int \frac{ada}{a}$$

$$= -3a + c_1$$

$$= -3\sqrt{5-2x-x^2} + c_1$$

$$\therefore \int \frac{2}{\sqrt{5-2x-x^2}}dx$$

$$= 2\int \frac{dx}{\sqrt{\left(\sqrt{6}\right)^2 - (x+1)^2}}$$

$$= 2\sin^{-1}\frac{(x+1)}{\sqrt{6}} + c_2$$

$$\therefore \int \frac{3(x+1)}{\sqrt{5-2x-x^2}}dx - \int \frac{2}{\sqrt{5-2x-x^2}}dx$$

$= -3\sqrt{5-2x-x^2} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$, c is the integrating constant

42. Question

Evaluate:

$$\int \frac{(6x+5)}{\sqrt{6+x-2x^2}}dx$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ where c is the integrating constant

$$\therefore \int \frac{(6x+5)}{\sqrt{6+x-2x^2}}dx$$

$$= \int \frac{\frac{6}{4}(4x-1) + \frac{13}{2}}{\sqrt{6+x-2x^2}}dx$$

$$= \frac{3}{2}\int \frac{(4x-1)}{\sqrt{6+x-2x^2}}dx + \frac{13}{2}\int \frac{1}{\sqrt{6+x-2x^2}}dx$$

Tip – Assuming $6 + x - 2x^2 = a^2$, $(1 - 4x)dx = 2ada$ i.e. $(4x - 1)dx = - 2ada$

$$\therefore \frac{3}{2}\int \frac{(4x-1)}{\sqrt{6+x-2x^2}}dx$$

$$= -\frac{3}{2}\int \frac{2ada}{a}$$

$$= -3a + c_1$$

$$= -3\sqrt{6+x-2x^2} + c_1$$

$$\therefore \frac{13}{2}\int \frac{1}{\sqrt{6+x-2x^2}}dx$$

$$= \frac{13}{2}\int \frac{dx}{\sqrt{\left(\frac{7}{2\sqrt{2}}\right)^2 - 2\left(x-\frac{1}{4}\right)^2}}$$

$$= \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{\left(\frac{7}{4}\right)^2 - \left(x-\frac{1}{4}\right)^2}}$$

$$= \frac{13}{2\sqrt{2}}\sin^{-1}\frac{\left(x-\frac{1}{4}\right)}{\left(\frac{7}{4}\right)} + c_2$$

$$= \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + c_2$$

$$\therefore \frac{3}{2}\int \frac{(4x-1)}{\sqrt{6+x-2x^2}}dx + \frac{13}{2}\int \frac{1}{\sqrt{6+x-2x^2}}dx$$

$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + c$, c is the integrating constant

43. Question

Evaluate:

$$\int\sqrt{\frac{1+x}{x}}dx$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$$\int \sqrt{\frac{1+x}{x}}dx$$

$$= \int \sqrt{\frac{(1+x)^2}{x(1+x)}}dx$$

$$= \int \frac{1+x}{\sqrt{x^2+x}}dx$$

$$= \int \frac{\frac{1}{2}(2x+1)+\frac{1}{2}}{\sqrt{x^2+x}}dx$$

$= \frac{1}{2}\int \frac{2x + 1}{\sqrt{x^2 + x}}dx + \frac{1}{2}\int \frac{dx}{\sqrt{x^2 + x}}$

Tip – Taking $x^2 + x = a^2$, $(2x + 1)dx = 2ada$

$\therefore \frac{1}{2}\int \frac{2x + 1}{\sqrt{x^2 + x}}dx$

$= \frac{1}{2}\int \frac{2ada}{a}$

$= a + c_1$

$= \sqrt{x^2 + x} + c_1$

$\therefore \frac{1}{2}\int \frac{1}{\sqrt{x^2 + x}}dx$

$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$

$= \frac{1}{2}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x}\right| + c_2$

$\therefore \frac{1}{2}\int \frac{2x + 1}{\sqrt{x^2 + x}}dx + \frac{1}{2}\int \frac{dx}{\sqrt{x^2 + x}}$

$= \sqrt{x^2 + x} + \frac{1}{2}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x}\right| + c$, c is the integrating constant

44. Question

Evaluate:

$$\int \frac{(x+2)}{\sqrt{x^2+5x+6}}dx$$

Answer

Formula to be used - $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) + c$ where c is the integrating constant

$\int \frac{(x + 2)}{\sqrt{x^2 + 5x + 6}}dx$

$= \int \frac{\frac{1}{2}(2x + 5) - \frac{1}{2}}{\sqrt{x^2 + 5x + 6}}dx$

$= \frac{1}{2}\int \frac{2x + 5}{\sqrt{x^2 + 5x + 6}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2 + 5x + 6}}$

Tip – Taking $x^2 + 5x + 6 = a^2$, $(2x + 5)dx = 2ada$

$\therefore \frac{1}{2}\int \frac{2x + 5}{\sqrt{x^2 + 5x + 6}}dx$

$= \frac{1}{2}\int \frac{2ada}{a}$

$= a + c_1$

$= \sqrt{x^2 + 5x + 6} + c_1$

$\therefore -\frac{1}{2}\int \frac{1}{\sqrt{x^2 + 5x + 6}}dx$

$= -\frac{1}{2}\int \frac{dx}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$

$= -\frac{1}{2}\log\left|\left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6}\right| + c_2$

$\therefore \frac{1}{2}\int \frac{2x + 5}{\sqrt{x^2 + 5x + 6}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2 + 5x + 6}}$

$= \sqrt{x^2 + 5x + 6} - \frac{1}{2}\log\left|\left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6}\right| + c$, c is the integrating constant

Exercise 14C

1. Question

Evaluate the following integrals:

$\int \sqrt{4 - x^2}dx$

Answer

To Find : $\int \sqrt{4 - x^2}\,dx$

Now, $\int \sqrt{4 - x^2}\,dx$ can be written as $\int \sqrt{2^2 - x^2}\,dx$

Formula Used: $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int \sqrt{2^2 - x^2}\,dx$ is of the form $\int \sqrt{a^2 - x^2}\,dx$,

Hence, $\int \sqrt{2^2 - x^2}\,dx = \frac{1}{2}x\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} + C$

$= \frac{1}{2}x\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} + C$

$= \frac{1}{2}x\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + C$

Therefore, $\int \sqrt{4 - x^2}\,dx = \frac{1}{2}x\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + C$

2. Question

Evaluate the following integrals:

$\int \sqrt{4 - 9x^2}dx$

Answer

To Find : $\int \sqrt{4 - 9x^2}\,dx$

Now, $\int \sqrt{4 - 9x^2}\,dx$ can be written as $\int \sqrt{2^2 - (3x)^2}\,dx$

Formula Used: $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int \sqrt{2^2 - (3x)^2}\,dx$ is of the form $\int \sqrt{a^2 - x^2}\,dx$, Hence, $\int \sqrt{2^2-(3x)^2}\,dx = \frac{1}{2}(3x)\sqrt{2^2-(3x)^2} + \frac{2^2}{2}\sin^{-1}\frac{3x}{2} + C$

$= \frac{x}{2}\sqrt{4-9x^2} + \frac{4}{6}\sin^{-1}\frac{3x}{2} + C$

$= \frac{x}{2}\sqrt{4-9x^2} + \frac{2}{3}\sin^{-1}\frac{3x}{2} + C$

Therefore, $\int \sqrt{4-9x^2}\,dx = \frac{x}{2}\sqrt{4-9x^2} + \frac{2}{3}\sin^{-1}\frac{3x}{2} + C$

3. Question

Evaluate the following integrals:

$\int\sqrt{x^2-2}dx$

Answer

To Find : $\int \sqrt{x^2-2}\,dx$

Now, $\int \sqrt{x^2-2}\,dx$ can be written as $\int \sqrt{x^2-(\sqrt{2})^2}\,dx$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$

Since $\int \sqrt{x^2-(\sqrt{2})^2}\,dx$ is of the form $\int \sqrt{x^2-a^2}\,dx$,

Hence, $\int \sqrt{x^2-(\sqrt{2})^2}\,dx = \frac{x}{2}\sqrt{x^2-(\sqrt{2})^2} - \frac{(\sqrt{2})^2}{2}\log|x+\sqrt{x^2-(\sqrt{2})^2}| + C$

$= \frac{x}{2}\sqrt{x^2-2} - \frac{2}{2}\log|x+\sqrt{x^2-2}| + C$

$= \frac{x}{2}\sqrt{x^2-2} - \log|x+\sqrt{x^2-2}| + C$

Therefore, $\int \sqrt{x^2-2}\,dx = \frac{x}{2}\sqrt{x^2-2} - \log|x+\sqrt{x^2-2}| + C$

4. Question

Evaluate the following integrals:

$\int\sqrt{2x^2-3}dx$

Answer

To Find : $\int\sqrt{2x^2-3}dx$

Now, $\int \sqrt{2x^2-3}\,dx$ can be written as $\int \sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2}\,dx$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$

Since $\int \sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2}\,dx$ is of the form $\int \sqrt{x^2-a^2}\,dx$,

Hence, $\int \sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2}\,dx = \frac{\sqrt{2}x}{2}\sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2} - \frac{(\sqrt{3})^2}{2}\log|\sqrt{2}x+\sqrt{(\sqrt{2}x)^2-(\sqrt{3})^2}| + C$

$= \frac{\sqrt{2}x}{2}\sqrt{2x^2-3} - \frac{3}{2}\log|\sqrt{2}x+\sqrt{2x^2-3}| + C$

$= \frac{x}{2}\sqrt{2x^2-3} - \frac{3}{2\sqrt{2}}$ log $|\sqrt{2}x + \sqrt{2x^2-3}|$+ C

Therefore, $\int \sqrt{2x^2-3}\,dx = \frac{x}{2}\sqrt{2x^2-3} - \frac{3}{2\sqrt{2}}$ log $|\sqrt{2}x + \sqrt{2x^2-3}|$+ C

5. Question

Evaluate the following integrals:

$$\int\sqrt{x^2+5}dx$$

Answer

To Find : $\int \sqrt{x^2+5}\,dx$

Now, $\int \sqrt{x^2+5}\,dx$ can be written as $\int \sqrt{x^2+(\sqrt{5})^2}\,dx$

Formula Used: $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}$ log $|x + \sqrt{x^2+a^2}|$+ C

Since $\int \sqrt{x^2+(\sqrt{5})^2}\,dx$ is of the form $\int \sqrt{x^2+a^2}\,dx$,

Hence, $\int \sqrt{x^2+(\sqrt{5})^2}\,dx = \frac{x}{2}\sqrt{x^2+(\sqrt{5})^2} + \frac{(\sqrt{5})^2}{2}$ log $|x + \sqrt{x^2+(\sqrt{5})^2}|$+ C

$= \frac{x}{2}\sqrt{x^2+5} + \frac{5}{2}$ log $|x + \sqrt{x^2+5}|$+ C

Therefore, $\int \sqrt{x^2+5}\,dx = \frac{x}{2}\sqrt{x^2+5} + \frac{5}{2}$ log $|x + \sqrt{x^2+5}|$+ C

6. Question

Evaluate the following integrals:

$$\int\sqrt{4x^2+9}dx$$

Answer

To Find : $\int \sqrt{4x^2+9}\,dx$

Now, $\int \sqrt{4x^2+9}\,dx$ can be written as $\int \sqrt{(2x)^2+3^2}\,dx$

Formula Used: $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}$ log $|x + \sqrt{x^2+a^2}|$+ C

Since $\int \sqrt{(2x)^2+3^2}\,dx$ is of the form $\int \sqrt{x^2+a^2}\,dx$,

Hence, $\int \sqrt{(2x)^2+3^2}\,dx = \frac{2x}{2}\sqrt{(2x)^2+3^2} + \frac{3^2}{2}$ log $|2x + \sqrt{(2x)^2+3^2}|$+ C

$= \frac{2x}{2}\sqrt{4x^2+9} + \frac{9}{2}$ log $|2x + \sqrt{4x^2+9}\ |$ + C

$= \frac{x}{2}\sqrt{4x^2+9} + \frac{9}{4}$ log $|2x + \sqrt{4x^2+9}|$+ C

Therefore, $\int \sqrt{4x^2+9}\,dx = \frac{x}{2}\sqrt{4x^2+9} + \frac{9}{4}$ log $|2x + \sqrt{4x^2+9}|$+ C

7. Question

Evaluate the following integrals:

$\int\sqrt{3x^2+4}dx$

Answer

To Find : $\int \sqrt{3x^2+4}\,dx$

Now, $\int \sqrt{3x^2+4}\,dx$ can be written as $\int \sqrt{(\sqrt{3}x)^2+2^2}\,dx$

Formula Used: $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$

Since $\int \sqrt{(\sqrt{3}x)^2+2^2}\,dx$ is of the form $\int \sqrt{x^2+a^2}\,dx$,

Hence, $\int \sqrt{(\sqrt{3}x)^2+2^2}\,dx = \frac{\sqrt{3}x}{2}\sqrt{(\sqrt{3}x)^2+2^2} + \frac{2^2}{2}\log|\sqrt{3}x+\sqrt{(\sqrt{3}x)^2+2^2}| + C$

$= \frac{\sqrt{3}x}{2}\sqrt{3x^2+4} + \frac{4}{2}\log|\sqrt{3}x+\sqrt{3x^2+4}\,| + C$

$= \frac{x}{2}\sqrt{3x^2+4} + \frac{2}{\sqrt{3}}\log|\sqrt{3}x+\sqrt{3x^2+4}| + C$

Therefore, $\int \sqrt{3x^2+4}\,dx = \frac{x}{2}\sqrt{3x^2+4} + \frac{2}{\sqrt{3}}\log|\sqrt{3}x+\sqrt{3x^2+4}| + C$

8. Question

Evaluate the following integrals:

$\int \cos x\sqrt{9-\sin^2 x}\,dx$

Answer

To Find : $\int cosx\sqrt{9-\sin^2 x}\,dx$

Now, let sin x = t

$\Rightarrow$cosx dx = dt

Therefore, $\int cosx\sqrt{9-\sin^2 x}\,dx$ can be written as $\int \sqrt{3^2-t^2}\,dt$

Formula Used: $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since , $\int \sqrt{3^2-t^2}\,dt$ is in the form of $\int \sqrt{a^2-x^2}\,dx$ with t as a variable instead of x .

$\Rightarrow \int \sqrt{3^2-t^2}\,dt = \frac{1}{2}t\sqrt{3^2-t^2} + \frac{3^2}{2}\sin^{-1}\frac{t}{3} + C$

$= \frac{t}{2}\sqrt{9-t^2} + \frac{9}{2}\sin^{-1}\frac{t}{3} + C$

Now since sin x = t and cosx dx = dt

$\Rightarrow \int cosx\sqrt{9-\sin^2 x}\,dx = \frac{\sin x}{2}\sqrt{9-\sin^2 x} + \frac{9}{2}\sin^{-1}(\frac{\sin x}{3}) + C$

9. Question

Evaluate the following integrals:

$\int\sqrt{x^2-4x+2}dx$

Answer

To Find : $\int \sqrt{x^2-4x+2}\,dx$

Now, $\int \sqrt{x^2-4x+2}\,dx$ can be written as $\int \sqrt{x^2-4x+2^2-2^2+2}\,dx$

i.e., $\int \sqrt{(x-2)^2-2}\,dx$

Here , let x - 2 = y ⇒dx = dy

Therefore, $\int \sqrt{(x-2)^2-2}\,dx$ can be written as $\int \sqrt{y^2-(\sqrt{2})^2}\,dy$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$

Since $\int \sqrt{y^2-(\sqrt{2})^2}\,dy$ is of the form $\int \sqrt{x^2-a^2}\,dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2-(\sqrt{2})^2}\,dy = \frac{y}{2}\sqrt{y^2-(\sqrt{2})^2} - \frac{(\sqrt{2})^2}{2}\log|y+\sqrt{y^2-(\sqrt{2})^2}| + C$

$= \frac{y}{2}\sqrt{y^2-2} - \frac{4}{2}\log|y+\sqrt{y^2-2}| + C$

$= \frac{y}{2}\sqrt{y^2-2} - 2\log|y+\sqrt{y^2-2}| + C$

Since , x - 2 = y and dx = dy

$\Rightarrow \int \sqrt{(x-2)^2-2}\,dx = \frac{(x-2)}{2}\sqrt{(x-2)^2-2} - 2\log|(x-2)+\sqrt{(x-2)^2-2}| + C$ Therefore,

$\int \sqrt{x^2-4x+2}\,dx = \frac{(x-2)}{2}\sqrt{x^2-4x+2} - 2\log|(x-2)+\sqrt{x^2-4x+2}| + C$

10. Question

Evaluate the following integrals:

$\int \sqrt{x^2+6x-4}\,dx$

Answer

To Find : $\int \sqrt{x^2+6x-4}\,dx$

Now, $\int \sqrt{x^2+6x-4}\,dx$ can be written as $\int \sqrt{x^2+6x+3^2-3^2-4}\,dx$

i.e, $\int \sqrt{(x+3)^2-13}\,dx$

Here , let x + 3 = y ⇒dx = dy

Therefore, $\int \sqrt{(x+3)^2-13}\,dx$ can be written as $\int \sqrt{y^2-(\sqrt{13})^2}\,dy$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$

Since $\int \sqrt{y^2-(\sqrt{13})^2}\,dy$ is of the form $\int \sqrt{x^2-a^2}\,dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2-(\sqrt{13})^2}\,dy = \frac{y}{2}\sqrt{y^2-(\sqrt{13})^2} - \frac{(\sqrt{13})^2}{2}\log|y+\sqrt{y^2-(\sqrt{13})^2}| + C$

$= \frac{y}{2}\sqrt{y^2-13} - \frac{13}{2}\log|y+\sqrt{y^2-13}| + C$

Since , x + 3 = y and dx = dy

$\Rightarrow \int\sqrt{(x+3)^2-13}\,dx = \frac{(x+3)}{2}\sqrt{(x+3)^2-13} - \frac{13}{2}\log|(x+3)+\sqrt{(x+3)^2-13}|+C$

Therefore,

$\int\sqrt{x^2+6x-4}\,dx = \frac{(x+3)}{2}\sqrt{x^2+6x-4} - \frac{13}{2}\log|(x+3)+\sqrt{x^2+6x-4}|+C$

11. Question

Evaluate the following integrals:

$$\int\sqrt{2x-x^2}dx$$

Answer

To Find : $\int\sqrt{2x-x^2}$ dx

Now, $\int\sqrt{2x-x^2}\,dx$ can be written as $\int\sqrt{2x-x^2-1^2+1^2}\,dx$

i.e, $\int\sqrt{1-(x-1)^2}\,dx$

Let x – 1 = y ⇒ dx = dy

Therefore , $\int\sqrt{1-(x-1)^2}\,dx$ becomes $\int\sqrt{1^2-y^2}\,dy$

Formula Used: $\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int\sqrt{1^2-y^2}\,dy$ is of the form $\int\sqrt{a^2-x^2}\,dx$ with change in variable,

Hence $\int\sqrt{1^2-y^2}\,dy = \frac{1}{2}y\sqrt{1^2-y^2} + \frac{1^2}{2}\sin^{-1}\frac{y}{1} + C$

$= \frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}\frac{y}{1} + C$

Here we have x – 1 = y and dx = dy

$\Rightarrow \int\sqrt{1-(x-1)^2}\,dx = \frac{(x-1)}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}\frac{(x-1)}{1} + C$

Therefore , $\int\sqrt{2x-x^2}\,dx = \frac{(x-1)}{2}\sqrt{2x-x^2} + \frac{1}{2}\sin^{-1}(x-1) + C$

12. Question

Evaluate the following integrals:

$$\int\sqrt{1-4x-x^2}dx$$

Answer

To Find : $\int\sqrt{1-4x-x^2}$ dx

Now, $\int\sqrt{1-4x-x^2}\,dx$ can be written as $\int\sqrt{1-4x-x^2-2^2+2^2}\,dx$

i.e, $\int\sqrt{5-(x+2)^2}\,dx$

Let x + 2 = y ⇒ dx = dy

Therefore , $\int\sqrt{5-(x+2)^2}\,dx$ becomes $\int\sqrt{(\sqrt{5})^2-y^2}\,dy$

Formula Used: $\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int \sqrt{(\sqrt{5})^2 - y^2}\, dy$ is of the form $\int \sqrt{a^2 - x^2}\, dx$ with change in variable,

Hence $\int \sqrt{(\sqrt{5})^2 - y^2}\, dy = \frac{1}{2} y \sqrt{(\sqrt{5})^2 - y^2} + \frac{(\sqrt{5})^2}{2} \sin^{-1} \frac{y}{\sqrt{5}}$ + C

$= \frac{y}{2}\sqrt{5 - y^2} + \frac{5}{2} \sin^{-1} \frac{y}{\sqrt{5}}$ + C

Here we have x + 2 = y and dx = dy

$\Rightarrow \int \sqrt{5 - (x+2)^2}\, dx = \frac{(x+2)}{2}\sqrt{5 - (x+2)^2} + \frac{5}{2} \sin^{-1}(\frac{x+2}{\sqrt{5}})$ + C

Therefore , $\int \sqrt{1 - 4x - x^2}\, dx = \frac{(x+2)}{2}\sqrt{1 - 4x - x^2} + \frac{5}{2} \sin^{-1}(\frac{x+2}{\sqrt{5}})$ + C

13. Question

Evaluate the following integrals:

$$\int \sqrt{2ax - x^2}\, dx$$

Answer

To Find : $\int \sqrt{2ax - x^2}$ dx

Now, $\int \sqrt{2ax - x^2}\, dx$ can be written as $\int \sqrt{2ax - x^2 - a^2 + a^2}\, dx$

i.e, $\int \sqrt{a^2 - (x-a)^2}\, dx$

Let x - a = y ⇒ dx = dy

Therefore , $\int \sqrt{a^2 - (x-a)^2}\, dx$ becomes $\int \sqrt{a^2 - y^2}\, dy$

Formula Used: $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$ + C

Since $\int \sqrt{a^2 - y^2}\, dy$ is of the form $\int \sqrt{a^2 - x^2}\, dx$ with change in variable,

Hence $\int \sqrt{a^2 - y^2}\, dy = \frac{1}{2} y\sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a}$ + C

$= \frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a}$ + C

Here we have x - a = y and dx = dy

$\Rightarrow \int \sqrt{a^2 - (x-a)^2}\, dx = \frac{(x-a)}{2}\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1}(\frac{x-a}{a})$ + C

Therefore , $\int \sqrt{2ax - x^2}\, dx = \frac{(x-a)}{2}\sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}(\frac{x-a}{a})$ + C

14. Question

Evaluate the following integrals:

$$\int \sqrt{2x^2 + 3x + 4}\, dx$$

Answer

To Find : $\int \sqrt{2x^2 + 3x + 4}\, dx$

Now , consider $\int \sqrt{2x^2 + 3x + 4}\, dx = \int \sqrt{2[x^2 + \frac{3}{2}x + 2]}\, dx$

$= \sqrt{2}\int\sqrt{x^2+\frac{3}{2}x+2}dx$

$= \sqrt{2}\int\sqrt{x^2+\frac{3}{2}x+(\frac{3}{4})^2-\left(\frac{3}{4}\right)^2+2}dx$

$= \sqrt{2}\int\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}dx$

Let x + $\frac{3}{4}$ = y ⇒dx = dy

Hence $\sqrt{2}\int\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}dx$ becomes $\sqrt{2}\int\sqrt{y^2+(\frac{\sqrt{23}}{4})^2}dy$

Formula Used: $\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$

Now consider $\int\sqrt{y^2+(\frac{\sqrt{23}}{4})^2}dy$ which is in the form of $\int\sqrt{x^2+a^2}\,dx$ with change in variable.

$\Rightarrow \int\sqrt{y^2+(\frac{\sqrt{23}}{4})^2}dy = \frac{y}{2}\sqrt{y^2+(\frac{\sqrt{23}}{4})^2} + \frac{(\frac{\sqrt{23}}{4})^2}{2}\log|y+\sqrt{y^2+(\frac{\sqrt{23}}{4})^2}| + C$

$= \frac{y}{2}\sqrt{y^2+\frac{23}{16}} + \frac{23}{32}\log|y+\sqrt{y^2+\frac{23}{16}}| + C$

Since x + $\frac{3}{4}$ = y and dx = dy

$\Rightarrow \int\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}dx = \frac{1}{8}(4x+3)\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}} + \frac{23}{32}\log|x+\frac{3}{4}+\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}| + C$

Now , $\sqrt{2}\int\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}dx = \frac{\sqrt{2}}{8}(4x+3)\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}} + \frac{23\sqrt{2}}{32}\log|x+\frac{3}{4}+\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}| + C$

Therefore,

$\int\sqrt{2x^2+3x+4}dx = \frac{1}{8}(4x+3)\sqrt{2x^2+3x+4} + \frac{23}{32}\log|(x+\frac{3}{4})+\sqrt{2x^2+3x+4}| + C$

15. Question

Evaluate the following integrals:

$\int\sqrt{x^2+x}dx$

Answer

To Find : $\int\sqrt{x^2+x}\,dx$

Now, $\int\sqrt{x^2+x}\,dx$ can be written as $\int\sqrt{x^2+x+(\frac{1}{2})^2-(\frac{1}{2})^2}dx$

i.e, $\int\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}dx$

Here , let x + $\frac{1}{2}$ = y ⇒dx = dy

Therefore, $\int\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}dx$ can be written as $\int\sqrt{y^2-(\frac{1}{2})^2}dy$

Formula Used: $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$

Since $\int \sqrt{y^2 - (\frac{1}{2})^2}\, dy$ is of the form $\int \sqrt{x^2 - a^2}\, dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2 - (\frac{1}{2})^2}\, dy = \frac{y}{2}\sqrt{y^2 - (\frac{1}{2})^2} - \frac{(\frac{1}{2})^2}{2} \log |y + \sqrt{y^2 - (\frac{1}{2})^2}| + C$

$= \frac{y}{2}\sqrt{y^2 - \frac{1}{4}} - \frac{1}{8} \log |y + \sqrt{y^2 - \frac{1}{4}}| + C$

Since , $x + \frac{1}{2} = y$ and $dx = dy$

$\Rightarrow \int \sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}}\, dx = \frac{1}{4}(2x+1)\sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}} - \frac{1}{8} \log |(x + \frac{1}{2}) + \sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}}| + C$

Therefore,

$\int \sqrt{x^2 + x}\, dx = \frac{1}{4}(2x+1)\sqrt{x^2 + x} - \frac{1}{8} \log |x + \frac{1}{2} + \sqrt{x^2 + x}| + C$

16. Question

Evaluate the following integrals:

$\int \sqrt{x^2 + x + 1} dx$

Answer

To Find : $\int \sqrt{x^2 + x + 1}\, dx$

Now, $\int \sqrt{x^2 + x + 1}\, dx$ can be written as $\int \sqrt{x^2 + x + (\frac{1}{2})^2 - \left(\frac{1}{2}\right)^2 + 1}\, dx$

i.e, $\int \sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}\, dx$

Here , let $x + \frac{1}{2} = y \Rightarrow dx = dy$

Therefore, $\int \sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}\, dx$ can be written as $\int \sqrt{y^2 + (\frac{\sqrt{3}}{2})^2}\, dy$

Formula Used: $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$

Since $\int \sqrt{y^2 + (\frac{\sqrt{3}}{2})^2}\, dy$ is of the form $\int \sqrt{x^2 + a^2}\, dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2 + (\frac{\sqrt{3}}{2})^2}\, dy = \frac{y}{2}\sqrt{y^2 + (\frac{\sqrt{3}}{2})^2} + \frac{(\frac{\sqrt{3}}{2})^2}{2} \log |y + \sqrt{y^2 + (\frac{\sqrt{3}}{2})^2}| + C$

$= \frac{y}{2}\sqrt{y^2 + \frac{3}{4}} + \frac{3}{8} \log |y + \sqrt{y^2 + \frac{3}{4}}| + C$

Since , $x + \frac{1}{2} = y$ and $dx = dy$

$\Rightarrow \int \sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}\, dx = \frac{1}{4}(2x+1)\sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{1}{8} \log |(x + \frac{1}{2}) + \sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}| + C$

Therefore,

$\int \sqrt{x^2 + x + 1}\, dx = \frac{1}{4}(2x+1)\sqrt{x^2 + x + 1} + \frac{3}{8} \log |x + \frac{1}{2} + \sqrt{x^2 + x + 1}| + C$

17. Question

Evaluate the following integrals:

$$\int (2x-5)\sqrt{x^2-4x+3}dx$$

Answer

To Find : $\int (2x-5)\sqrt{x^2-4x+3}\,dx$

Now, let $2x-5$ be written as (2x - 4) -1 and split

Therefore ,

$\int (2x-5)\sqrt{x^2-4x+3}\,dx = \int \{(2x-4)\sqrt{x^2-4x+3} - 1\sqrt{x^2-4x+3}\}dx$

$= \int (2x-4)\sqrt{x^2-4x+3}\,dx - \int \sqrt{x^2-4x+3}dx$

Now solving, $\int (2x-4)\sqrt{x^2-4x+3}\,dx$

Let $x^2-4x+3 = u \Rightarrow dx = \frac{du}{(2x-4)}$

Thus, $\int (2x-4)\sqrt{x^2-4x+3}\,dx$ becomes $\int \sqrt{u}\,du$

Now , $\int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}u^{\frac{3}{2}}$

$= \frac{2}{3}(x^2-4x+3)^{\frac{3}{2}}$

Now solving, $\int \sqrt{x^2-4x+3}dx$

$\int \sqrt{x^2-4x+3}dx = \int \sqrt{x^2-4x+2^2-2^2+3}dx$

$= \int \sqrt{(x-2)^2-1}dx$

Let x - 2 = y ⇒ dx = dy

Then $\int \sqrt{(x-2)^2-1}dx$ becomes $\int \sqrt{y^2-1^2}dy$

Formula Used: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$

Since $\int \sqrt{y^2-1^2}dy$ is in the form of $\int \sqrt{x^2-a^2}\,dx$ with change in variable.

Hence $\int \sqrt{y^2-1^2}dy = \frac{y}{2}\sqrt{y^2-1^2} - \frac{1^2}{2}\log|y+\sqrt{y^2-1^2}| + C$

$= \frac{y}{2}\sqrt{y^2-1} - \frac{1}{2}\log|y+\sqrt{y^2-1}| + C$

Now, since x - 2 = y and dx = dy

$\int \sqrt{(x-2)^2-1}dx = \frac{(x-2)}{2}\sqrt{(x-2)^2-1} - \frac{1}{2}\log|(x-2)+\sqrt{(x-2)^2-1}| + C$

Hence $\int \sqrt{x^2-4x+3}dx = \frac{(x-2)}{2}\sqrt{x^2-4x+3} - \frac{1}{2}\log|(x-2)+\sqrt{x^2-4x+3}| + C$

Therefore , $\int (2x-4)\sqrt{x^2-4x+3}\,dx - \int \sqrt{x^2-4x+3}dx = \frac{2}{3}(x^2-4x+3)^{\frac{3}{2}}$

$-\frac{(x-2)}{2}\sqrt{x^2-4x+3} + \frac{1}{2}\log|(x-2)+\sqrt{x^2-4x+3}| + C$

$i.e. \int (2x-5)\sqrt{x^2-4x+3}\,dx = \frac{2}{3}(x^2-4x+3)^{\frac{3}{2}}$

$-\frac{(x-2)}{2}\sqrt{x^2-4x+3}+\frac{1}{2}\log|x-2+\sqrt{x^2-4x+3}|+C$

18. Question

Evaluate the following integrals:

$$\int (x+2)\sqrt{x^2+x+1}dx$$

Answer

To Find : $\int (x+2)\sqrt{x^2+x+1}\,dx$

Now, let x + 2 be written as $\frac{1}{2}(2x+1)+\frac{3}{2}$ and split

Therefore ,

$\int (x+2)\sqrt{x^2+x+1}\,dx = \int\{\frac{(2x+1)\sqrt{x^2+x+1}}{2}+\frac{3}{2}\sqrt{x^2+x+1}\}dx$

$=\frac{1}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx+\frac{3}{2}\int\sqrt{x^2+x+1}\,dx$

Now solving, $\frac{1}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx$

Let $x^2+x+1=u \Rightarrow dx=\frac{du}{(2x+1)}$

Thus, $\frac{1}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx$ becomes $\frac{1}{2}\int\sqrt{u}\,du$

Now , $\frac{1}{2}\int\sqrt{u}\,du=\frac{1}{2}\int u^{\frac{1}{2}}du=\frac{1}{2}(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1})=\frac{1}{3}u^{\frac{3}{2}}$

$=\frac{1}{3}(x^2+x+1)^{\frac{3}{2}}$

Now solving , $\int\sqrt{x^2+x+1}\,dx$

Now, $\int\sqrt{x^2+x+1}\,dx$ can be written as $\int\sqrt{x^2+x+(\frac{1}{2})^2-\left(\frac{1}{2}\right)^2+1}\,dx$

i.e, $\int\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$

Here , let $x+\frac{1}{2}=y \Rightarrow dx=dy$

Therefore, $\int\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$ can be written as $\int\sqrt{y^2+(\frac{\sqrt{3}}{2})^2}\,dy$

Formula Used: $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+C$

Since $\int\sqrt{y^2+(\frac{\sqrt{3}}{2})^2}\,dy$ is of the form $\int\sqrt{x^2+a^2}\,dx$ with change in variable.

$\Rightarrow \int\sqrt{y^2+(\frac{\sqrt{3}}{2})^2}\,dy=\frac{y}{2}\sqrt{y^2+(\frac{\sqrt{3}}{2})^2}+\frac{(\frac{\sqrt{3}}{2})^2}{2}\log|y+\sqrt{y^2+(\frac{\sqrt{3}}{2})^2}|+C$

$=\frac{y}{2}\sqrt{y^2+\frac{3}{4}}+\frac{3}{8}\log|y+\sqrt{y^2+\frac{3}{4}}|+C$

Since , $x+\frac{1}{2}=y$ and dx = dy

$\Rightarrow \int \sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}dx = \frac{1}{4}(2x+1)\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{1}{8}\log|(x+\frac{1}{2}) + \sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}|+C$

Therefore,

$\int \sqrt{x^2+x+1}\,dx = \frac{1}{4}(2x+1)\sqrt{x^2+x+1} + \frac{3}{8}\log|x+\frac{1}{2}+\sqrt{x^2+x+1}|+C$

Hence ,

$\frac{1}{2}\int(2x+1)\sqrt{x^2+x+1}dx + \frac{3}{2}\int\sqrt{x^2+x+1}dx = \frac{1}{3}(x^2+x+1)^{\frac{3}{2}} + \frac{3}{8}(2x+1)\sqrt{x^2+x+1} + \frac{9}{16}\log|(x+\frac{1}{2}) + \sqrt{x^2+x+1}|+C$

Therefore , $\int(x+2)\sqrt{x^2+x+1}\,dx = \frac{1}{3}(x^2+x+1)^{\frac{3}{2}} + \frac{3}{8}(2x+1)\sqrt{x^2+x+1} + \frac{9}{16}\log|(x+\frac{1}{2}) + \sqrt{x^2+x+1}|+C$

19. Question

Evaluate the following integrals:

$$\int (x-5)\sqrt{x^2+x}dx$$

Answer

To Find : $\int(x-5)\sqrt{x^2+x}\,dx$

Now, let x - 5 be written as $\frac{1}{2}(2x+1) - \frac{11}{2}$ and split

Therefore ,

$\int(x-5)\sqrt{x^2+x}\,dx = \int\{\frac{(2x+1)\sqrt{x^2+x}}{2} - \frac{11}{2}\sqrt{x^2+x}\}dx$

$= \frac{1}{2}\int(2x+1)\sqrt{x^2+x}dx - \frac{11}{2}\int\sqrt{x^2+x}dx$

Now solving, $\frac{1}{2}\int(2x+1)\sqrt{x^2+x}dx$

Let $x^2+x = u \Rightarrow dx = \frac{du}{(2x+1)}$

Thus, $\frac{1}{2}\int(2x+1)\sqrt{x^2+x}dx$ becomes $\frac{1}{2}\int\sqrt{u}\,du$

Now , $\frac{1}{2}\int\sqrt{u}\,du = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) = \frac{1}{3}u^{\frac{3}{2}}$

$= \frac{1}{3}(x^2+x)^{\frac{3}{2}}$

Now solving, $\int\sqrt{x^2+x}dx$

Now, $\int\sqrt{x^2+x}\,dx$ can be written as $\int\sqrt{x^2+x+(\frac{1}{2})^2-(\frac{1}{2})^2}\,dx$

i.e, $\int\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}\,dx$

Here , let $x+\frac{1}{2} = y \Rightarrow dx = dy$

Therefore, $\int\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}\,dx$ can be written as $\int\sqrt{y^2-(\frac{1}{2})^2}\,dy$

Formula Used: $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}$ log |x + $\sqrt{x^2 - a^2}$|+ C

Since $\int \sqrt{y^2 - (\frac{1}{2})^2}\, dy$ is of the form $\int \sqrt{x^2 - a^2}\, dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2 - (\frac{1}{2})^2}\, dy = \frac{y}{2}\sqrt{y^2 - (\frac{1}{2})^2} - \frac{(\frac{1}{2})^2}{2}$ log |y + $\sqrt{y^2 - (\frac{1}{2})^2}$|+ C

$= \frac{y}{2}\sqrt{y^2 - \frac{1}{4}} - \frac{1}{8}$ log |y + $\sqrt{y^2 - \frac{1}{4}}$|+ C

Since , x + $\frac{1}{2}$ = y and dx = dy

$\Rightarrow \int \sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}}\, dx = \frac{1}{4}(2x + 1)\sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}} - \frac{1}{8}$ log |(x + $\frac{1}{2}$) + $\sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}}$|+C

Therefore,

$\int \sqrt{x^2 + x}\, dx = \frac{1}{4}(2x + 1)\sqrt{x^2 + x} - \frac{1}{8}$ log |x + $\frac{1}{2}$ + $\sqrt{x^2 + x}$|+C

Now ,

$\frac{1}{2}\int (2x + 1)\sqrt{x^2 + x}\, dx - \frac{11}{2}\int \sqrt{x^2 + x}\, dx = \frac{1}{3}(x^2 + x)^{\frac{3}{2}} - \frac{11}{8}(2x + 1)\sqrt{x^2 + x} + \frac{11}{16}$ log |x + $\frac{1}{2}$ + $\sqrt{x^2 + x}$|+C

Therefore ,

$\int (x - 5)\sqrt{x^2 + x}\, dx = \frac{1}{3}(x^2 + x)^{\frac{3}{2}} - \frac{11}{8}(2x + 1)\sqrt{x^2 + x} + \frac{11}{16}$ log |x + $\frac{1}{2}$ + $\sqrt{x^2 + x}$|+C

20. Question

Evaluate the following integrals:

$$\int (4x + 1)\sqrt{x^2 - x - 2}\, dx$$

Answer

To Find : $\int (4x + 1)\sqrt{x^2 - x - 2}\, dx$

Now, let 4x + 1 be written as 2(2x - 1) + 3 and split

Therefore ,

$\int (4x + 1)\sqrt{x^2 - x - 2}\, dx = \int \{2(2x - 1)\sqrt{x^2 - x - 2} + 3\sqrt{x^2 - x - 2}\}\, dx$

$= 2\int (2x - 1)\sqrt{x^2 - x - 2}\, dx + 3\int \sqrt{x^2 - x - 2}\, dx$

Now solving, $2\int (2x - 1)\sqrt{x^2 - x - 2}\, dx$

Let $x^2 - x - 2$ = u ⇒ dx = $\frac{du}{(2x-1)}$

Thus, $2\int (2x - 1)\sqrt{x^2 - x - 2}\, dx$ becomes $2\int \sqrt{u}\, du$

Now , $2\int \sqrt{u}\, du = 2\int u^{\frac{1}{2}}\, du = 2(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}) = \frac{4}{3}u^{\frac{3}{2}}$

$= \frac{4}{3}(x^2 - x - 2)^{\frac{3}{2}}$

Now solving, $\int \sqrt{x^2 - x - 2}\, dx$

Now, $\int \sqrt{x^2 - x - 2}\, dx$ can be written as $\int \sqrt{x^2 - x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 2}\, dx$

i.e, $\int \sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}}\, dx$

Here , let x - $\frac{1}{2}$ = y ⇒dx = dy

Therefore, $\int \sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}}\, dx$ can be written as $\int \sqrt{y^2 - (\frac{3}{2})^2}\, dy$

Formula Used: $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$

Since $\int \sqrt{y^2 - (\frac{3}{2})^2}\, dy$ is of the form $\int \sqrt{x^2 - a^2}\, dx$ with change in variable.

$\Rightarrow \int \sqrt{y^2 - (\frac{3}{2})^2}\, dy = \frac{y}{2}\sqrt{y^2 - (\frac{3}{2})^2} - \frac{(\frac{3}{2})^2}{2} \log |y + \sqrt{y^2 - (\frac{3}{2})^2}| + C$

$= \frac{y}{2}\sqrt{y^2 - \frac{9}{4}} - \frac{9}{8} \log |y + \sqrt{y^2 - \frac{9}{4}}| + C$

Since , x - $\frac{1}{2}$ = y and dx = dy

$\Rightarrow \int \sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}}\, dx = \frac{1}{4}(2x - 1)\sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}} - \frac{9}{8} \log |(x - \frac{1}{2}) + \sqrt{(x - \frac{1}{2})^2 - \frac{9}{4}}| + C$

Therefore,

$\int \sqrt{x^2 - x - 2}\, dx = \frac{1}{4}(2x - 1)\sqrt{x^2 - x - 2} - \frac{9}{8} \log |x - \frac{1}{2} + \sqrt{x^2 - x - 2}| + C$

Hence ,

$2\int (2x - 1)\sqrt{x^2 - x - 2}dx + 3\int \sqrt{x^2 - x - 2}dx = \frac{4}{3}(x^2 - x - 2)^{\frac{3}{2}} + \frac{3}{4}(2x - 1)\sqrt{x^2 - x - 2} - \frac{27}{8} \log |x - \frac{1}{2} + \sqrt{x^2 - x - 2}| + C$

Therefore ,

$\int (4x + 1)\sqrt{x^2 - x - 2}\, dx = \frac{4}{3}(x^2 - x - 2)^{\frac{3}{2}} + \frac{3}{4}(2x - 1)\sqrt{x^2 - x - 2} - \frac{27}{8} \log |x - \frac{1}{2} + \sqrt{x^2 - x - 2}| + C$

21. Question

Evaluate the following integrals:

$$\int (x + 1)\sqrt{2x^2 + 3}dx$$

Answer

To Find : $\int (x + 1)\sqrt{2x^2 + 3}\, dx$

Now, $\int (x + 1)\sqrt{2x^2 + 3}\, dx$ can be written as

$\int (x + 1)\sqrt{2x^2 + 3}\, dx = \int \{x\sqrt{2x^2 + 3} + \sqrt{2x^2 + 3}\}\, dx$

$= \int x\sqrt{2x^2 + 3}dx + \int \sqrt{2x^2 + 3}dx$

Now solving, $\int x\sqrt{2x^2 + 3}dx$

Let $2x^2 + 3$ = u ⇒ dx = $\frac{1du}{4x}$

Thus, $\int x\sqrt{2x^2+3}dx$ becomes $\frac{1}{4}\int \sqrt{u}\,du$

Now , $\frac{1}{4}\int \sqrt{u}\,du = \frac{1}{4}\int u^{\frac{1}{2}}du = \frac{1}{4}(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}) = \frac{1}{6}u^{\frac{3}{2}}$

$=\frac{1}{6}(2x^2+3)^{\frac{3}{2}}$

Now solving, $\int \sqrt{2x^2+3}dx$

Now, $\int \sqrt{2x^2+3}\,dx$ can be written as $\int \sqrt{(\sqrt{2}x)^2+(\sqrt{3})^2}\,dx$

Formula Used: $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$

Since $\int \sqrt{2x^2+3}\,dx$ is of the form $\int \sqrt{x^2+a^2}\,dx$.

$\Rightarrow \int \sqrt{2x^2+3}\,dx = \frac{\sqrt{2}x}{2}\sqrt{(\sqrt{2}x)^2+(\sqrt{3})^2} + \frac{(\sqrt{3})^2}{2}\log|\sqrt{2}x + \sqrt{(\sqrt{2}x)^2+(\sqrt{3})^2}| + C$

$= \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\log|\sqrt{2}x + \sqrt{2x^2+3}| + C$

Therefore,

$\int x\sqrt{2x^2+3}dx + \int \sqrt{2x^2+3}dx = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\log|\sqrt{2}x + \sqrt{2x^2+3}| + C$

Hence ,

$\int (x+1)\sqrt{2x^2+3}\,dx = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\log|\sqrt{2}x + \sqrt{2x^2+3}| + C$

22. Question

Evaluate the following integrals:

$\int x\sqrt{1+x-x^2}dx$

Answer

To Find : $\int x\sqrt{1+x-x^2}\,dx$

Now, let x be written as $\frac{1}{2} - \frac{1}{2}(1-2x)$ and split

Therefore ,

$\int x\sqrt{1+x-x^2}\,dx = \int\{\frac{\sqrt{-x^2+x+1}}{2} - \frac{(1-2x)\sqrt{-x^2+x+1}}{2}\}dx$

$= \frac{1}{2}\int(2x-1)\sqrt{-x^2+x+1}\,dx + \frac{1}{2}\int \sqrt{-x^2+x+1}\,dx$

Now solving, $\frac{1}{2}\int(2x-1)\sqrt{-x^2+x+1}\,dx$

Let $-x^2+x+1 = u \Rightarrow dx = \frac{du}{(1-2x)}$

Thus, $\frac{1}{2}\int(2x-1)\sqrt{-x^2+x+1}\,dx$becomes $-\frac{1}{2}\int \sqrt{u}\,du$

Now , $-\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{2}\int u^{\frac{1}{2}}du = -\frac{1}{2}(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}) = -\frac{1}{3}u^{\frac{3}{2}}$

$=-\frac{1}{3}(-x^2+x+1)^{\frac{3}{2}}$

Now solving, $\int \sqrt{-x^2+x+1}\,dx$

$\int \sqrt{-x^2+x+1}\,dx$ can be written as $\int \sqrt{-x^2+x-\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+1}\,dx$

i.e, $\int \sqrt{\frac{5}{4}-(x-\frac{1}{2})^2}\,dx = \frac{1}{2}\int \sqrt{5-(2x-1)^2}\,dx$

let 2x - 1 = y ⇒ dx = $\frac{1dy}{2}$

Therefore , $\frac{1}{4}\int \sqrt{5-(2x-1)^2}\,dx$ becomes $\frac{1}{4}\int \sqrt{(\sqrt{5})^2-y^2}\,dy$

Formula Used: $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$ + C

Since $\int \sqrt{(\sqrt{5})^2-y^2}\,dy$ is of the form $\int \sqrt{a^2-x^2}\,dx$ with change in variable .

Hence, $\int \sqrt{(\sqrt{5})^2-y^2}\,dy = \frac{1}{2}y\sqrt{(\sqrt{5})^2-y^2}+\frac{(\sqrt{5})^2}{2}\sin^{-1}\frac{y}{\sqrt{5}}$ + C

$=\frac{1}{2}y\sqrt{5-y^2}+\frac{5}{2}\sin^{-1}\frac{y}{\sqrt{5}}$ + C

Since , 2x – 1 = y and dx = $\frac{1dy}{2}$

Therefore,

$\frac{1}{4}\int \sqrt{5-(2x-1)^2}\,dx = \frac{1}{8}(2x-1)\sqrt{5-(2x-1)^2}+\frac{5}{8}\sin^{-1}\frac{(2x-1)}{\sqrt{5}}$ + C

i.e, $\int \sqrt{-x^2+x+1}\,dx = \frac{1}{8}(2x-1)\sqrt{-x^2+x+1}+\frac{5}{8}\sin^{-1}\frac{(2x-1)}{\sqrt{5}}$ + C

hence , $\int x\sqrt{1+x-x^2}\,dx = \frac{1}{2}\int (2x-1)\sqrt{-x^2+x+1}\,dx + \frac{1}{2}\int \sqrt{-x^2+x+1}\,dx = =-\frac{1}{3}(-x^2+x+1)^{\frac{3}{2}} + \frac{1}{16}(2x-1)\sqrt{-x^2+x+1}+\frac{5}{16}\sin^{-1}(\frac{2x-1}{\sqrt{5}})$ + C

23. Question

Evaluate the following integrals:

Answer

To Find : $\int (2x-5)\sqrt{2+3x-x^2}\,dx \int (2x-5)\sqrt{2+3x-x^2}\,dx$

Now, let 2x - 5 be written as (2x – 3) -2 and split

Therefore ,

$\int (2x-5)\sqrt{2+3x-x^2}\,dx = \int \{(2x-3)\sqrt{-x^2+3x+2}-2\sqrt{-x^2+3x+2}\}\,dx$

= $\int (2x-3)\sqrt{-x^2+3x+2}\,dx$ - 2 $\int \sqrt{-x^2+3x+2}\,dx$

Now solving, $\int (2x-3)\sqrt{-x^2+3x+2}\,dx$

Let $-x^2+3x+2$= u ⇒ dx = $\frac{du}{(3-2x)}$

Thus, $\int (2x-3)\sqrt{-x^2+3x+2}\,dx$ becomes $-\int \sqrt{u}\,du$

Now , $-\int \sqrt{u}\,du = -\int u^{\frac{1}{2}}\,du = -\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) = -\frac{2}{3}u^{\frac{3}{2}}$

$= -\frac{2}{3}(-x^2+3x+2)^{\frac{3}{2}}$

Now solving, $\int \sqrt{-x^2+3x+2}\,dx$

$\int \sqrt{-x^2+3x+2}\,dx$ can be written as $\int \sqrt{-x^2+3x-\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2+2}\,dx$

i.e, $\int \sqrt{\frac{17}{4}-(x-\frac{3}{2})^2}\,dx$

let x - $\frac{3}{2}$ = y ⇒ dx = dy

Therefore , $\int \sqrt{\frac{17}{4}-(x-\frac{3}{2})^2}\,dx$ becomes $\int \sqrt{(\frac{\sqrt{17}}{2})^2-y^2}\,dy$

Formula Used: $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$ + C

Since $\int \sqrt{(\frac{\sqrt{17}}{2})^2-y^2}\,dy$ is of the form $\int \sqrt{a^2-x^2}\,dx$ with change in variable .

Hence, $\int \sqrt{(\frac{\sqrt{17}}{2})^2-y^2}\,dy = \frac{1}{2}y\sqrt{(\frac{\sqrt{17}}{2})^2-y^2}+\frac{(\frac{\sqrt{17}}{2})^2}{2}\sin^{-1}\frac{y}{\frac{\sqrt{17}}{2}}$ + C

$= \frac{1}{2}y\sqrt{\frac{17}{4}-y^2}+\frac{17}{8}\sin^{-1}\frac{y}{\frac{\sqrt{17}}{2}}$ + C

Since , x - $\frac{3}{2}$ = y and dx = dy

Therefore,

$\int \sqrt{\frac{17}{4}-(x-\frac{3}{2})^2}\,dx = \frac{1}{4}(2x-3)\sqrt{\frac{17}{4}-(x-\frac{3}{2})^2}+\frac{17}{8}\sin^{-1}(\frac{2x-3}{\sqrt{17}})$ + C

i.e, $\int \sqrt{-x^2+3x+2}\,dx = \frac{1}{4}(2x-3)\sqrt{-x^2+3x+2}+\frac{17}{8}\sin^{-1}(\frac{2x-3}{\sqrt{17}})$ + C

hence ,

$\int(2x-5)\sqrt{2+3x-x^2}\,dx = \int(2x-3)\sqrt{-x^2+3x+2}\,dx$ - 2 $\int \sqrt{-x^2+3x+2}\,dx = -\frac{2}{3}(-x^2+3x+2)^{\frac{3}{2}}$ - $\frac{1}{2}(2x-3)\sqrt{-x^2+3x+2}$ - $\frac{17}{4}\sin^{-1}(\frac{2x-3}{\sqrt{17}})$ + C

24. Question

Evaluate the following integrals:

$$\int(6x+5)\sqrt{6+x-2x^2}\,dx$$

Answer

To Find : $\int(6x+5)\sqrt{6+x-2x^2}\,dx$

Now, let 6x + 5 be written as $\frac{13}{2}$ - $\frac{3}{2}$(1 - 4x)and split

Therefore ,

$\int(6x+5)\sqrt{6+x-2x^2}\,dx = \int\{\frac{13\sqrt{-2x^2+x+6}}{2}-\frac{3(1-4x)\sqrt{-2x^2+x+6}}{2}\}dx$

$= \frac{3}{2}\int(4x-1)\sqrt{-2x^2+x+6}\,dx + \frac{13}{2}\int\sqrt{-2x^2+x+6}\,dx$

Now solving, $\int(4x-1)\sqrt{-2x^2+x+6}dx$

Let $-2x^2 + x + 6 = u \Rightarrow dx = \frac{du}{(1-4x)}$

Thus, $\int(4x-1)\sqrt{-2x^2+x+6}\,dx$ becomes $-\int\sqrt{u}\,du$

Now , $-\int\sqrt{u}\,du = -\int u^{\frac{1}{2}}du = -(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}) = -\frac{2}{3}u^{\frac{3}{2}}$

$= -\frac{2}{3}(-2x^2+x+6)^{\frac{3}{2}}$

Now solving, $\int\sqrt{-2x^2+x+6}\,dx$

$\int\sqrt{-2x^2+x+6}\,dx$ can be written as $\int\sqrt{-(\sqrt{2}x)^2+x-\left(\frac{1}{2\sqrt{2}}\right)^2+\left(\frac{1}{2\sqrt{2}}\right)^2+6}\,dx$

i.e, $\int\sqrt{\frac{49}{8}-(\sqrt{2}x-\frac{1}{2\sqrt{2}})^2}\,dx$

let $\sqrt{2}x - \frac{1}{2\sqrt{2}} = y \Rightarrow dx = \frac{dy}{\sqrt{2}}$

Therefore , $\int\sqrt{\frac{49}{8}-(\sqrt{2}x-\frac{1}{2\sqrt{2}})^2}\,dx$ becomes $\int\sqrt{(\frac{7}{2\sqrt{2}})^2-y^2}\,dy$

Formula Used: $\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

Since $\int\sqrt{(\frac{7}{2\sqrt{2}})^2-y^2}\,dy$ is of the form $\int\sqrt{a^2-x^2}\,dx$ with change in variable .

Hence, $\int\sqrt{(\frac{7}{2\sqrt{2}})^2-y^2}\,dy = \frac{1}{2}y\sqrt{(\frac{7}{2\sqrt{2}})^2-y^2} + \frac{(\frac{7}{2\sqrt{2}})^2}{2}\sin^{-1}\frac{y}{\frac{7}{2\sqrt{2}}} + C$

$= \frac{1}{2}y\sqrt{\frac{49}{8}-y^2} + \frac{7}{16}\sin^{-1}\frac{y}{\frac{\sqrt{17}}{2}} + C$

Since , $\sqrt{2}x - \frac{1}{2\sqrt{2}} = y$ and $dx = \frac{dy}{\sqrt{2}}$

Therefore,

$\int\sqrt{\frac{49}{8}-(\sqrt{2}x-\frac{1}{2\sqrt{2}})^2}\,dx = \frac{1}{4\sqrt{2}}(4x-1)\sqrt{\frac{49}{8}-(\sqrt{2}x-\frac{1}{2\sqrt{2}})^2} + \frac{49}{16}\sin^{-1}(\frac{4x-1}{7}) + C$

i.e, $\int\sqrt{-2x^2+x+6}\,dx = \frac{1}{4\sqrt{2}}(4x-1)\sqrt{-2x^2+x+6} + \frac{49}{16}\sin^{-1}(\frac{4x-1}{7}) + C$

hence ,

$\int(6x+5)\sqrt{6+x-2x^2}\,dx = \frac{3}{2}\int(4x-1)\sqrt{-2x^2+x+6}\,dx + \frac{13}{2}\int\sqrt{-2x^2+x+6}\,dx = -(-2x^2+x+6)^{\frac{3}{2}} + \frac{13}{16}(4x-1)\sqrt{-2x^2+x+6} + \frac{637}{32\sqrt{2}}\sin^{-1}(\frac{4x-1}{7}) + C$

25. Question

Evaluate the following integrals:

$\int(x+1)\sqrt{1-x-x^2}\,dx$

Answer

To Find : $\int (x+1)\sqrt{1-x-x^2}\,dx$

Now, let x + 1 be written as $\frac{1}{2}$ - $\frac{1}{2}$(-2x - 1) and split

Therefore ,

$\int (x+1)\sqrt{1-x-x^2}\,dx = \int \{\frac{\sqrt{-x^2-x+1}}{2} - \frac{(-2x-1)\sqrt{-x^2-x+1}}{2}\}\,dx$

$= \frac{1}{2}\int (2x-1)\sqrt{-x^2-x+1}\,dx + \frac{1}{2}\int \sqrt{-x^2-x+1}\,dx$

Now solving, $\int (2x-1)\sqrt{-x^2-x+1}\,dx$

Let $-x^2 - x + 1$ = u ⇒ dx = $\frac{du}{-2x-1}$

Thus, $\int (2x-1)\sqrt{-x^2-x+1}\,dx$ becomes $-\int \sqrt{u}\,du$

Now , $-\int \sqrt{u}\,du = -\int u^{\frac{1}{2}}\,du = -(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}) = -\frac{2}{3}u^{\frac{3}{2}}$

$= -\frac{2}{3}(-x^2-x+1)^{\frac{3}{2}}$

Now solving, $\int \sqrt{-x^2-x+1}\,dx$

$\int \sqrt{-x^2-x+1}\,dx$ can be written as $\int \sqrt{-x^2-x-\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+1}\,dx$

i.e, $\int \sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}\,dx$

let $x+\frac{1}{2}$= y ⇒ dx = dy

Therefore , $\int \sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}\,dx$ becomes $\int \sqrt{(\frac{\sqrt{5}}{2})^2-y^2}\,dy$

Formula Used: $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$ + C

Since $\int \sqrt{(\frac{\sqrt{5}}{2})^2-y^2}\,dy$ is of the form $\int \sqrt{a^2-x^2}\,dx$ with change in variable .

Hence, $\int \sqrt{(\frac{\sqrt{5}}{2})^2-y^2}\,dy = \frac{1}{2}y\sqrt{(\frac{\sqrt{5}}{2})^2-y^2} + \frac{(\frac{\sqrt{5}}{2})^2}{2}\sin^{-1}\frac{y}{\frac{\sqrt{5}}{2}}$ + C

$= \frac{1}{2}y\sqrt{\frac{5}{4}-y^2} + \frac{5}{8}\sin^{-1}\frac{y}{\frac{\sqrt{5}}{2}}$ + C

Since , $x+\frac{1}{2}$= y and dx = dy

Therefore,

$\int \sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}\,dx = \frac{1}{4}(2x+1)\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2} + \frac{5}{8}\sin^{-1}(\frac{2x+1}{\sqrt{5}})$ + C

i.e, $\int \sqrt{-x^2-x+1}\,dx = \frac{1}{4}(2x+1)\sqrt{-x^2-x+1} + \frac{5}{8}\sin^{-1}(\frac{2x+1}{\sqrt{5}})$ + C

hence ,

$\int (x+1)\sqrt{1-x-x^2}\,dx = \frac{1}{2}\int (2x-1)\sqrt{-x^2-x+1}\,dx + \frac{1}{2}\int \sqrt{-x^2-x+1}\,dx = -\frac{1}{3}(-x^2-x+1)^{\frac{3}{2}} +$

$\frac{1}{8}(2x+1)\sqrt{-x^2-x+1}+\frac{5}{16}\sin^{-1}(\frac{2x+1}{\sqrt{5}})$ + C

26. Question

Evaluate the following integrals:

$$\int (x-3)\sqrt{x^2+3x-18}dx$$

Answer

To Find : $\int (x-3)\sqrt{x^2+3x-18}\,dx$

Now, let x - 3 be written as $\frac{1}{2}$(2x + 3) - $\frac{9}{2}$ and split

Therefore ,

$\int (x-3)\sqrt{x^2+3x-18}\,dx = \int\{\frac{(2x+3)\sqrt{x^2+3x-18}}{2} - \frac{9\sqrt{x^2+3x-18}}{2}\}\,dx$

$= \frac{1}{2}\int(2x+3)\sqrt{x^2+3x-18}\,dx$ - $\frac{9}{2}\int\sqrt{x^2+3x-18}\,dx$

Now solving, $\int(2x+3)\sqrt{x^2+3x-18}dx$

Let $x^2+3x-18$= u ⇒ dx = $\frac{du}{2x+3}$

Thus, $\int(2x+3)\sqrt{x^2+3x-18}\,dx$ becomes $\int\sqrt{u}\,du$

Now , $\int\sqrt{u}\,du = \int u^{\frac{1}{2}}\,du$ =($\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}$) = $\frac{2}{3}u^{\frac{3}{2}}$

=$\frac{2}{3}(x^2+3x-18)^{\frac{3}{2}}$

Now solving, $\int\sqrt{x^2+3x-18}\,dx$

$\int\sqrt{x^2+3x-18}\,dx$ can be written as $\int\sqrt{x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-18}\,dx$

i.e, $\int\sqrt{(x+\frac{3}{2})^2-\frac{81}{4}}\,dx$

let $x+\frac{3}{2}$= y ⇒ dx = dy

Therefore, $\int\sqrt{(x+\frac{3}{2})^2-\frac{81}{4}}\,dx$ can be written as $\int\sqrt{y^2-(\frac{9}{2})^2}\,dy$

Formula Used: $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}$ log |x + $\sqrt{x^2-a^2}$|+ C

Since $\int\sqrt{y^2-(\frac{9}{2})^2}\,dy$ is of the form $\int\sqrt{x^2-a^2}\,dx$ with change in variable.

⇒ $\int\sqrt{y^2-(\frac{9}{2})^2}\,dy = \frac{y}{2}\sqrt{y^2-(\frac{9}{2})^2} - \frac{(\frac{9}{2})^2}{2}$ log |y + $\sqrt{y^2-(\frac{9}{2})^2}$|+ C

$= \frac{y}{2}\sqrt{y^2-\frac{81}{4}} - \frac{81}{8}$ log |y + $\sqrt{y^2-\frac{81}{4}}$|+ C

Since , x + $\frac{3}{2}$ = y and dx = dy

$\Rightarrow \int \sqrt{(x+\frac{3}{2})^2-\frac{81}{4}}\,dx = \frac{1}{4}(2x+3)\sqrt{(x+\frac{3}{2})^2-\frac{81}{4}} - \frac{81}{8}\log|(x+\frac{3}{2})+\sqrt{(x+\frac{3}{2})^2-\frac{81}{4}}|+C$

Therefore,

$\int \sqrt{x^2+3x-18}\,dx = \frac{1}{4}(2x+3)\sqrt{x^2+3x-18} - \frac{81}{8}\log|x+\frac{3}{2}+\sqrt{x^2+3x-18}|+C$

Hence ,

$\int (x-3)\sqrt{x^2+3x-18}\,dx = \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\,dx - \frac{9}{2}\int \sqrt{x^2+3x-18}\,dx =$
$\frac{1}{3}(x^2+3x-18)^{\frac{3}{2}} - \frac{9}{8}(2x+3)\sqrt{x^2+3x-18} + \frac{726}{16}\log|x+\frac{3}{2}+\sqrt{x^2+3x-18}|+C$