

# NCERT Solutions for Class-XII Math

## Chapter-6 Exercise- Miscellaneous NCERT Math Class 12

1. Using differentials, find the approximate value of each of the following:

A.  $\left(\frac{17}{81}\right)^{\frac{1}{4}}$

B.  $(33)^{\frac{1}{5}}$

1. (A) Let us consider  $y = x^{1/4}$  and  $x = \left(\frac{16}{81}\right)$  and  $\Delta x = \frac{1}{81}$

$$\text{Then, } \Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$

$$\text{Therefore, } \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

Now,  $dy$  is approximately equal to  $\Delta y$  and is equal to

$$dy = \left(\frac{dy}{dx}\right) \Delta x = \frac{1}{4(x)^{\frac{3}{4}}} (\Delta x)$$

$$= \frac{1}{4 \left(\frac{16}{81}\right)^{\frac{3}{4}}} \left(\frac{1}{81}\right) = \frac{27}{4 \times 8} = \frac{1}{32 \times 3} = \frac{1}{96} = 0.010$$

Therefore, the approximate value of  $\left(\frac{17}{81}\right)^{\frac{1}{4}}$  is  $\frac{2}{3} + 0.010 = 0.677$ .

- (B) Let us consider  $y = x^{\frac{1}{5}}$  and  $x=32$  and  $\Delta x = 1$

$$\text{Then, } \Delta y = (x + \Delta x)^{\frac{1}{5}} - x^{\frac{1}{5}}$$

$$= (33)^{\frac{1}{5}} - (32)^{\frac{1}{5}}$$

$$= (33)^{\frac{1}{5}} - \frac{1}{2}$$

Therefore,  $(33)^{\frac{1}{5}} = \frac{1}{2} + \Delta y$

Now,  $dy$  is approximately equal to  $\Delta y$  and is equal to

$$dy = \left( \frac{dy}{dx} \right) \Delta x = \frac{-1}{5(x)^{\frac{6}{5}}} (\Delta x)$$
$$= \frac{-1}{5(2)^6} (1) = \frac{-1}{320} = -0.003$$

**Therefore, the approximate value of  $(33)^{\frac{1}{5}}$  is  $\frac{1}{2} + (-0.003) = 0.497$**

2. Show that the function given by  $f(x) = \frac{\log x}{x}$  has maximum at  $x = e$ .
2. The given function is  $f(x) = \frac{\log x}{x}$ .

$$f'(x) = \frac{x \left( \frac{1}{x} \right) - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

Now,  $f'(x) = 0$

$$\Rightarrow 1 - \log x = 0$$

$$\Rightarrow \log x = 1$$

$$\Rightarrow \log x = \log e$$

$$\Rightarrow x = e$$

$$f''(x) = \frac{x^2 \left( -\frac{1}{x} \right) - (1 - \log x)(2x)}{x^4}$$

Now,  $= \frac{-x - 2x(1 - \log x)}{x^4}$

$$= \frac{-3 + 2 \log x}{x^3}$$

Now,  $f''(e) = \frac{-3 + 2 \log e}{e^3} = \frac{-3 + 2}{e^3} = \frac{-1}{e^3} < 0$

Therefore, by second derivative test,  $f$  is the maximum at  $x = e$ .

3. The two equal sides of an isosceles triangle with fixed base  $b$  are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

3. Let  $\triangle ABC$  be isosceles where  $BC$  is the base of fixed length  $b$ .

Let the length of the two equal sides of  $\triangle ABC$  be  $a$ .

Now, draw a perpendicular  $AD$  to  $BC$ . Then, we have

In  $\triangle ADC$ , by using Pythagoras theorem,

$$AD = \sqrt{a^2 - \frac{b^2}{4}}$$

$$\text{Then, Area of triangle (A)} = \frac{1}{2} b \sqrt{a^2 - \frac{b^2}{4}}$$

The rate of change of the area with respect to time ( $t$ ) is given by:

$$\frac{dA}{dt} = \frac{1}{2} b \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}} \frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}} \frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3cm per second.

$$\frac{da}{dt} = -3 \text{ cm/s}$$

$$\text{Then } \frac{dA}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

And when  $a = b$  we have,

$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}b$$

Therefore, if the two sides are equal to the base, then the area of the triangle is decreasing at the rate of  $\sqrt{3}b \text{ cm}^2/\text{s}$ .

4. Find the equation of the normal to curve  $x^2 = 4y$  which passes through the point  $(1, 2)$ .

4. The equation of the given curve is  $y^2 = 4x$ . Differentiating with respect to  $x$ , we have:

$$\Rightarrow 2y \frac{dy}{dx} = 4$$

$$\Rightarrow \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$$\Rightarrow \therefore \left. \frac{dy}{dx} \right|_{(1,2)} = \frac{2}{2} = 1$$

Now, the slope of the normal at point  $(1,2)$  is  $\left. \frac{dy}{dx} \right|_{(1,2)} = \frac{-1}{1} = -1$ .

Equation of the normal at  $(1,2)$  is  $y - 2 = -1(x - 1)$ .

$$y - 2 = -x + 1$$

$$x + y - 3 = 0$$

5. Show that the normal at any point  $\theta$  to the curve  $x = a \cos \theta + a \theta \sin \theta$ ,  $y = a \sin \theta - a \theta \cos \theta$  is at a constant distance from the origin.
5. We have  $x = a \cos \theta + a \theta \sin \theta$ ,

$$\Rightarrow \frac{dx}{d\theta} = -a \sin \theta + a \sin \theta + a \theta \cos \theta = a \theta \cos \theta$$

$$\text{And } y = a \sin \theta - a \theta \cos \theta$$

$$\Rightarrow \frac{dy}{d\theta} = a \cos \theta - a \cos \theta + a \theta \sin \theta = a \theta \sin \theta$$

$$\text{So, } \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a \theta \sin \theta}{a \theta \cos \theta} = \tan \theta$$

Then, Slope of the normal at any point  $\theta$  is  $-\frac{1}{\tan \theta}$ .

The equation of the normal at a given point  $(x,y)$  is:

$$y - a \sin \theta + a \theta \cos \theta = -\frac{1}{\tan \theta} (x - a \cos \theta - a \theta \sin \theta)$$

$$\Rightarrow y \sin \theta - a \sin^2 \theta + a \theta \sin \theta \cos \theta = -x \cos \theta + a \cos^2 \theta + a \theta \sin \theta \cos \theta$$

$$\Rightarrow x \cos \theta + y \sin \theta - a(\sin^2 \theta + \cos^2 \theta) = 0$$

$$\Rightarrow x \cos \theta + y \sin \theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$$\frac{|-a|}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = \frac{|-a|}{\sqrt{1}} = |-a|, \text{ which is independent of } \theta.$$

**Therefore, the perpendicular distance of the normal from the origin is constant.**

6. Find the intervals in which the function  $f$  given by

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x} \text{ is}$$

(i) strictly increasing

(ii) strictly decreasing.

6.  $f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$

$$\begin{aligned}
\therefore f'(x) &= \frac{(2 + \cos x)(4\cos x - 2 - \cos x + x\sin x) - (4\sin x - 2x - x\cos x)(-\sin x)}{(2 + \cos x)^2} \\
&= \frac{(2 + \cos x)(3\cos x - 2 + x\sin x) + \sin x(4\sin x - 2x - x\cos x)}{(2 + \cos x)^2} \\
&= \frac{6\cos x - 4 + 2x\sin x + 3\cos^2 x - 2\cos x + x\sin x\cos x + 4\sin^2 x - 2x\sin x - x\sin x\cos x}{(2 + \cos x)^2} \\
&= \frac{4\cos x - 4 + 3\cos^2 x + 4\sin^2 x}{(2 + \cos^2 x)^2} \\
&= \frac{4\cos x - 4 + 3\cos x + 4 - 4\cos^2 x}{(2 + \cos x)^2} \\
&= \frac{4\cos x - \cos^2 x}{(2 + \cos x)^2} = \frac{\cos x(4 - \cos x)}{(2 + \cos x)^2}
\end{aligned}$$

Now,  $f'(x) = 0$

$$\Rightarrow \cos x = 0 \text{ or } \cos x = 4$$

But,  $\cos x \neq 4$

$$\Rightarrow \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Now,  $x = \frac{\pi}{2}$  and  $x = \frac{3\pi}{2}$  divides  $(0, 2\pi)$  into three disjoint intervals i.e.,

$$\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \text{ and } \left(\frac{3\pi}{2}, 2\pi\right)$$

In intervals  $\left(0, \frac{\pi}{2}\right)$  and  $\left(\frac{3\pi}{2}, 2\pi\right)$ ,  $f'(x) > 0$ .

Thus,  $f(x)$  is increasing for  $0 < x < \frac{\pi}{2}$  and  $\frac{3\pi}{2} < x < 2\pi$ .

In the interval  $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ ,  $f'(x) < 0$ .

Thus,  $f(x)$  is decreasing for  $\frac{\pi}{2} < x < \frac{3\pi}{2}$ .

7. Find the intervals in which the function  $f$  given by  $f(x) = x^3 + \frac{1}{x^3}$ ,  $x \neq 0$  is

- (i) Increasing
- (ii) decreasing.

7. It is given that  $f(x) = x^3 + \frac{1}{x^3}$

$$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$

Then,  $f'(x) = 0$

$$\Rightarrow 3x^6 - 3 = 0$$

$$\Rightarrow x^6 = 1$$

$$\Rightarrow x = \pm 1$$

Now, the points  $x = 1$  and  $x = -1$  divide the real line into three disjoint intervals  $(-\infty, -1)$ ,  $(-1, 1)$  and  $(1, \infty)$ .

In interval  $(-\infty, -1)$  and  $(1, \infty)$  when  $x < -1$  and  $x > 1$  then  $f'(x) > 0$

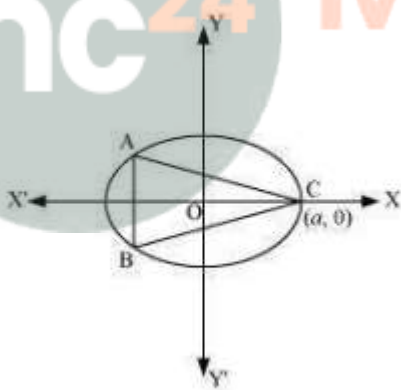
**Therefore, when  $x < -1$  and  $x > 1$ ,  $f$  is increasing.**

And, in interval  $(-1, 1)$  when  $-1 < x < 1$  then  $f'(x) < 0$ .

**Therefore, when  $-1 < x < 1$ ,  $f$  is decreasing.**

8. Find the maximum area of an isosceles triangle inscribed in the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with its vertex at one end of the major axis.

8.



The given ellipse is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let the major axis be along the  $x$ -axis.

Let  $ABC$  be the triangle inscribed in the ellipse where vertex  $C$  is at  $(a, 0)$ .

Since the ellipse is symmetrical with respect to the  $x$ -axis and  $y$ -axis, we can assume the coordinates of  $A$  to be  $(-x_1, y_1)$  and the coordinates of  $B$  to be  $(-x_1, -y_1)$ .

Now, we have  $y_1 = \pm \frac{b}{a} \sqrt{a^2 - x_1^2}$

Coordinates of A are  $\left(-x_1, \frac{b}{a}\sqrt{a^2 - x_1^2}\right)$  and the coordinates of B are

$$\Rightarrow \left(x_1, -\frac{b}{a}\sqrt{a^2 - x_1^2}\right)$$

As the point  $(x_1, y_1)$  lies on the ellipse, the area of triangle  $ABC(A)$  is given by,

$$A = \frac{1}{2} \left| a \left( \frac{2b}{a}\sqrt{a^2 - x_1^2} \right) + (-x_1) \left( -\frac{b}{a}\sqrt{a^2 - x_1^2} \right) + (-x_1) \left( -\frac{b}{a}\sqrt{a^2 - x_1^2} \right) \right|$$

$$\Rightarrow A = b\sqrt{a^2 - x_1^2} + x_1 \frac{b}{a}\sqrt{a^2 - x_1^2}$$

$$\therefore \frac{dA}{dx_1} = \frac{-2x_1 b}{2\sqrt{a^2 - x_1^2}} + \frac{b}{a}\sqrt{a^2 - x_1^2} - \frac{2bx_1^2}{a2\sqrt{a^2 - x_1^2}}$$

$$= \frac{b}{a\sqrt{a^2 - x_1^2}} \left[ -x_1 a + (a^2 - x_1^2) - x_1^2 \right]$$

$$= \frac{b(-2x_1^2 - x_1 a + a^2)}{a\sqrt{a^2 - x_1^2}}$$

Now,  $\frac{dA}{dx_1} = 0$

$$\Rightarrow -2x_1^2 - x_1 a + a^2 = 0$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{a^2 - 4(-2)(a^2)}}{2(-2)}$$

$$= \frac{a \pm \sqrt{9a^2}}{-4}$$

$$= \frac{a \pm 3a}{-4}$$

$$\Rightarrow x_1 = -a, \frac{a}{2}$$

But,  $x_1$  cannot be equal to  $a$ .

$$\therefore x_1 = \frac{a}{2} \Rightarrow y_1 = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a}\sqrt{3} = \frac{\sqrt{3}b}{2}$$

$$\begin{aligned} \text{Now, } \frac{d^2A}{dx_1^2} &= \frac{b}{a} \left\{ \frac{\sqrt{a^2 - x_1^2}(-4x_1 - a) - (-2x_1^2 - x_1a + a^2) \frac{(-2x_1)}{2\sqrt{a^2 - x_1^2}}}{a^2 - x_1^2} \right\} \\ &= \frac{b}{a} \left\{ \frac{(a^2 - x_1^2)(-4x_1 - a) + x_1(-2x_1^2 - x_1a + a^2)}{(a^2 - x_1^2)^{\frac{3}{2}}} \right\} \\ &= \frac{b}{a} \left\{ \frac{2x^3 - 3a^2x - a^3}{(a^2 - x_1^2)^{\frac{3}{2}}} \right\} \end{aligned}$$

Also, when  $x_1 = \frac{a}{2}$ , then

$$\begin{aligned} \frac{d^2A}{dx_1^2} &= \frac{b}{a} \left\{ \frac{2\frac{a^3}{8} - 3\frac{a^3}{2} - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}} \right\} = \frac{b}{a} \left\{ \frac{\frac{a^3}{4} - \frac{3}{2}a^3 - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}} \right\} \\ &= -\frac{b}{a} \left\{ \frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}} \right\} < 0 \end{aligned}$$

Thus, the area is the maximum when  $x_1 = \frac{a}{2}$ .

Maximum area of the triangle is given by,

$$\begin{aligned} A &= b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}} \\ &= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2} \\ &= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab \end{aligned}$$

9. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is  $8 \text{ m}^3$ . If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?
9. Let  $l$ ,  $b$ , and  $h$  be the length, breadth and height of the tank respectively.

then, we have  $h = 2\text{m}$

$$\text{Volume of the tank} = 8 \text{ m}^3$$

$$\text{Volume of the tank} = l \times b \times h$$

$$\Rightarrow 8 = l \times b \times 2$$

$$\Rightarrow lb = 4$$

$$\Rightarrow b = \frac{4}{l}$$

$$\text{Now, area of the base} = lb = 4$$

$$\text{Area of the 4 walls (A)} = 2h(l + b)$$

$$\therefore A = 4 \left( 1 + \frac{4}{l^2} \right)$$

$$\text{Now, } \frac{dA}{dl} = 0$$

$$\Rightarrow 1 - \frac{4}{l^2} = 0$$

$$\Rightarrow l^2 = 4$$

$$\Rightarrow l = \pm 2$$

Since, length cannot be negative therefore  $l = 2$ .

$$\Rightarrow b = 2$$

$$\text{Now, } \frac{d^2A}{dl^2} = \frac{32}{l^3}$$

$$\text{When } l = 2, \frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0$$

Then, by second derivative test, the area is the minimum when  $l = 2$ .

We have,  $l = b = h = 2$

Therefore, Cost of building the base = Rs  $70 \times (lb) = \text{Rs } 70 (4) = \text{Rs } 280$ .

Cost of building the walls = Rs  $2h (l + b) \times 45 = \text{Rs } 90(2)(2+2)$

= Rs  $8(90) = \text{Rs } 720$ .

Required total cost = Rs  $(280 + 720) = \text{Rs } 1000$ .

**Therefore, the total cost of the tank will be Rs 1000.**

10. The sum of the perimeter of a circle and square is  $k$ , where  $k$  is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.
10. Let  $r$  be the radius of the circle and  $a$  be the side of the square.  
Then, we have:

$$2\pi r + 4a = k \text{ (where } k \text{ is constant)}$$

$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square ( $A$ ) is given by,

$$\Rightarrow A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$

$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

$$\text{Now, } \frac{dA}{dr} = 0$$

$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$

$$8r = k - 2\pi r$$

$$\Rightarrow (8 + 2\pi)r = k$$

$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$

$$\text{Now, } \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$

$$\therefore \text{When } r = \frac{k}{2(4\pi)}, \frac{d^2A}{dr^2} > 0.$$

The sum of the areas is least when  $r = \frac{k}{2(4\pi)}$ .

$$\text{When } r = \frac{k}{2(4\pi)}, a = \frac{k - 2\pi \left[ \frac{k}{2(4\pi)} \right]}{4} = \frac{k(4\pi)\pi - k}{44(\pi)} = \frac{4k}{4(\pi)} = \frac{k}{\pi} = 2r.$$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

- 11.** A window is in the form of a rectangle surmounted by a semi-circular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening. **Ans.**

Let  $x$  and  $y$  be the length and breadth of the rectangular window.

- 11.** Radius of the semi-circular opening =  $\frac{x}{2}$

It is given that the perimeter of the window is 10m.

$$\Rightarrow x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x\left(1 + \frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x\left(1 + \frac{\pi}{2}\right)$$

$$\Rightarrow y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

Therefore, Area of the window (A) is given by

$$= xy + \frac{\pi}{2}\left(\frac{x}{y}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right] + \frac{\pi x^2}{8}$$

$$= 5x - x^2\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi x^2}{8}$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{\pi}{4} + \frac{1}{2}\right) + \frac{\pi}{4}x$$

$$= 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x$$

$$\therefore \frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

Now,  $\frac{dA}{dx} = 0$ , then

$$\Rightarrow 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$

$$\Rightarrow x\left(1 + \frac{\pi}{4}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}$$

Then, when  $x = \frac{20}{\pi + 4}$  then  $\frac{d^2A}{dx^2} < 0$ .

Therefore, by second derivative test, the area is maximum when length

$$x = \frac{20}{\pi + 4} \text{ m.}$$

$$\text{Now, } y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4} \text{ m}$$

Therefore, the required dimensions of the window to admit maximum light is given by length

$$= \frac{20}{\pi + 4} \text{ m and breadth} = \frac{10}{\pi + 4} \text{ m.}$$

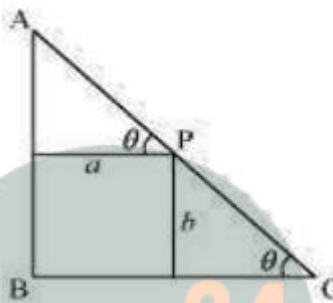
12. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.

Show that the maximum length of the hypotenuse is  $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$

12. Let  $\triangle ABC$  be right-angled at B. Let  $AB = x$  and  $BC = y$ .

Let P be a point on the hypotenuse of the triangle such that P is at a distance of a and b from the sides AB and BC respectively.

Let  $\therefore C = \theta$



We have,

$$AC = \sqrt{x^2 + y^2}$$

Now,

$$PC = b \operatorname{cosec} \theta$$

$$\text{And, } AP = a \sec \theta$$

$$AC = AP + PC$$

$$AC = b \operatorname{cosec} \theta + a \sec \theta \dots (1)$$

$$\therefore \frac{d(AC)}{d\theta} = -b \operatorname{cosec} \theta \cot \theta + a \sec \theta \tan \theta$$

$$\therefore \frac{d(AC)}{d\theta} = 0$$

$$\Rightarrow a \sec \theta \tan \theta = b \operatorname{cosec} \theta \cot \theta$$

$$\Rightarrow \frac{a}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta} = \frac{b}{\sin \theta} \frac{\cos \theta}{\sin \theta}$$

$$\Rightarrow a \sin^3 \theta = b \cos^3 \theta$$

$$\Rightarrow (a)^{\frac{1}{3}} \sin \theta = (b)^{\frac{1}{3}} \cos \theta$$

$$\Rightarrow \tan \theta = \left( \frac{b}{a} \right)^{\frac{1}{3}}$$

$$\therefore \sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \text{ and } \cos\theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}$$

It can be clearly shown that  $\frac{d^2(AC)}{d\theta^2} < 0$  when  $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$ .

Therefore, by second derivative test, the length of the hypotenuse is the maximum when

$$\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

Now, when  $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$ , we have:

$$\Rightarrow AC = \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{\frac{1}{b^{\frac{1}{3}}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{\frac{1}{a^{\frac{1}{3}}}} \text{ Using}$$

$$\Rightarrow = \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}} \left( b^{\frac{2}{3}} + a^{\frac{2}{3}} \right)$$

$$= \left( a^{\frac{2}{3}} + b^{\frac{2}{3}} \right)^{\frac{3}{2}}$$

[Using (1) and (2)]

Hence, the maximum length of the hypotenuses is  $\left( a^{\frac{2}{3}} + b^{\frac{2}{3}} \right)^{\frac{3}{2}}$ .

**13.** Find the points at which the function  $f$  given by  $f(x) = (x-2)^4(x+1)^3$  has

(i) local maxima (ii) local minima

(iii) point of inflexion

**13.** It is given that function is  $f(x) = (x-2)^4(x+1)^3$

$$\Rightarrow f'(x) = 4(x-2)^3(x+1)^3 + 3(x+1)^2(x-2)^4$$

$$= (x-2)^3(x+1)^2[4(x+1) + 3(x-2)]$$

$$= (x-2)^3(x+1)^2(7x-2)$$

Now,  $f'(x) = 0$

$$\Rightarrow x = -1 \text{ and } x = \frac{2}{7} \text{ or } x = 2$$

Now, for values of  $x$  close to  $\frac{2}{7}$  and to the left of  $\frac{2}{7}$ ,

$$f'(x) > 0.$$

Also, for values of  $x$  close to  $\frac{2}{7}$  and to the right of  $\frac{2}{7}$ ,  $f'(x) < 0$ .

**Then,  $x = \frac{2}{7}$  is the point of local maxima.**

Now, for values of  $x$  close to 2 and to the left of 2,  $f'(x) < 0$ .

Also, for values of  $x$  close to 2 and to the right of 2,  $f'(x) > 0$ .

**Then,  $x = 2$  is the point of local minima.**

Now, as the value of  $x$  varies through -1,  $f'(x)$  does not change its sign.

**Then,  $x = -1$  is the point of inflexion.**

14. Find the absolute maximum and minimum values of the function  $f$  given by  $f(x) = \cos^2 x + \sin x$ ,  $x \in [0, \pi]$

14.  $f(x) = \cos^2 x + \sin x$

$$f'(x) = 2\cos x(-\sin x) + \cos x$$

$$= -2\sin x \cos x + \cos x$$

$$\text{Now, } f'(x) = 0$$

$$\Rightarrow 2\sin x \cos x = \cos x \Rightarrow \cos x(2\sin x - 1) = 0$$

$$\Rightarrow \sin x = \frac{1}{2} \text{ or } \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{6}, \text{ or } \frac{\pi}{2} \text{ as } x \in [0, \pi]$$

Now, evaluating the value of  $f$  at critical points  $x = \frac{\pi}{2}$  and  $x = \frac{\pi}{6}$  and at the end points of the interval  $[0, \pi]$  (i.e., at  $x = 0$  and  $x = \pi$ ), we have:

$$f\left(\frac{\pi}{6}\right) = \cos^2 \frac{\pi}{6} + \sin \frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$$

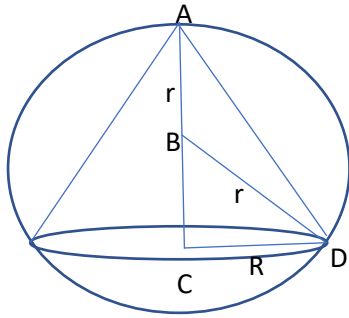
$$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$$

$$f(\pi) = \cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1$$

$$f\left(\frac{\pi}{2}\right) = \cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2} = 0 + 1 = 1$$

Hence, the absolute maximum value of  $f$  is  $\frac{5}{4}$  occurring at  $x = \frac{\pi}{6}$  and the absolute minimum value of  $f$  is 1 occurring at  $x = 0, \frac{\pi}{2}$ , and  $\pi$ .

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius  $r$  is  $\frac{4r}{3}$ .
15. Let  $R$  and  $h$  be the radius and the height of the cone respectively.



The volume ( $V$ ) of the cone is given by;

$$V = \frac{1}{3} \pi r^2 h$$

Now, from the right triangle  $BCD$ , we get,

$$BC = \sqrt{r^2 - R^2}$$

$$\therefore h = r + \sqrt{r^2 - R^2}$$

$$V = \frac{1}{3} \pi R^2 \left( r + \sqrt{r^2 - R^2} \right) = \frac{1}{3} \pi R^2 r + \frac{1}{3} \pi R^2 \sqrt{r^2 - R^2}$$

$$\frac{dV}{dR} = \frac{2}{3} \pi R r + \frac{2}{3} \pi R \sqrt{r^2 - R^2} + \frac{R^2}{3} \cdot \frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3} \pi R r + \frac{2}{3} \pi R \sqrt{r^2 - R^2} + \frac{R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3} \pi R r + \frac{2\pi R(r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3} \pi R r + \frac{2\pi R r^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

Now, if  $\frac{dV}{dR} = 0$ , then,

$$\frac{2}{3} \pi R r = -\frac{2\pi R r^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2(r^2 - R^2) = (3R^2 - 2r^2)^2$$

$$\Rightarrow 4r^4 - 4r^2 R^2 = 9R^4 + 4r^4 - 12R^2 r^2$$

$$\Rightarrow 9R^4 - 8r^2R^2 = 0$$

$$\Rightarrow 9R^2 = 8r^2$$

$$\Rightarrow R^2 = \frac{8r^2}{9}$$

$$\text{Now, } \frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}(2\pi r^2 - 9\pi R^2) - (2\pi Rr^2 - 3\pi R^3)(-6R)}{9(r^2 - R^2)} \frac{1}{2\sqrt{r^2 - R^2}}$$

Now, when  $R^2 = \frac{8r^2}{9}$ , it can be shown that  $\frac{d^2V}{dR^2} < 0$ .

Therefore, the volume is the maximum when  $R^2 = \frac{8r^2}{9}$ .

$$\text{When } R^2 = \frac{8r^2}{9},$$

$$\text{Height of the cone} = r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}.$$

Therefore, it can be seen that the altitude of the circular cone of maximum volume that can be inscribed in a sphere of radius  $r$  is  $\frac{4r}{3}$ .

16. Let  $f$  be a function defined on  $[a, b]$  such that  $f'(x) > 0$ , for all  $x \in (a, b)$ . Then prove that  $f$  is an increasing function on  $(a, b)$ .

16. Since,  $f'(x) > 0$  on  $(a, b)$

Then,  $f$  is a differentiating function  $(a, b)$

Also, every differentiable function is continuous,

Therefore,  $f$  is continuous on  $[a, b]$

Let  $x_1, x_2 \in (a, b)$  and  $x_2 > x_1$  then by LMV theorem, there exists  $c \in (a, b)$  s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$$

$$\Rightarrow f(x_2) - f(x_1) > 0 \text{ as } x_2 > x_1 \text{ and } f'(x) > 0$$

$$\Rightarrow f(x_2) > f(x_1)$$

$$\therefore \text{for } x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Therefore,  $f$  is an increasing function.

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius  $R$  is  $\frac{2R}{\sqrt{3}}$ . Also find the maximum volume.

17. Let  $r$  and  $h$  be the radius and the height of the cylinder respectively.

$$\text{Now, } h = 2\sqrt{R^2 - r^2}$$

The volume ( $V$ ) of the cylinder is given by:

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

$$\frac{dV}{dr} = 4\pi r \sqrt{R^2 - r^2} + \frac{2\pi r^2 (-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r \sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r (R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

$$\text{Now, if } \frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\text{Now, } \frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2} (4\pi R^2 - 18\pi r^2) - (4\pi r R^2 - 6\pi r^3) \frac{(-2r)}{2\sqrt{R^2 - r^2}}}{(R^2 - r^2)}$$

$$= \frac{(R^2 - r^2)(4\pi R^2 - 18\pi r^2) + (4\pi r R^2 - 6\pi r^3)}{(R^2 - r^2)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{(R^2 - r^2)^{\frac{3}{2}}}$$

Now, we can see that at  $r^2 = \frac{2R^2}{3}$ ,  $\frac{d^2V}{dr^2} < 0$ .

Therefore, the volume is the maximum when  $r^2 = \frac{2R^2}{3}$ .

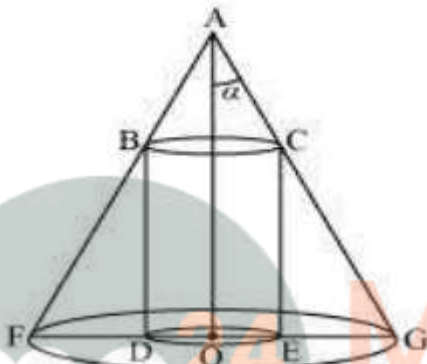
When  $r^2 = \frac{2R^2}{3}$ , the height of the cylinder is  $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$ .

Therefore, the volume of the cylinder is the maximum when the height of the cylinder is  $\frac{2R}{\sqrt{3}}$ .

Hence Proved.

18. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height  $h$  and semi vertical angle  $\alpha$  is one-third that of the cone and the greatest volume of cylinder is  $\frac{4}{27}\pi h^3 \tan^2 \alpha$ .

18. The given right circular cone of fixed height ( $h$ ) and semi-vertical angle ( $\alpha$ ) can be drawn as:



Here, a cylinder of radius  $R$  and height  $H$  is inscribed in the cone.

Then,  $GAO = \alpha$ ,  $OG = r$ ,  $OA = h$ ,  $OE = R$ , and  $CE = H$ .

We have,

$$r = h \tan \alpha$$

Now, since  $\triangle VAOG$  is similar to  $\triangle VCEG$ , we have:

$$\Rightarrow \frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r - R} \quad [EG = OG - OE]$$

$$\Rightarrow H = \frac{h}{r}(r - R) = \frac{h}{h \tan \alpha}(h \tan \alpha - R) = \frac{1}{\tan \alpha}(h \tan \alpha - R)$$

Now, the volume ( $V$ ) of the cylinder is given by,

$$V = \pi R^2 H = \frac{\pi R^2}{\tan \alpha}(h \tan \alpha - R) = \pi R^2 h - \frac{\pi R^3}{\tan \alpha}$$

$$\therefore \frac{dV}{dR} = 2\pi R h - \frac{3\pi R^2}{\tan \alpha}$$

$$\text{Now, } \frac{dV}{dR} = 0$$

$$\Rightarrow 2\pi R h = \frac{3\pi R^2}{\tan\alpha}$$

$$\Rightarrow 2h \tan\alpha = 3R$$

$$\Rightarrow R = \frac{2h}{3} \tan\alpha$$

$$\text{Now, } \frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan\alpha}$$

And, for  $R = \frac{2h}{3} \tan\alpha$ , we have:

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan\alpha} \left( \frac{2h}{3} \tan\alpha \right) = 2\pi h - 4\pi h = -2\pi h < 0$$

By second derivative test, the volume of the cylinder is the greatest when  $R = \frac{2h}{3} \tan\alpha$

$$\text{When } R = \frac{2h}{3} \tan\alpha, H = \frac{1}{\tan\alpha} \left( h \tan\alpha - \frac{2h}{3} \tan\alpha \right) = \frac{1}{\tan\alpha} \left( \frac{h \tan\alpha}{3} \right) = \frac{h}{3}.$$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$$\pi \left( \frac{2h}{3} \tan\alpha \right)^2 \left( \frac{h}{3} \right) = \pi \left( \frac{4h^2}{9} \tan^2\alpha \right) \left( \frac{h}{3} \right) = \frac{4}{27} \pi h^3 \tan^2\alpha$$

Hence, the given result is proved.

### Choose the correct answer in the Exercises from 19 to 24.

19. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of
- (A) 1 m/h (B) 0.1 m/h  
(C) 1.1 m/h (D) 0.5 m/h

19. **The correct option is (A)**

**Explanation:** Let V be the volume of the cylinder

$$V = \pi r^2 h$$

$$= \pi(10)^2 h$$

$$\Rightarrow V = 100\pi h$$

Differentiating w.r.t. t we get,

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

The tank is being filled with wheat at the rate of 314 cubic meter per hour.

$$\frac{dV}{dt} = 314 \text{ m}^3 / \text{h}$$

Then, we have,

$$314 = 100\pi \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{314}{100 \times 3.14} = 1$$

**Therefore, the depth of wheat is increasing at the rate of 1 m/h.**

20. The slope of the tangent to the curve  $x = t^2 + 3t - 8$ ,  $y = 2t^2 - 2t - 5$  at the point  $(2, -1)$  is

(A)  $\frac{22}{7}$

(B)  $\frac{6}{7}$

(C)  $\frac{7}{6}$

(D)  $\frac{-6}{7}$

20. The given curve is  $x = t^2 + 3t - 8$  and  $y = 2t^2 - 2t - 5$ .  $\therefore \frac{dx}{dt} = 2t + 3$  and  $\frac{dy}{dt} = 4t - 2$

$$\Rightarrow \therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{4t - 2}{2t + 3}$$

The given point is  $(2, -1)$ .

At  $x = 2$ , we have:

$$\Rightarrow t^2 + 3t - 8 = 2$$

$$\Rightarrow t^2 + 3t - 10 = 0$$

$$\Rightarrow (t - 2)(t + 5) = 0$$

$$\Rightarrow t = 2 \text{ or } t = -5$$

At  $y = -1$ , we have:

$$2t^2 - 2t - 5 = -1$$

$$\Rightarrow 2t^2 - 2t - 4 = 0$$

$$\Rightarrow 2(t^2 - t - 2) = 0$$

$$\Rightarrow (t - 2)(t + 1) = 0$$

$$\Rightarrow t = 2 \text{ or } t = -1$$

The common value of  $t$  is 2.

Hence, the slope of the tangent to the given curve at point  $(2, -1)$  is

$$\left. \frac{dy}{dx} \right|_{t=2} = \frac{4(2) - 2}{2(2) + 3} = \frac{8 - 2}{4 + 3} = \frac{6}{7}$$

The correct answer is B.

21. The line  $y = mx + 1$  is a tangent to the curve  $y^2 = 4x$  if the value of  $m$  is

- (A) 1 (B) 2  
(C) 3 (D)  $\frac{1}{2}$

21. **Explanation:** It is given that the equation of the tangent to the given curve is

$$y = mx + 1$$

Now, substituting the value of  $y$  in  $y^2 = 4x$ , we get

$$\Rightarrow (mx+1)^2 = 4x$$

$$\Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$$

$$\Rightarrow m^2x^2 + x(2m - 4) + 1 = 0 \dots\dots\dots(1)$$

Since, a tangent touches the curve at one point, the root of equation (1) must be equal.

Thus, we get

$$\text{Discriminant} = 0$$

$$(2m-4)^2 - 4(m^2)(1) = 0$$

$$\Rightarrow 4m^2 + 16 - 16m - 4m^2 = 0$$

$$\Rightarrow 16 - 16m = 0$$

$$\Rightarrow m = 1$$

Therefore, the required value of  $m$  is 1.

22. The normal at the point  $(1,1)$  on the curve  $2y + x^2 = 3$  is

- (A)  $x + y = 0$  (B)  $x - y = 0$   
(C)  $x + y + 1 = 0$  (D)  $x - y = 1$

22. The equation of the given curve is  $2y + x^2 = 3$ .

Differentiating with respect to  $x$ , we have:

$$\frac{2dy}{dx} + 2x = 0$$

$$\Rightarrow \frac{dy}{dx} = -x$$

$$\therefore \left. \frac{dy}{dx} \right|_{(1,1)} = -1$$

The slope of the normal to the given curve at point  $(1,1)$  is  $\frac{-1}{\left. \frac{dy}{dx} \right|_{(1,1)}} = 1$ .

Hence, the equation of the normal to the given curve at  $(1,1)$  is given as:

$$\Rightarrow y - 1 = 1(x - 1)$$

$$\Rightarrow y - 1 = x - 1$$

$$\Rightarrow x - y = 0$$

The correct answer is B.

23. The normal to the curve  $x^2 = 4y$  passing (1,2) is  
(A)  $x + y = 3$  (B)  $x - y = 3$   
(C)  $x + y = 1$  (D)  $x - y = 1$

23. **Explanation:**

It is given that the equation of curve is  $x^2 = 4y$   
Differentiating w.r.t.  $x$ , we get,

$$2x = 4 \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

The slope of the normal to the given curve at point (h,k) is

$$\left. \frac{dy}{dx} \right|_{(h,k)} = \frac{-2}{h}$$

Then, the equation of the normal to the curve at (h,k) is

$$\Rightarrow y - k = \frac{-2}{h}(x - h)$$

Now, it is given that the normal passes through the point (1,2)

Thus, we get,

$$\Rightarrow 2 - k = \frac{-2}{h}(1 - h)$$

$$\Rightarrow k = 2 + \frac{2}{h}(1 - h) \dots \dots \dots (1)$$

Since (h,k) lies on the curve  $x^2 = 4y$ , we have  $h^2 = 4k$

$$\Rightarrow k = \frac{h^2}{4}$$

Now putting the value of  $k$  in (1), we get

$$\frac{h^2}{4} = 2 + \frac{2}{h}(1 - h)$$

$$\Rightarrow \frac{h^3}{4} = 2h + 2 - 2h = 2$$

$$\Rightarrow h^3 = 8$$

$$\Rightarrow h = 2$$

**Therefore, the equation of the normal is given as:**

$$\Rightarrow y - 1 = \frac{-2}{2}(x - 2)$$

$$\Rightarrow y - 1 = -(x - 2)$$

$$\Rightarrow x + y = 3$$

24. The points on the curve  $9y^2 = x^3$ , where the normal to the curve makes equal intercepts with the axes are

(A)  $\left(\pm 4, \frac{3}{8}\right)$                       (B)  $\left(4, \frac{-8}{3}\right)$   
 (C)  $\left(4, \pm \frac{3}{8}\right)$                       (D)  $\left(\pm 4, \frac{3}{8}\right)$

24. The equation of the given curve is  $9y^2 = x^3$ .

Differentiating with respect to  $x$ , we have:

$$9(2y) \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

The slope of the normal to the given curve at point  $(x_1, y_1)$  is

$$\left. \frac{dy}{dx} \right|_{(x_1, y_1)} = -\frac{6y_1}{x_1^2}$$

The equation of the normal to the curve at  $(x_1, y_1)$  is

$$y - y_1 = \frac{-6y_1}{x_1^2}(x - x_1).$$

$$\Rightarrow x_1^2 y - x_1^2 y_1 = -6xy_1 + 6x_1 y_1$$

$$\Rightarrow 6xy_1 + x_1^2 y = 6x_1 y_1 + x_1^2 y_1$$

$$\Rightarrow \frac{6xy_1}{6x_1 y_1 + x_1^2 y_1} + \frac{x_1^2 y}{6x_1 y_1 + x_1^2 y_1} = 1$$

$$\Rightarrow \frac{x}{x_1(6 + x_1)} + \frac{y}{y_1(6 + x_1)} = 1$$

It is given that the normal makes equal intercepts with the axes. Therefore, We have:

$$\therefore \frac{x_1(6 + x_1)}{6} = \frac{y_1(6 + x_1)}{x_1}$$

$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$

$$\Rightarrow x_1^2 = 6y_1$$

Also, the point  $(x_1, y_1)$  lies on the curve, so we have

$$9y_1^2 = x_1^3$$

From (i) and (ii), we have:

$$9\left(\frac{x_1^2}{6}\right)^2 = x_1^3 \Rightarrow \frac{x_1^4}{4} = x_1^3 \Rightarrow x_1 = 4$$

From (ii), we have:

$$9y_1^2 = (4)^3 = 64$$

$$\Rightarrow y_1^2 = \frac{64}{9}$$

$$\Rightarrow y_1 = \pm \frac{8}{3}$$

Hence, the required points are  $\left(4, \pm \frac{8}{3}\right)$ .

The correct answer is A.