EXERCISE 5.1

Prove the following identities:

1. $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

Solution:

Let us consider LHS: $\sec^4 x - \sec^2 x$

$(\sec^2 x)^2 - \sec^2 x$

By using the formula, $\sec^2 \theta = 1 + \tan^2 \theta$.

$(1 + \tan^2 x)^2 - (1 + \tan^2 x)$

$1 + 2\tan^2 x + \tan^4 x - 1 - \tan^2 x$

$\tan^4 x + \tan^2 x$

= RHS

∴ LHS = RHS

Hence proved.

2. $\sin^6 x + \cos^6 x = 1 - 3 \sin^2 x \cos^2 x$

Solution:

Let us consider LHS: $\sin^6 x + \cos^6 x$

$(\sin^2 x)^3 + (\cos^2 x)^3$

By using the formula, $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$(\sin^2 x + \cos^2 x) [(\sin^2 x)^2 + (\cos^2 x)^2 - \sin^2 x \cos^2 x]$

By using the formula, $\sin^2 x + \cos^2 x = 1$ and $a^2 + b^2 = (a + b)^2 - 2ab$

$1 \times [(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x - \sin^2 x \cos^2 x$

$1^2 - 3\sin^2 x \cos^2 x$

$1 - 3\sin^2 x \cos^2 x$

= RHS

∴ LHS = RHS

Hence proved.

3. (cosec x – sin x) (sec x – cos x) (tan x + cot x) = 1

Solution:

Let us consider LHS: (cosec x – sin x) (sec x – cos x) (tan x + cot x)

By using the formulas

$\text{cosec } \theta = 1/\sin \theta$;

$\sec \theta = 1/\cos \theta$;

$\tan \theta = \sin \theta / \cos \theta$;

$\cot \theta = \cos \theta / \sin \theta$

Now,

$$\left(\frac{1}{\sin x} - \sin x\right)\left(\frac{1}{\cos x} - \cos x\right)\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)$$

$$\frac{1-\sin^2 x}{\sin x} \times \frac{1-\cos^2 x}{\cos x} \times \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$

By using the formula,

$\sin^2 x + \cos^2 x = 1$;

$$\frac{\cos^2 x}{\sin x} \times \frac{\sin^2 x}{\cos x} \times \frac{1}{\sin x \cos x}$$

1 = RHS

∴ LHS = RHS

Hence proved.

4. cosec x (sec x – 1) – cot x (1 – cos x) = tan x – sin x

Solution:

Let us consider LHS: cosec x (sec x – 1) – cot x (1 – cos x)

By using the formulas

cosec θ = 1/sin θ;

sec θ = 1/cos θ;

tan θ = sin θ / cos θ;

cot θ = cos θ / sin θ

Now,

$$\frac{1}{\sin x}\left(\frac{1}{\cos x} - 1\right) - \frac{\cos x}{\sin x}(1 - \cos x)$$

$$\frac{1}{\sin x}\left(\frac{1-\cos x}{\cos x}\right) - \frac{\cos x}{\sin x}(1 - \cos x)$$

$$\left(\frac{1-\cos x}{\sin x}\right)\left(\frac{1}{\cos x} - \cos x\right)$$

$$\left(\frac{1-\cos x}{\sin x}\right)\left(\frac{1-\cos^2 x}{\cos x}\right)$$

By using the formula, $1 - \cos^2 x = \sin^2 x$;

$$\left(\frac{1-\cos x}{\sin x}\right)\left(\frac{\sin^2 x}{\cos x}\right)$$

$$(1-\cos x)\left(\frac{\sin x}{\cos x}\right)$$

$$\frac{\sin x}{\cos x}-\sin x$$

$$\tan x-\sin x$$

= RHS

∴ LHS = RHS

Hence Proved.

5.

$$\frac{1-\sin x\cos x}{\cos x(\sec x-\operatorname{cosec} x)}\cdot\frac{\sin^2 x-\cos^2 x}{\sin^3 x+\cos^3 x}=\sin x$$

Solution:

Let us consider the LHS:

$$\frac{1-\sin x\cos x}{\cos x(\sec x-\operatorname{cosec} x)}\cdot\frac{\sin^2 x-\cos^2 x}{\sin^3 x+\cos^3 x}$$

By using the formula,

cosec θ = 1/sin θ;

sec θ = 1/cos θ;

Now,

$$\frac{1-\sin x\cos x}{\cos x\left(\frac{1}{\cos x}-\frac{1}{\sin x}\right)}\times\frac{(\sin x)^2-(\cos x)^2}{(\sin x)^3+(\cos x)^3}$$

By using the formula, $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$$\frac{1-\sin x\cos x}{\cos x\left(\frac{\sin x-\cos x}{\cos x\sin x}\right)}\times\frac{(\sin x+\cos x)(\sin x-\cos x)}{(\sin x+\cos x)\,[(\sin x)^2+(\cos x)^2-\sin x\cos x]}$$

$$\frac{\sin x(1-\sin x\cos x)}{\sin x-\cos x}\times\frac{(\sin x+\cos x)(\sin x-\cos x)}{(\sin x+\cos x)\,[(\sin x)^2+(\cos x)^2-\sin x\cos x]}$$

$$\frac{\sin x\,(1-\sin x\cos x)}{1} \times \frac{1}{[(\sin x)^2 + (\cos x)^2 - \sin x\ \cos x]}$$

By using the formula, $\sin^2 x + \cos^2 x = 1$.

$$\sin x\,(1-\sin x\cos x) \times \frac{1}{(1-\sin x\cos x)}$$

sin x

= RHS

∴ LHS = RHS

Hence Proved.

6.

$$\frac{\tan x}{1-\cot x} + \frac{\cot x}{1-tanx} = (\sec x\ cosec\ x + 1)$$

Solution:

Let us consider the LHS:

$$\frac{\tan x}{1-\cot x} + \frac{\cot x}{1-tanx}$$

By using the formula,

tan θ = sin θ / cos θ;

cot θ = cos θ / sin θ

Now,

$$\frac{\frac{\sin x}{\cos x}}{1-\frac{\cos x}{\sin x}} + \frac{\frac{\cos x}{\sin x}}{1-\frac{\sin x}{\cos x}}$$

$$\frac{\frac{\sin x}{\cos x}}{\frac{\sin x-\cos x}{\sin x}} + \frac{\frac{\cos x}{\sin x}}{\frac{\cos x-\sin x}{\cos x}}$$

$$\frac{\sin^2 x}{\cos x\,(\sin x-\cos x)} - \frac{\cos^2 x}{\sin x\,(\sin x-\cos x)}$$

$$\frac{\sin^3 x-\cos^3 x}{\sin x\cos x\,(\sin x-\cos x)}$$

By using the formula, $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$

$$\frac{(\sin x - \cos x)\,[(\sin x)^2 + (\cos x)^2 + \sin x \cos x]}{\sin x \cos x(\sin x - \cos x)}$$

We know, $\sin^2 x + \cos^2 x = 1$.

$$\frac{[1 + \sin x \cos x]}{\sin x \cos x}$$

$$\frac{1}{\sin x \cos x} + \frac{\sin x \cos x}{\sin x \cos x}$$

$$\frac{1}{\sin x} \times \frac{1}{\cos x} + 1$$

By using the formula,
cosec θ = 1/sin θ,
sec θ = 1/cos θ;
cosec x × sec x + 1
sec x cosec x + 1
=RHS
∴ LHS = RHS
Hence Proved.

7.

$$\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} + \frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 2$$

Solution:
Let us consider LHS:

$$\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} + \frac{\sin^3 x - \cos^3 x}{\sin x - \cos x}$$

By using the formula $a^3 \pm b^3 = (a \pm b)(a^2 + b^2 \mp ab)$

$$\frac{(\sin x + \cos x)\,[(\sin x)^2 + (\cos x)^2 - \sin x \cos x]}{\sin x + \cos x} + \frac{(\sin x - \cos x)\,[(\sin x)^2 + (\cos x)^2 + \sin x \cos x]}{\sin x - \cos x}$$

We know, $\sin^2 x + \cos^2 x = 1$.

1 - sinx cosx + 1 + sinx cosx

2

= RHS

∴ LHS = RHS

Hence Proved.

8. $(\sec x \sec y + \tan x \tan y)^2 - (\sec x \tan y + \tan x \sec y)^2 = 1$

Solution:

Let us consider LHS:

$(\sec x \sec y + \tan x \tan y)^2 - (\sec x \tan y + \tan x \sec y)^2$

Expanding the above equation we get,

$[(\sec x \sec y)^2 + (\tan x \tan y)^2 + 2 (\sec x \sec y) (\tan x \tan y)] - [(\sec x \tan y)^2 + (\tan x \sec y)^2 + 2 (\sec x \tan y) (\tan x \sec y)]$

$[\sec^2 x \sec^2 y + \tan^2 x \tan^2 y + 2 (\sec x \sec y) (\tan x \tan y)] - [\sec^2 x \tan^2 y + \tan^2 x \sec^2 y + 2 (\sec^2 x \tan^2 y) (\tan x \sec y)]$

$\sec^2 x \sec^2 y - \sec^2 x \tan^2 y + \tan^2 x \tan^2 y - \tan^2 x \sec^2 y$

$\sec^2 x (\sec^2 y - \tan^2 y) + \tan^2 x (\tan^2 y - \sec^2 y)$

$\sec^2 x (\sec^2 y - \tan^2 y) - \tan^2 x (\sec^2 y - \tan^2 y)$

We know, $\sec^2 x - \tan^2 x = 1$.

$\sec^2 x \times 1 - \tan^2 x \times 1$

$\sec^2 x - \tan^2 x$

1 = RHS

∴ LHS = RHS

Hence proved.

9.

$$\frac{\cos x}{1-\sin x} = \frac{1+\cos x+\sin x}{1+\cos x-\sin x}$$

Solution:

Let us Consider RHS:

$$\frac{1+\cos x+\sin x}{1+\cos x-\sin x}$$

$$\frac{(1+\cos x)+(\sin x)}{(1+\cos x)-(\sin x)}$$

$$\frac{(1+\cos x)+(\sin x)}{(1+\cos x)-(\sin x)} \times \frac{(1+\cos x)+(\sin x)}{(1+\cos x)+(\sin x)}$$

$$\frac{[(1+\cos x)+(\sin x)]^2}{(1+\cos x)^2-(\sin x)^2}$$

$$\frac{[(1+\cos x)+(\sin x)]^2}{(1+\cos x)^2-(\sin x)^2}$$

$$\frac{(1+\cos x)^2+(\sin x)^2+2(1+\cos x)(\sin x)}{(1+\cos^2 x+2\cos x)-(\sin^2 x)}$$

$$\frac{(1+\cos x)^2+(\sin x)^2+2(1+\cos x)(\sin x)}{(1+\cos^2 x+2\cos x)-(\sin^2 x)}$$

$$\frac{(1+\cos x)^2+(\sin x)^2+2(1+\cos x)(\sin x)}{(1+\cos^2 x+2\cos x)-(\sin^2 x)}$$

$$\frac{1+\cos^2 x+2\cos x+\sin^2 x+2\sin x+2\sin x\cos x}{1+\cos^2 x+2\cos x-\sin^2 x}$$

We know, $\sin^2x + \cos^2x = 1$.

$$\frac{1+1+2\cos x+2\sin x+2\sin x\cos x}{(1-\sin^2 x)+\cos^2 x+2\cos x}$$

We know, $1 - \cos^2x = \sin^2x$.

$$\frac{2+2\cos x+2\sin x+2\sin x\cos x}{\cos^2 x+\cos^2 x+2\cos x}$$

$$\frac{2+2\cos x+2\sin x+2\sin x\cos x}{2\cos^2 x+2\cos x}$$

$$\frac{2+2\cos x+2\sin x+2\sin x\cos x}{\cos^2 x+\cos^2 x+2\cos x}$$

$$\frac{1+\cos x+\sin x+\sin x\cos x}{\cos x(\cos x+1)}$$

$$\frac{1(1+\cos x)+\sin x(\cos x+1)}{\cos x(\cos x+1)}$$

$$\frac{(1+\sin x)(\cos x+1)}{\cos x(\cos x+1)}$$

$$\frac{1+\sin x}{\cos x} \times \frac{\cos x}{\cos x}$$

$$\frac{(1+\sin x)\cos x}{\cos^2 x}$$

We know, $1 - \sin^2 x = \cos^2 x$.

$$\frac{(1+\sin x)\cos x}{1-\sin^2 x}$$

$$\frac{(1+\sin x)\cos x}{(1-\sin x)(1+\sin x)}$$

$$\frac{\cos x}{1-\sin x}$$

= LHS

∴ LHS = RHS

Hence Proved.

10.

$$\frac{\tan^3 x}{1+\tan^2 x} + \frac{\cot^3 x}{1+\cot^2 x} = \frac{1-2\sin^2 x \cos^2 x}{\sin x \cos x}$$

Solution:

Let us consider LHS:

$$\frac{\tan^3 x}{1+\tan^2 x} + \frac{\cot^3 x}{1+\cot^2 x}$$

By using the formulas,

$1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \text{cosec}^2 x$

$$\frac{\tan^3 x}{\sec^2 x} + \frac{\cot^3 x}{\text{cosec}^2 x}$$

$$\frac{\frac{\sin^3 x}{\cos^3 x}}{\frac{1}{\cos^2 x}} + \frac{\frac{\cos^3 x}{\sin^3 x}}{\frac{1}{\sin^2 x}}$$

$$\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x}$$

$$\frac{\sin^4 x + \cos^4 x}{\cos x \sin x}$$

$$\frac{(\sin^2 x)^2 + (\cos^2 x)^2}{\cos x \sin x}$$

We know, $a^2 + b^2 = (a + b)^2 - 2ab$

$$\frac{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x}{\sin x \cos x}$$

$$\frac{1^2 - 2\sin^2 x \cos^2 x}{\sin x \cos x}$$

$$\frac{1 - 2\sin^2 x \cos^2 x}{\sin x \cos x}$$

= RHS

∴ LHS = RHS

Hence Proved.

11.

$$1 - \frac{\sin^2 x}{1 + \cot x} - \frac{\cos^2 x}{1 + \tan x} = \sin x \cos x$$

Solution:

Let us consider LHS:

$$1 - \frac{\sin^2 x}{1 + \cot x} - \frac{\cos^2 x}{1 + \tan x}$$

By using the formula,

$\tan \theta = \sin \theta / \cos \theta$;

$\cot \theta = \cos \theta / \sin \theta$

Now,

$$1 - \frac{\sin^2 x}{1 + \frac{\cos x}{\sin x}} - \frac{\cos^2 x}{1 + \frac{\sin x}{\cos x}}$$

$$1 - \frac{\sin^3 x}{\sin x + \cos x} - \frac{\cos^3 x}{\sin x + \cos x}$$

$$\frac{\sin x + \cos x - (\sin^3 x + \cos^3 x)}{\sin x + \cos x}$$

By using the formula, $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$$\frac{\sin x + \cos x - ((\sin x + \cos x)(\sin x)^2 + (\cos x)^2 - \sin x \cos x))}{\sin x + \cos x}$$

$$\frac{(\sin x + \cos x)(1 - \sin^2 x - \cos^2 x + \sin x \cos x)}{\sin x + \cos x}$$

$1 - (\sin^2 x + \cos^2 x) + \sin x \cos x$

We know, $\sin^2 x + \cos^2 x = 1$.

$1 - 1 + \sin x \cos x$

Sin x cos x

= RHS

∴ LHS = RHS

Hence proved.

12.

$$\left(\frac{1}{\sec^2 x - \cos^2 x} + \frac{1}{cosec^2 x - \sin^2 x}\right)\sin^2 x \cos^2 x = \frac{1 - \sin^2 x \cos^2 x}{2 + \sin^2 x \cos^2 x}$$

Solution:

Let us consider LHS:

$$\left(\frac{1}{\sec^2 x - \cos^2 x} + \frac{1}{cosec^2 x - \sin^2 x}\right)\sin^2 x \cos^2 x$$

By using the formula,

cosec θ = 1/sin θ,

sec θ = 1/cos θ;

$$\left(\frac{1}{\frac{1}{\cos^2 x} - \cos^2 x} + \frac{1}{\frac{1}{\sin^2 x} - \sin^2 x}\right)\sin^2 x \cos^2 x$$

$$\left(\frac{\cos^2 x}{1 - \cos^4 x} + \frac{\sin^2 x}{1 - \sin^4 x}\right)\sin^2 x \cos^2 x$$

$$\left(\frac{\cos^2 x(1 - \sin^4 x) + \sin^2 x(1 - \cos^4 x)}{(1 - \cos^4 x)(1 - \sin^4 x)}\right)\sin^2 x \cos^2 x$$

$$\left(\frac{\cos^2 x - \cos^2 x \sin^4 x + \sin^2 x - \sin^2 x \cos^4 x}{(1+\sin^2 x)(1-\sin^2 x)(1+\cos^2 x)(1-\cos^2 x)}\right)\sin^2 x \cos^2 x$$

We know, $\sin^2 x + \cos^2 x = 1$.

$$\left(\frac{1 - \cos^2 x \sin^4 x - \sin^2 x \cos^4 x}{(1+\sin^2 x)\cos^2 x(1+\cos^2 x)\sin^2 x}\right)\sin^2 x \cos^2 x$$

$$\left(\frac{1 - \cos^2 x \sin^2 x(\sin^2 x + \cos^2 x)}{(1+\sin^2 x)(1+\cos^2 x)}\right)$$

$$\frac{1-\sin^2 x \cos^2 x}{2+\sin^2 x \cos^2 x}$$

= RHS

∴ LHS = RHS

Hence proved.

13. $(1 + \tan\alpha \tan\beta)^2 + (\tan\alpha - \tan\beta)^2 = \sec^2\alpha \sec^2\beta$

Solution:

Let us consider LHS: $(1 + \tan\alpha \tan\beta)^2 + (\tan\alpha - \tan\beta)^2$

$1 + \tan^2\alpha \tan^2\beta + 2\tan\alpha\tan\beta + \tan^2\alpha + \tan^2\beta - 2\tan\alpha\tan\beta$

$1 + \tan^2\alpha \tan^2\beta + \tan^2\alpha + \tan^2\beta$

$\tan^2\alpha(\tan^2\beta + 1) + 1(1 + \tan^2\beta)$

$(1 + \tan^2\beta)(1 + \tan^2\alpha)$

We know, $1 + \tan^2\theta = \sec^2\theta$

So,

$\sec^2\alpha \sec^2\beta$

= RHS

∴ LHS = RHS

Hence proved.