

EXERCISE 19.23

Evaluate the following integrals:

$$1. \int \frac{1}{5 + 4 \cos x} dx$$

Solution:

$$\text{Given } I = \int \frac{1}{5 + 4 \cos x} dx$$

$$\text{We know that } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\Rightarrow \int \frac{1}{5 + 4 \cos x} dx = \int \frac{1}{5 + 4 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{5 \left(1 + \tan^2 \frac{x}{2} \right) + 4 \left(1 - \tan^2 \frac{x}{2} \right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{5 \left(1 + \tan^2 \frac{x}{2} \right) + 4 \left(1 - \tan^2 \frac{x}{2} \right)} dx = \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2} + 9} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2) dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2} + 9} dx = \int \frac{2dt}{t^2 + 9}$$

$$= 2 \int \frac{1}{t^2 + 9} dt$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow 2 \int \frac{1}{t^2 + 9} dt = 2 \left(\frac{1}{3} \right) \tan^{-1} \left(\frac{t}{3} \right) + c$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{\tan x/2}{3} \right) + c$$

2. $\int \frac{1}{5 - 4 \sin x} dx$

Solution:

Given $I = \int \frac{1}{5 - 4 \sin x} dx$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5 - 4 \sin x} dx = \int \frac{1}{5 - 4 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{5 \left(1 + \tan^2 \frac{x}{2} \right) - 4 \left(2 \tan \frac{x}{2} \right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{5 \left(1 + \tan^2 \frac{x}{2} \right) - 4 \left(2 \tan \frac{x}{2} \right)} dx = \int \frac{\sec^2 \frac{x}{2}}{5 + 5 \tan^2 \frac{x}{2} - 8 \tan \frac{x}{2}} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2) dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{5 + 5 \tan^2 \frac{x}{2} - 8 \tan \frac{x}{2}} dx = \int \frac{2dt}{5 + 5t^2 - 8t}$$

$$= \frac{2}{5} \int \frac{1}{t^2 - \frac{8}{5}t + 1} dt$$

$$= \frac{2}{5} \int \frac{1}{\left(t - \frac{4}{5} \right)^2 + \left(\frac{3}{5} \right)^2} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\begin{aligned} \Rightarrow \frac{2}{5} \int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt &= \frac{2}{5} \left(\frac{1}{\frac{3}{5}} \right) \tan^{-1} \left(\frac{t - \frac{4}{5}}{\frac{3}{5}} \right) + c \\ &= \frac{2}{3} \tan^{-1} \left(\frac{5 \tan x/2 - 4}{3} \right) + c \end{aligned}$$

3. $\int \frac{1}{1 - 2 \sin x} dx$

Solution:

Given $I = \int \frac{1}{1 - 2 \sin x} dx$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1 - 2 \sin x} dx = \int \frac{1}{1 - 2 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{1 \left(1 + \tan^2 \frac{x}{2} \right) - 2 \left(2 \tan \frac{x}{2} \right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{1 \left(1 + \tan^2 \frac{x}{2} \right) - 2 \left(2 \tan \frac{x}{2} \right)} dx = \int \frac{\sec^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2}} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2) dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2}} dx = \int \frac{2dt}{1 + t^2 - 4t}$$

$$= 2 \int \frac{1}{t^2 - 4t + 1} dt$$

$$= 2 \int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt$$

We know that $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

$$\Rightarrow 2 \int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt = 2 \left(\frac{1}{2\sqrt{3}} \right) \log \left| \frac{(t-2-\sqrt{3})}{(t-2+\sqrt{3})} \right| + c$$

$$= \frac{1}{\sqrt{3}} \log \left| \frac{(\tan x - (2 + \sqrt{3}))}{(\tan x + (2 + \sqrt{3}))} \right| + c$$

$$= \frac{1}{\sqrt{3}} \log \left| \frac{\tan \frac{x}{2} - 2 - \sqrt{3}}{\tan \frac{x}{2} - 2 + \sqrt{3}} \right| + c$$

4. $\int \frac{1}{4 \cos x - 1} dx$

Solution:

Given $I = \int \frac{1}{4 \cos x - 1} dx$

We know that $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{-1 + 4 \cos x} dx = \int \frac{1}{-1 + 4 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{-1 \left(1 + \tan^2 \frac{x}{2} \right) + 4 \left(1 - \tan^2 \frac{x}{2} \right)} dx$$

Replacing $1 + \tan^2 \frac{x}{2}$ in numerator by $\sec^2 \frac{x}{2}$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{-1 \left(1 + \tan^2 \frac{x}{2} \right) + 4 \left(1 - \tan^2 \frac{x}{2} \right)} dx = \int \frac{\sec^2 \frac{x}{2}}{-5 \tan^2 \frac{x}{2} + 3} dx$$

Putting $\tan \frac{x}{2} = t$ and $\sec^2(\frac{x}{2}) dx = dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{-5 \tan^2 \frac{x}{2} + 3} dx = \int \frac{2 dt}{3 - 5t^2}$$

$$= \frac{2}{5} \int \frac{1}{\frac{3}{5} - t^2} dt$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

$$\Rightarrow \frac{2}{5} \int \frac{1}{\frac{3}{5} - t^2} dt = \frac{2}{5} \left(\frac{1}{2\sqrt{\frac{3}{5}}} \right) \log \left| \frac{\sqrt{\frac{3}{5}} + t}{\sqrt{\frac{3}{5}} - t} \right| + c$$

$$= \frac{1}{\sqrt{15}} \log \left| \frac{\sqrt{3} + \sqrt{5} \tan \frac{x}{2}}{\sqrt{3} - \sqrt{5} \tan \frac{x}{2}} \right| + c$$

$$\therefore I = \int \frac{1}{4 \cos x - 1} dx = \frac{1}{\sqrt{15}} \log \left| \frac{\sqrt{3} + \sqrt{5} \tan \frac{x}{2}}{\sqrt{3} - \sqrt{5} \tan \frac{x}{2}} \right| + c$$

5. $\int \frac{1}{1 - \sin x + \cos x} dx$

Solution:

Given $I = \int \frac{1}{1 - \sin x + \cos x} dx$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1 - \sin x + \cos x} dx = \int \frac{1}{1 - \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{2 - 2 \tan \frac{x}{2}} dx$$

$$= \int \frac{2dt}{2 - 2t}$$

$$= \int \frac{1}{1 - t} dt$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{1 - t} dt = -\log|1 - t| + c$$

$$= -\log\left|1 - \tan \frac{x}{2}\right| + c$$

$$\therefore I = \int \frac{1}{1 - \sin x + \cos x} dx = -\log\left|1 - \tan \frac{x}{2}\right| + c$$

