Find ‘c’ of Lagrange’s mean-value theorem for

$$f(x) = \sqrt{25 - x^2} \text{ on } [1, 5]$$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [1,5].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sqrt{25 - 25} - \sqrt{25 - 1}}{5 - 1}$$

$$= \frac{-\sqrt{24}}{4}$$

$$f'(c) = \frac{1}{2\sqrt{25 - c^2}}(-2c)$$

$$\Rightarrow \frac{-c}{\sqrt{25 - c^2}} = \frac{-\sqrt{24}}{4}$$

$$\Rightarrow 4c = \sqrt{24(25 - c^2)}$$

$\Rightarrow 16c^2=600-24c^2$

$\Rightarrow 40c^2=600$

$\Rightarrow c^2=15$

$$\Rightarrow c = \sqrt{15}$$

14 C. Question

Find ‘c’ of Lagrange’s mean-value theorem for

$$f(x) = \sqrt{x + 2} \text{ on } [4, 6]$$

Answer

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sqrt{8} - \sqrt{6}}{6 - 4}$$

$$= \frac{\sqrt{8} - \sqrt{6}}{2}$$

$$\mathbf{f'(c)} = \frac{1}{2\sqrt{c + 2}}$$

$$\Rightarrow \frac{1}{2\sqrt{c + 2}} = \frac{\sqrt{8} - \sqrt{6}}{2}$$

$\Rightarrow \frac{1}{\sqrt{c+2}} = \frac{\sqrt{8}-\sqrt{6}}{1}$

$\Rightarrow \sqrt{c+2} = \frac{1}{\sqrt{8}-\sqrt{6}} \times \frac{\sqrt{8}+\sqrt{6}}{\sqrt{8}+\sqrt{6}}$

$\Rightarrow \sqrt{c+2} = \frac{\sqrt{8}+\sqrt{6}}{2}$

$\Rightarrow \mathbf{c+2} = \frac{1}{4}\left(8+6+2\sqrt{48}\right)$

$\Rightarrow \mathbf{c} = \frac{\mathbf{3}}{\mathbf{2}} + \mathbf{2\sqrt{3}}$

⇒ **c=4.964**

15. Question

Using Lagrange's mean-value theorem, find a point on the curve $y = x^2$, where the tangent is parallel to the line joining the point (1, 1) and (2, 4)

Answer

Given:

$y=x^2$

Since y is a polynomial function.

It is continuous and differentiable in [1,2]

So, there exists a c such that:

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$= \frac{4-1}{2-1}$

$= 3$

⇒ f' (c)=2c

⇒ 2c=3

$c = \frac{3}{2}$

So, the point is $\left(\frac{3}{2}, \frac{9}{4}\right)$

16. Question

Find a point on the curve $y = x^3$, where the tangent to the curve is parallel to the chord joining the points (1, 1) and (3, 27).

Answer

Given:

$y = x^3$

Since y is a polynomial function.

It is continuous and differentiable in [1,3]

So, there exists a c such that:

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$= \frac{\mathbf{27 - 1}}{\mathbf{3 - 1}}$

$= 13$

$\Rightarrow$ f' (c)=3c^2

$\Rightarrow$ 3c^2=13

$\Rightarrow \ c = \sqrt{\frac{13}{3}}$

$\Rightarrow \ c = \frac{\sqrt{39}}{3}$

So the point is $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$

17. Question

Find the points on the curve $y = x^3 - 3x$, where the tangent to the curve is parallel to the chord joining (1, −2) and (2, 2).

Answer

Given:

y=x^3-3x

Since y is a polynomial function.

It is continuous and differentiable in [1,2]

So, there exists a c such that:

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$= \frac{\mathbf{(8 - 6) - (1 - 3)}}{\mathbf{2 - 1}}$

=4

$\Rightarrow$ f' (c)=3c^2-3

$\Rightarrow$ 3 c^2-3=4

$\Rightarrow$ 3c^2=7

$\Rightarrow \ c^2 = \frac{7}{3}$

$\Rightarrow \ \mathbf{c} = \pm\sqrt{\frac{\mathbf{7}}{\mathbf{3}}}$

So, the points are $\left(\sqrt{\frac{7}{3}}, \frac{-2}{3}\sqrt{\frac{7}{3}}\right), \left(\frac{-7}{3}\sqrt{\frac{7}{3}}, \frac{2}{3}\sqrt{\frac{7}{3}}\right)$

18. Question

If $f(x) = x(1 - \log x)$, where c > 0, show that $(a - b)\log c = b(1 - \log b) - a\,(1 - \log a)$, where 0 < a < c < b.

Answer

Given:

f(x)=x(1-log x)

Since the function is continuous as well as differentiable

So, there exists c such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow\ (1-\log c)-c\times\frac{1}{c}=\frac{b(1-\log b)-a(1-\log a)}{b-a}$$

$$\Rightarrow\ \log c=\frac{b(1-\log b)-a(1-\log a)}{b-a}$$

(b-a) log c=b(1-log b)-a(1-log a)

Hence proved.

Exercise 11E

1. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$$(5x-1)^2+4\cdot$$

Answer

min. value = 4

Since the square of any no. Is positive, the given function has no maximum value.

The minimum value exists when the quantity $(5x-1)^2=0$

Therefore, minimum value=4

2. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$$-(x-3)^2+9$$

Answer

max. value = 9

Since the quantity $(x-3)^2$ has a –ve sign, the max. Value it can have is 9.

Also hence it has no minimum value.

3. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$$-|x+4|+6$$

Answer

max. value = 6

Since |x+4| is non-negative for all x belonging to R.

Therefore the least value it can have is 0 .

Hence value of function is 6.

It has no minimumvalue as it can have infinitely many.

4. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$$\sin\ 2x + 5$$

Answer

max. value = 4, min. value = 6

f(x)=sin2x+5

We know that,

$-1 \leq \sin\theta \leq 1$

$-1 \leq \sin 2x \leq 1$

Adding 5 on both sides,

$-1+5 \leq \sin 2x+5 \leq 1+5$

$4 \leq \sin 2x+5 \leq 6$

Hence

max value of f(x)=2x+5 will be 6

Min value of f(x) =2x+5 will be 4

5. Question

Find the maximum or minimum values, if any, without using derivatives, of the function:

$$|\sin\ 4x + 3|$$

Answer

max. value = 4, min. value = 2

We know that

$-1 \leq \sin\theta \leq 1$

$-1 \leq \sin 4x \leq 1$

Adding 3 on both sides,

We get

$-1+3 \leq \sin 4x+3 \leq 1+3$

$2 \leq |\sin 4x+3| \leq 4$

Hence min.Value is 2 and max value is 4

6. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = (x-3)^4$$

Answer

local max. value is 0 at x = 3

$F'(x)=4(x-3)^3=0$

$\Rightarrow X=3$

 local max. Vaue is 0.

7. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x)=x^2$$

Answer

local min. value is 0 at x = 0

$F'(x)=2x=0$

x=0

 local min.value is 0

8. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x)=2x^3-21x^2+36x-20$$

Answer

local max. value is −3 at x = 1 and local min. value is −128 at x = 6

$F'(x)=6x^2-42x+36=0$

⇒ 6(x-1)(x-6)=0

⇒x=1,6

$F''(x)=12x-42$

$F''(1)<0$,1 is the pont of local max.

$F''(6)>0$, 6 is the point of localmin.

F(1)=2-21+36-20=-3

F(6)=-128

9. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x)=x^3-6x^2+9x+15$$

Answer

local max. value is 19 at x = 1 and local min. value is 15 at x = 3

$F'(x)=3x^2-12x+9=0$

⇒3(x-3)(x-1)=0

⇒ x=3,1

$F''(x)=6x-12$

$F''(3)=18-12=6>0$, 3 is the of local min.

$F''(1)<0$, 1 is the point of local max.

F(3)=15

F(1)=19

10. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = x^4 - 62x^2 + 120x + 9$$

Answer

local max. value is 68 at x = 1 and local min. values are −1647 at x = −6 and −316 at x = 5

F'(x)=4x^3-124x+120=0

⇒ 4(x^3-31x+30)=0

For x=1, the given eq is 0

x-1 is a factor,

4(x-1)(x+6)(x-5)=0

⇒X=1,-6,5

F$''$(1)<0, 1is the point of max.

F$''$(-6) and f$''$(5)>0, -6 and 5 are point of min.

F(1)=68

F(-6)=-1647

F(5)=-316

11. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = -x^3 + 12x^2 - 5$$

Answer

local max. value is 251 at x = 8 and local min. value is −5 at x = 0

'}(x)=-3x^2+24x=0

⇒ -3x(x-8)=0

⇒ x=0,8

F$''$(x)=-6x+24

F$''$(0)>0, 0 is the point of local min.

F$''$(8)<0, 8 is the point of local max.

F(8)=251 and f(0)=-5

12. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x) = (x-1)(x+2)^2$$

Answer

local max. value is 0 at x = −2 and local min. value is −4 at x = 0

$f'(x)=(x-1)2(x+2)+(x+2)^2=0$

x=0,-2

$f''(0)>0$, 0 is the point of local min.

$f''(-2)<0$, -2 is the point of local max.

f(0)=-4

f(-2)=0

13. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x)=-(x-1)^3(x+1)^2$$

Answer

local max. value is 0 at each of the points x = 1 and x = −1 and local min. value is $\frac{-3456}{3125}$ at $x=-\frac{1}{5}$

$F'(x)=-(x-1)^3 2(x+1)-3(x-1)^2(x+1)^2=0$

$\Rightarrow x=1,-1,-\frac{1}{5}$

Since, $f''(1)$ and $f''(-1)<0$, 1 and -1 are the points of local max.

$F''(-\frac{1}{5})>0$, $-\frac{1}{5}$ is the point of local min.

F(1)=f(-1)=0

Also, $f\left(-\frac{1}{5}\right)=-\frac{3456}{3125}$

14. Question

Find the point of local maxima or local minima or local minima and the corresponding local maximum and minimum values of each of the following functions:

$$f(x)=\frac{x}{2}+\frac{2}{x}, x>0$$

Answer

local min. value is 2 at x = 2

$F'(x)=\frac{1}{2}-\frac{2}{x^2}=0$

$\Rightarrow x^2-4=0$

$\Rightarrow x=\pm 2$

But since x>0, x=2

$F''(2)=\frac{2}{x^3}$

$=\frac{2}{8}<0$

point of local mini. is 2

$F(2)=\frac{2}{2}+\frac{2}{2}=2$

15. Question

Find the maximum and minimum values of $2x^3 - 24x + 107$ on the interval $[-3, 3]$.

Answer

max. value is 139 at x = −2 and min. value is 89 at x = 3

$F'(x)=6x^2-24=0$

$6(x^2-4)=0$

$6(x^2-2^2)=0$

$6(x-2)(x+2)=0$

X=2,-2

Now, we shall evaluate the value of f at these points and the end points

$F(2)=2(2)^3-24(2)+107=75$

$F(-2)=2(-2)^3-24(-2)+107=139$

$F(-3)=2(-3)^3-24(-3)+107=125$

$F(3)=2(3)^3-24(3)+107=89$

16. Question

Find the maximum and minimum values of $3x^4 - 8x^3 + 12x^2 - 48x + 1$ on the interval $[1, 4]$.

Answer

max. value is 257 at x = 4 and min. value is −63 at x = 2

$F'(x)=12x^3-24x^2+24x-48=0$

$12(x^3-2x^2+2x-4)=0$

Since for x=2, $x^3-2x^2+2x-4=0$, x-2 is a factor

On dividing x^3-2x^2+2x-4 by x-2, we get,

$12(x-2)(x^2+2)=0$

X=2,4

Now, we shall evaluate the value of f at these points and the end points

$F(1)=3(1)^4-8(1)^3+12(1)^2-48(1)+1=-40$

$F(2)= 3(2)^4-8(2)^3+12(2)^2-48(2)+1=-63$

$F(4)= 3(4)^4-8(4)^3+12(4)^2-48(4)+1=257$

17. Question

Find the maximum and minimum of

$$f(x)=\left(\sin x+\frac{1}{2}\cos x\right) \text{ in } 0 \le x \le \frac{\pi}{2}$$

Answer

max. value is $\frac{3}{4}$ at $x = \frac{\pi}{6}$ and min. value is $\frac{1}{2}$ at $x = \frac{\pi}{2}$

$F^{|}(x) = \cos x - \frac{1}{2}\sin x = 0$

ð 2 cos x=sin x

$\Rightarrow \frac{\pi}{6} = \frac{\pi}{3}$

$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\cos\frac{\pi}{2} = \frac{1}{2}$

$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{6} = \frac{1}{2} + \frac{\sqrt{3}}{4}$

$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \frac{1}{2}\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{4}$

18. Question

Show that the maximum value of $x^{1/x}$ is $e^{1/e}$

Answer

The given function is

$Y = x^{\frac{1}{x}}$

Now, taking logarithm from both sides, we get..

$\log y = \frac{1}{x}\log x$

Differentiating both sides w.r.t x....

$\frac{1}{y}y' = -\frac{1}{x^2}\ln(x) + \frac{1}{x^2}$

$\Rightarrow y' = \frac{y}{x^2}(1 - \ln(x))$

(1-ln(x))=0

ln(x)=1

x=e

hence the max. point is x=e

max value is $e^{\frac{1}{e}}$.

19. Question

Show that $\left(x + \frac{1}{x}\right)$ has a maximum and minimum, but the maximum value is less than the minimum value.

Answer

$F(x) = x + \frac{1}{x}$

Taking first derivative and equating it to zero to find extreme points.

$F'(x) = 1 - \frac{1}{x^2} = 0$

$X^2=1$

x=1,x=-1

now to determine which of these is min. And max. We use second derivative.

$f^{||}(x)=\frac{2}{x^3}$

$f^{||}(1)=2$ and $f^{||}(-1)=-2$

since $f^{||}(1)$ is +ve it is minimum point while $f^{||}(-1)$ is –ve it is maximum point

max value-> $f(-1)=-1+\frac{1}{-1}=-2$

min vaue-> $f(1)=1+\frac{1}{1}=2$

hence maximum value is less than minimum value

20. Question

Find the maximum profit that a company can make, if the profit function is given by $p(x)=41+24x-18x^2$

.

Answer

49

$\frac{dp}{dx}=-24-36x$

=0

⇒x=−23

Step 2

$\frac{d^2p}{dx^2}=-36$ is negative

Step 3

maximum profit $=p\left(-\frac{2}{3}\right)$

=49

21. Question

An enemy jet is flying along the curve $y=(x^2+2)$. A soldier is placed at the point (3, 2). Find the nearest point between the soldier and the jet.

Answer

(1, 3)

Let P(x,y) be the position of the jet and the soldier is placed at A(3,2)

AP=$\sqrt{(x-3)^2+(y-2)^2}$

As $y=x^2+2$ or $y-2=x^2$

 $AP^2=(x-3)^2+x^4=z$ (say)

$\frac{dz}{dx}=2(x-3)+4x^3$

$\frac{dz}{dx}=0$

2 □x-6+4x^3=0

Put x=1

2-6+4=0

□ x-1 is a factor

And $\frac{d^2z}{dx^2} = 12x^2 + 2$

$\frac{dz}{dx} = 0$ or x=1

and $\frac{d^2z}{dx^2}$(at x=1)>0

□ z is minimum when x=1,y=1+2=3

Point is (1,3)

22. Question

Find the maximum and minimum values of

$f(x) = (-x + 2\sin x)$ on $[0, 2\pi]$.

Answer

max. value is $\left(-\frac{\pi}{3} + \sqrt{3}\right)$ at $x = \frac{\pi}{3}$ and min. value is $\left(\frac{5\pi}{3} + \sqrt{3}\right)$ at $x = \frac{5\pi}{3}$

f$'$(x)=-1+2cos x=0

$\Rightarrow \cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}$

By finding the general solution, we get $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$

Now, by finding the second derivative, we get that $f''(\frac{\pi}{3}) < 0$ and $f''(\frac{5\pi}{3}) > 0$

Therefore, max. value is $\left(-\frac{\pi}{3} + \sqrt{3}\right)$ at $x = \frac{\pi}{3}$ and min. value is $\left(\frac{5\pi}{3} + \sqrt{3}\right)$ at $x = \frac{5\pi}{3}$

Exercise 11F

1. Question

Find two positive number whose product is 49 and the sum is minimum.

Answer

Given,

• The two numbers are positive.

• the product of two numbers is 49.

• the sum of the two numbers is minimum.

Let us consider,

• x and y are the two numbers, such that x > 0 and y > 0

• Product of the numbers : x × y = 49

• Sum of the numbers : S = x + y

Now as,

x × y = 49

$y = \frac{49}{x}$ ------ (1)

Consider,

S = x + y

By substituting (1), we have

$S = x + \frac{49}{x}$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}\left(x + \frac{49}{x}\right)$$

$$\frac{dS}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{49}{x}\right)$$

$\frac{dS}{dx} = 1 + 49\left(\frac{-1}{x^2}\right)$ ----- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 1 + 49\left(\frac{-1}{x^2}\right) = 0$$

$$= 1 - \left(\frac{49}{x^2}\right) = 0$$

$$= 1 = \left(\frac{49}{x^2}\right)$$

$= x^2 = 49$

$= x = \pm\sqrt{49}$

As x > 0, then x = 7

Now, for checking if the value of S is maximum or minimum at x=7, we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 7.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[1 + 49\left(\frac{-1}{x^2}\right)\right]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[1] + \frac{d}{dx}\left[49\left(\frac{-1}{x^2}\right)\right]$$

$$\frac{d^2S}{dx^2} = 0 + \left[49\left(\frac{-1 \times -2}{x^3}\right)\right]$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{d^2S}{dx^2} = 49\left(\frac{2}{x^3}\right) = \frac{98}{x^3}$$

Now when x = 7,

$$\left[\frac{d^2S}{dx^2}\right]_{x=7} = \frac{98}{7^3} = \frac{98}{343} > 0$$

As second differential is positive, hence the critical point x = 7 will be the minimum point of the function S.

Therefore, the function S = sum of the two numbers is minimum at x = 7.

From Equation (1), if x= 7

$$y = \frac{49}{7} = 7$$

Therefore, x = 7 and y = 7 are the two positive numbers whose product is 49 and the sum is minimum.

2. Question

Find two positive numbers whose sum is 16 and the sum of whose squares is minimum.

Answer

Given,

• The two numbers are positive.

• the sum of two numbers is 16.

• the sum of the squares of two numbers is minimum.

Let us consider,

• x and y are the two numbers, such that x > 0 and y > 0

• Sum of the numbers : x + y = 16

• Sum of squares of the numbers : $S = x^2 + y^2$

Now as,

x + y = 16

y = (16-x) ------ (1)

Consider,

$S = x^2 + y^2$

By substituting (1), we have

$S = x^2 + (16-y)^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}[x^2 + (16-x)^2]$$

$$\frac{dS}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}[(16-x)^2]$$

$\frac{dS}{dx} = 2x + 2(16-x)(-1)$ ----- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 2x + 2(16 - x)(-1) = 0$$

⇒ 2x – 2(16 -x) = 0

⇒ 2x – 32 + 2x = 0

= 4x = 32

$$\Rightarrow x = \frac{32}{4}$$

⇒ x = 8

As x > 0, x = 8

Now, for checking if the value of S is maximum or minimum at x=8, we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 8.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[2x + 2(16 - x)(-1)]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[2x] - 2\frac{d}{dx}[16 - x]$$

$$\frac{d^2S}{dx^2} = 2 - 2[0 - 1]$$

[Since $\frac{d}{dx}$ (constant) = 0 and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{d^2S}{dx^2} = 2 - 0 + 2 = 4$$

Now when x = 8,

$$\left[\frac{d^2S}{dx^2}\right]_{x=8} = 4 > 0$$

As second differential is positive, hence the critical point x = 8 will be the minimum point of the function S.

Therefore, the function S = sum of the squares of the two numbers is minimum at x = 8.

From Equation (1), if x= 8

y = 16 – 8 = 8

Therefore, x = 8 and y = 8 are the two positive numbers whose su is 16 and the sum of the squares is minimum.

3. Question

Divide 15 into two parts such that the square of one number multiplied with the cube of the other number is maximum.

Answer

Given,

• the number 15 is divided into two numbers.

• the product of the square of one number and cube of another number is maximum.

Let us consider,

• x and y are the two numbers

• Sum of the numbers : x + y = 15

• Product of square of the one number and cube of anther number : P = x^3 y^2

Now as,

x + y = 15

y = (15-x) ------ (1)

Consider,

P = x^3y^2

By substituting (1), we have

P = x^3 × $(15\text{-}x)^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with x

$$\frac{dP}{dx} = \frac{d}{dx}[x^3 \times (15-x)^2]$$

$$\frac{dP}{dx} = (15-x)^2\frac{d}{dx}(x^3) + x^3\frac{d}{dx}[(15-x)^2]$$

$$\frac{dP}{dx} = (15-x)^2(3x^2) + x^3[2(15-x)(-1)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$$\frac{dP}{dx} = (15-x)^2(3x^2) + x^3[-30+2x]$$

= 3×[15^2 – 2× (15)×(x) + x^2] x^2 + x^3(2x-30)

= x^2[3× (225 – 30x + x^2)+ x (2x - 30)]

= x^2[675– 90x + 3x^2+ 2x^2 – 60x]

= x^2[5x^2 – 120x + 675]

= 5x^2 [x^2 – 24x + 135] ----- (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dP}{dx} = 5x^2[x^2 - 24x + 135] = 0$$

Hence 5x^2 = 0 (or) x^2 – 24x + 135 = 0

x = 0 (or) $x = \frac{-(-24)\pm\sqrt{(-24)^2-4(1)(135)}}{2\times1}$

x = 0 (or) $x = \frac{24\pm\sqrt{576-540)}}{2}$

x = 0 (or) $x = \frac{24\pm\sqrt{36}}{2}$

x = 0 (or) $x = \frac{24\pm6}{2}$

x = 0 (or) $x = \frac{24+6}{2}$ (or) $x = \frac{24-6}{2}$

x = 0 (or) $x = \frac{30}{2}$ (or) $x = \frac{18}{2}$

x = 0 (or) x = 15 (or) x = 9

Now considering the critical values of x = 0,9,15

Now, for checking if the value of P is maximum or minimum at x=0,9,15, we will perform the second differentiation and check the value of $\frac{d^2P}{dx^2}$ at the critical value x = 0,9,15.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[5x^2\left(x^2 - 24x + 135\right)\right]$$

$$\frac{d^2P}{dx^2} = \left(x^2 - 24x + 135\right)\frac{d}{dx}\left[5x^2\right] + 5x^2\frac{d}{dx}\left[x^2 - 24x + 135\right]$$

$= (x^2 - 24x + 135)(5 \times 2x) + 5x^2(2x - 24 + 0)$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$= (x^2 - 24x + 135)(10x) + 5x^2(2x - 24)$

$= 10x^3 - 240x^2 + 1350x + 10x^3 - 120x^2$

$= 20x^3 - 360x^2 + 1350x$

$= 5x(4x^2 - 72x + 270)$

$$\frac{d^2P}{dx^2} = 5x\left(4x^2 - 72x + 270\right)$$

Now when x = 0,

$$\left[\frac{d^2P}{dx^2}\right]_{x=0} = 5 \times 0\left[4(0)^2 - 72(0) + 270\right]$$

= 0

So, we reject x = 0

Now when x = 15,

$$\left[\frac{d^2P}{dx^2}\right]_{x=15} = 5 \times 15\left[4(15)^2 - 72(15) + 270\right]$$

$= 65[(4 \times 225) - 1080 + 270]$

$= 65[900 - 1080 + 270]$

$= 65[1170 - 1080]$

$= 65 \times (90) > 0$

Hence $\left[\frac{d^2P}{dx^2}\right]_{x=15} > 0$, so at x = 15, the function P is minimum

Now when x = 9,

$$\left[\frac{d^2P}{dx^2}\right]_{x=9} = 5 \times 9\left[4(9)^2 - 72(9) + 270\right]$$

$= 45[(4 \times 81) - 648 + 270]$

$= 45[324 - 648 + 270]$

= 45 [594 – 648]

= 45 × (-54)

= -2430 < 0

As second differential is negative, hence at the critical point x = 9 will be the maximum point of the function P.

Therefore, the function P is maximum at x = 9.

From Equation (1), if x= 9

y = 15 – 9 = 6

Therefore, x = 9 and y = 6 are the two positive numbers whose sum is 15 and the product of the square of one number and cube of another number is maximum.

4. Question

Divide 8 into two positive parts such that the sum of the square of one and the cube of the other is minimum.

Answer

Given,

• the number 8 is divided into two numbers.

• the product of the square of one number and cube of another number is minimum.

Let us consider,

• x and y are the two numbers

• Sum of the numbers : x + y = 8

• Product of square of the one number and cube of anther number : $S = x^3 + y^2$

Now as,

x + y = 8

y = (8-x) ------ (1)

Consider,

$S = x^3 + y^2$

By substituting (1), we have

$S = x^3 + (8-x)^2$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.</u>

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}[x^3 + (8-x)^2]$$

$$\frac{dS}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}[(8-x)^2]$$

$$\frac{dS}{dx} = (3x^2) + 2(8-x)(-1)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dS}{dx} = 3x^2 - 16 + 2x$$

$= 3x^2 + 2x - 16$ ------ (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 3x^2 + 2x - 16 = 0$$

Hence $3x^2 + 2x - 16 = 0$

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-16)}}{2 \times 3}$$

$$= \frac{-2 \pm \sqrt{4 + 192}}{6}$$

$$= \frac{-2 \pm \sqrt{196}}{6}$$

$$x = \frac{-2 \pm 14}{6}$$

$x = \frac{-2+14}{6}$ (or) $x = \frac{-2-14}{6}$

$x = \frac{12}{6}$ (or) $x = \frac{-16}{6}$

x = 2 (or) x = -2.67

Now considering the critical values of x = 2,-2.67

Now, for checking if the value of P is maximum or minimum at x=2,-2.67, <u>we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 2,-2.67.</u>

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[3x^2 + 2x - 16\right]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[3x^2\right] + \frac{d}{dx}[2x] - \frac{d}{dx}[16]$$

= 3 (2x) + 2 (1) - 0

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

= 6x + 2

$$\frac{d^2S}{dx^2} = 6x + 2$$

Now when x = -2.67,

$$\left[\frac{d^2S}{dx^2}\right]_{x=-2.67} = 6(-2.67) + 2$$

= - 16.02 + 2 = -14.02

At x = -2.67 $\frac{d^2S}{dx^2} = -14.02 < 0$ hence, the function S will be maximum at this point.

Now consider x = 2,

$$\left[\frac{d^2S}{dx^2}\right]_{x=2} = 6(2) + 2$$

= 12 + 2 = 14

Hence $\left[\frac{d^2S}{dx^2}\right]_{x=2} = 14 > 0$, so at x = 2, the function S is minimum

As second differential is positive, hence at the critical point x = 2 will be the maximum point of the function S.

Therefore, the function S is maximum at x = 2.

From Equation (1), if x= 2

y = 8 – 2 = 6

Therefore, x = 2 and y = 6 are the two positive numbers whose sum is 8 and the sum of the square of one number and cube of another number is maximum.

5. Question

Divide a into two parts such that the product of the pth power of one part and the qth power of the second part may be maximum.

Answer

Given,

• the number 'a' is divided into two numbers.

• the product of the pth power of one number and qth power of another number is maximum.

Let us consider,

• x and y are the two numbers

• Sum of the numbers : x + y = a

• Product of square of the one number and cube of anther number : $P = x^p y^q$

Now as,

x + y = a

y = (a-x) ------ (1)

Consider,

$P = x^p y^q$

By substituting (1), we have

$P = x^p \times (a-x)^q$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with x

$$\frac{dP}{dx} = \frac{d}{dx}\left[x^p \times (a-x)^q\right]$$

$$\frac{dP}{dx} = (a-x)^q \frac{d}{dx}(x^p) + x^p \frac{d}{dx}\left[(a-x)^q\right]$$

$$\frac{dP}{dx} = (a-x)^q\,(px^{p-1}) + x^p\left[q(a-x)^{q-1}(-1)\right]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$$\frac{dP}{dx} = x^{p-1}(a-x)^{q-1}[(a-x)p - xq]$$

$= x^{p-1}(a-x)^{q-1}$[ap-xp-xq]

$= x^{p-1}(a-x)^{q-1}[ap - x(p+q)]$ ----- (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dP}{dx} = x^{p-1}(a-x)^{q-1}[ap - x(p+q)] = 0$$

Hence $x^{p-1} = 0$ (or) $(a-x)^{q-1}$ (or) $ap - x(p+q) = 0$

$x = 0$ (or) $x = a$ (or) $x = \frac{ap}{p+q}$

Now considering the critical values of $x = 0, a$ and $x = \frac{ap}{p+q}$

Now, using the First Derivative test,

For f, a continuous function which has a critical point c, then, function has the local maximum at c, if f'(x) changes the sign from positive to negative as x increases through c, i.e. f'(x)>0 at every point close to the left of c and f'(x)<0 at every point close to the right of c.

Now when x = 0,

$$\left[\frac{dP}{dx}\right]_{x=0} = 0$$

So, we reject x = 0

Now when x = a,

$$\left[\frac{dP}{dx}\right]_{x=a} = 0$$

Hence we reject x = a

Now when $x < \frac{ap}{p+q}$,

$$\left[\frac{dP}{dx}\right]_{x<\frac{ap}{p+q}} = \left(\frac{ap}{p+q}\right)^{p-1}\left(a - \frac{ap}{p+q}\right)^{q-1}\left[ap - \frac{ap}{p+q}(p+q)\right] > 0 \text{ ---- (4)}$$

Now when $x > \frac{ap}{p+q}$,

$$\left[\frac{dP}{dx}\right]_{x>\frac{ap}{p+q}} = \left(\frac{ap}{p+q}\right)^{p-1}\left(a - \frac{ap}{p+q}\right)^{q-1}\left[ap - \frac{ap}{p+q}(p+q)\right] < 0 \text{ ---- (5)}$$

By using first derivative test, from (4) and (5), we can conclude that, the function P has local maximum at $x = \frac{ap}{p+q}$

From Equation (1), if $x = \frac{ap}{p+q}$

$$y = a - \frac{ap}{p+q} = \frac{a(p+q) - ap}{p+q} = \frac{ap}{p+q}$$

Therefore, $x = \frac{ap}{p+q}$ and $y = \frac{aq}{p+q}$ are the two positive numbers whose sum together to give the number 'a' and whose product of the pth power of one number and qth power of the other number is maximum.

6. Question

The rate of working of an engine is given by.

$$R = 15\upsilon + \frac{6000}{\upsilon}, \text{ where } 0 < \upsilon < 30$$

and υ is the speed of the engine. Show that R is the least when $\upsilon = 20$.

Answer

Given:

Rate of working of an engine R, v is the speed of the engine:

$R = 15v + \frac{6000}{v}$, where 0<v<30

For finding the maximum/ minimum of given function, we can find it by differentiating it with v and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Now, differentiating the function R with respect to v.

$$\frac{dR}{dv} = \frac{d}{dv}\left[15v + \frac{6000}{v}\right]$$

$$\frac{dR}{dv} = \frac{d}{dv}[15v] + \frac{d}{dv}\left[\frac{6000}{v}\right]$$

$$\frac{dR}{dv} = 15 + \left[\frac{6000}{v^2}\right](-1) = 15 - \frac{6000}{v^2} \text{ ----- (1)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Equating equation (1) to zero to find the critical value.

$$\frac{dR}{dv} = 15 - \frac{6000}{v^2} = 0$$

$$15 = \frac{6000}{v^2}$$

$$v^2 = \frac{6000}{15} = 400$$

$v^2 = 400$

$v = \pm\sqrt{400}$

v = 20 (or) v = -20

As given in the question 0<v<30, v = 20

Now, for checking if the value of R is maximum or minimum at v=20, we will perform the second differentiation and check the value of $\frac{d^2R}{dv^2}$ at the critical value v = 20.

Differentiating Equation (1) with respect to v again:

$$\frac{d^2R}{dv^2} = \frac{d}{dx}\left[15 - \frac{6000}{v^2}\right]$$

$$= \frac{d}{dx}[15] - \frac{d}{dx}\left[\frac{6000}{v^2}\right]$$

$$= 0 - (-2)\left[\frac{6000}{v^3}\right]$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$= 2\left[\frac{6000}{v^3}\right]$$

$$\frac{d^2R}{dv^2} = \left[\frac{12000}{v^3}\right] \text{ ----- (2)}$$

Now find the value of $\left(\frac{d^2R}{dv^2}\right)_{v=20}$

$$\left(\frac{d^2R}{dv^2}\right)_{v=20} = \left[\frac{12000}{(20)^3}\right] = \frac{12000}{20 \times 20 \times 20} = \frac{3}{2} > 0$$

So, at critical point v = 20. The function R is at its minimum.

Hence, the function R is at its minimum at v = 20.

7. Question

Find the dimensions of the rectangle of area 96 cm^2 whose perimeter is the least. Also, find the perimeter of the rectangle.

Answer

Given,

• Area of the rectangle is 93 cm^2.

• The perimeter of the rectangle is also fixed.

Let us consider,

• x and y be the lengths of the base and height of the rectangle.

• Area of the rectangle = A = x × y = 96 cm^2

• Perimeter of the rectangle = P = 2 (x + y)

As,

x × y = 96

$y = \frac{96}{x}$ ------ (1)

Consider the perimeter function,

P = 2 (x + y)

Now substituting (1) in P,

$P = 2\left(x + \frac{96}{x}\right)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[2\left(x + \frac{96}{x}\right)\right]$$

$$\frac{dP}{dx} = \frac{d}{dx}(2x) + 2\frac{d}{dx}\left(\frac{96}{x}\right)$$

$$\frac{dP}{dx} = 2(1) + 2\left(\frac{96}{x^2}\right)(-1)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{dP}{dx} = 2 - \left(\frac{192}{x^2}\right)$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 2 - \left(\frac{192}{x^2}\right) = 0$$

$$2 = \left(\frac{192}{x^2}\right)$$

$$x^2 = \left(\frac{192}{2}\right) = 96$$

$$x = \sqrt{96}$$

$$x = \pm 4\sqrt{6}$$

As the length and breadth of a rectangle cannot be negative, hence $x = 4\sqrt{6}$

Now to check if this critical point will determine the least perimeter, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2 - \left(\frac{192}{x^2}\right)\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(2) - \frac{d}{dx}\left(\frac{192}{x^2}\right)$$

$$\frac{d^2P}{dx^2} = 0 - (-2)\left(\frac{192}{x^3}\right)$$

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$\frac{d^2P}{dx^2} = \left(\frac{2 \times 192}{x^3}\right)$ ------ (4)

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=4\sqrt{6}}$

$$\frac{d^2P}{dx^2} = \left(\frac{2 \times 192}{(4\sqrt{6})^3}\right)$$

$$= \left(\frac{2 \times 192}{4\sqrt{6} \times 4\sqrt{6} \times 4\sqrt{6}}\right)$$

$$= \left(\frac{2 \times 192}{4\sqrt{6} \times 4\sqrt{6} \times 4\sqrt{6}}\right) = \frac{1}{\sqrt{6}}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=4\sqrt{6}} = \frac{1}{\sqrt{6}} > 0$, so the function P is minimum at $x = 4\sqrt{6}$.

Now substituting $x = 4\sqrt{6}$ in equation (1):

$$y = \frac{96}{4\sqrt{6}}$$

$$y = \frac{96\sqrt{6}}{4 \times 6}$$

[By rationalizing he numerator and denominator with $\sqrt{6}$]

$$\therefore y = 4\sqrt{6}$$

Hence, area of the rectangle with sides of a rectangle with $x = 4\sqrt{6}$ and $y = 4\sqrt{6}$ is 96cm^2 and has the least perimeter.

Now the perimeter of the rectangle is

$P = 2(4\sqrt{6} + 4\sqrt{6}) = 2(8\sqrt{6}) = 16\sqrt{6}$ cms

The least perimeter is $16\sqrt{6}$ cms.

8. Question

Prove that the largest rectangle with a given perimeter is a square.

Answer

Given,

• Rectangle with given perimeter.

Let us consider,

• 'p' as the fixed perimeter of the rectangle.

• 'x' and 'y' be the sides of the given rectangle.

• Area of the rectangle, A = x × y.

Now as consider the perimeter of the rectangle,

p = 2(x +y)

p = 2x + 2y

$y = \frac{p-2x}{2}$ ----- (1)

Consider the area of the rectangle,

A = x × y

Substituting (1) in the area of the rectangle,

$A = x \times \left(\frac{p-2x}{2}\right)$

$A = \frac{1}{2} \times (px - 2x^2)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{2}(px - 2x^2)\right]$

$\frac{dA}{dx} = \frac{1}{2}\frac{d}{dx}(px) - \frac{1}{2}\frac{d}{dx}(2x^2)$

$\frac{dA}{dx} = \frac{1}{2}(p) - \frac{2}{2}(2x)$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dA}{dx} = \frac{p}{2} - (2x)$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$\frac{dA}{dx} = \frac{p}{2} - (2x) = 0$

$2x = \frac{p}{2}$

$x = \frac{p}{4}$

Now to check if this critical point will determine the largest rectangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{p}{2} - (2x)\right]$

$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{p}{2}\right) - \frac{d}{dx}(2x)$

$\frac{d^2A}{dx^2} = 0 - 2 = -2$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dx^2} = -2$ ------ (4)

Now, consider the value of $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{p}{4}}$

$\frac{d^2A}{dx^2} = -2 < 0$

As $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{p}{4}} = -2 < 0$, so the function P is maximum at $x = \frac{p}{4}$.

Now substituting $x = \frac{p}{4}$ in equation (1):

$y = \frac{p - 2\left(\frac{p}{4}\right)}{2}$

$y = \frac{p - \frac{p}{2}}{2} = \frac{p}{4}$

$\therefore\ y = \frac{p}{4}$

As $= y = \frac{p}{4}$ the sides of the taken rectangle are equal, we can clearly say that a largest rectangle which has a given perimeter is a square.

9. Question

Given the perimeter of a rectangle, show that its diagonal is minimum when it is a square.

Answer

Given,

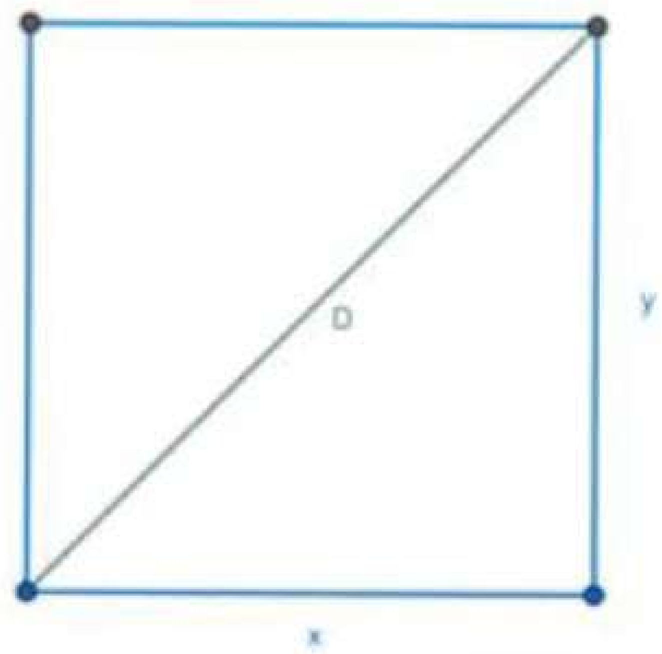
• Rectangle with given perimeter.

Let us consider,

• ‘p’ as the fixed perimeter of the rectangle.

• ‘x’ and ‘y’ be the sides of the given rectangle.

• Diagonal of the rectangle, $D = \sqrt{x^2 + y^2}$. (using the hypotenuse formula)



Now as consider the perimeter of the rectangle,

p = 2(x +y)

p = 2x + 2y

$y = \frac{p-2x}{2}$ ----- (1)

Consider the diagonal of the rectangle,

$D = \sqrt{x^2 + y^2}$

Substituting (1) in the diagonal of the rectangle,

$$D = \sqrt{x^2 + \left(\frac{p-2x}{2}\right)^2}$$

[squaring both sides]

$Z = D^2 = x^2 + \left(\frac{p-2x}{2}\right)^2$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[x^2 + \left(\frac{p-2x}{2}\right)^2\right]$$

$$\frac{dZ}{dx} = \frac{d}{dx}(x^2) + \frac{1}{4}\frac{d}{dx}[(p-2x)^2]$$

$$\frac{dZ}{dx} = 2x + \frac{1}{4}[2(p-2x)(-2)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

= 2x - p + 2x

$\frac{dZ}{dx} = 4x - p$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 4x - p = 0$$

4x –p = 0

4x = p

$$x = \frac{p}{4}$$

Now to check if this critical point will determine the minimum diagonal, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[4x - p]$$

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}(4x) - \frac{d}{dx}(p)$$

= 4 + 0

[Since $\frac{d}{dx}$ (constant) = 0 and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2Z}{dx^2} = 4$ ------ (4)

Now, consider the value of $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{p}{4}}$

$$\frac{d^2Z}{dx^2} = 4 > 0$$

As $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{p}{4}} = 4 > 0$, so the function Z is minimum at $x = \frac{p}{4}$.

Now substituting $x = \frac{p}{4}$ in equation (1):

$$y = \frac{p - 2\left(\frac{p}{4}\right)}{2}$$

$$y = \frac{p - \frac{p}{2}}{2} = \frac{p}{4}$$

$$\therefore y = \frac{p}{4}$$

As $x = y = \frac{p}{4}$ the sides of the taken rectangle are equal, we can clearly say that a rectangle with minimum diagonal which has a given perimeter is a square.

10. Question

Show that a rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}\,a$.

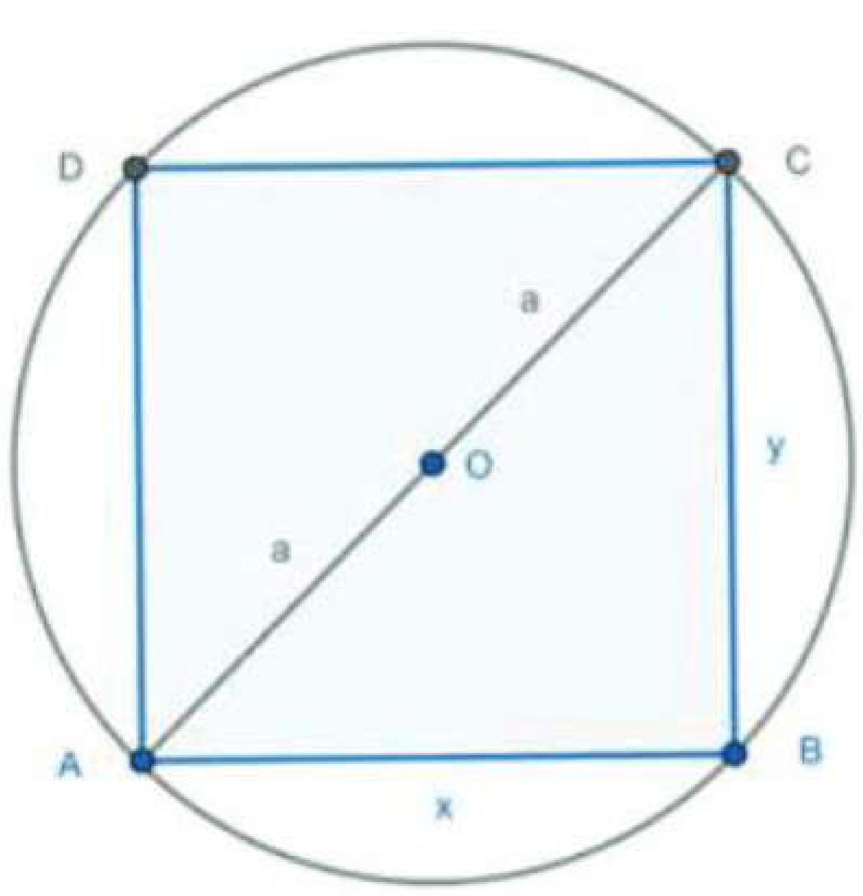
Answer

Given,

• Rectangle is of maximum perimeter.

• The rectangle is inscribed inside a circle.

• The radius of the circle is ‘a’.



Let us consider,

• 'x' and 'y' be the length and breadth of the given rectangle.

• Diagonal $AC^2 = AB^2 + BC^2$ is given by $4a^2 = x^2+y^2$ (as AC = 2a)

• Perimeter of the rectangle, P = 2(x+y)

Consider the diagonal,

$4a^2 = x^2 + y^2$

$y^2 = 4a^2 - x^2$

$y = \sqrt{4a^2 - x^2}$ ---- (1)

Now, perimeter of the rectangle, P

P = 2x + 2y

Substituting (1) in the perimeter of the rectangle.

$P = 2x + 2\sqrt{4a^2 - x^2}$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[2x + 2\sqrt{4a^2 - x^2}\right]$$

$$\frac{dP}{dx} = \frac{d}{dx}(2x) + 2\frac{d}{dx}[\sqrt{4a^2 - x^2}]$$

$$\frac{dP}{dx} = 2 + 2[\frac{1}{2}(4a^2 - x^2)^{-\frac{1}{2}}(-2x)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4a^2 - x^2}}$$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4a^2 - x^2}} = 0$$

$$2 = \frac{2x}{\sqrt{4a^2 - x^2}}$$

$$\sqrt{4a^2 - x^2} = x$$

[squaring on both sides]

$4a^2 - x^2 = x^2$

$2x^2 = 4a^2$

$x^2 = 2a^2$

$x = \pm a\sqrt{2}$

$x = a\sqrt{2}$

[as x cannot be negative]

Now to check if this critical point will determine the maximum diagonal, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2 - \frac{2x}{\sqrt{4a^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(2) - \frac{d}{dx}\left(\frac{2x}{\sqrt{4a^2 - x^2}}\right)$$

$$\frac{d^2P}{dx^2} = 0 - \left[\frac{\sqrt{4a^2 - x^2}\,\frac{d}{dx}(2x) - (2x)\frac{d}{dx}(\sqrt{4a^2 - x^2})}{(\sqrt{4a^2 - x^2})^2}\right]$$

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{4a^2 - x^2}\,(2) - (2x)\frac{1}{2}(4a^2 - x^2)^{-\frac{1}{2}}(-2x)}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{4a^2 - x^2}\,(2) + (2x^2)(4a^2 - x^2)^{-\frac{1}{2}}}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{2\sqrt{4a^2 - x^2} + \frac{2x^2}{\sqrt{4a^2 - x^2}}}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{2(4a^2 - x^2) + 2x^2}{(4a^2 - x^2)^{\frac{3}{2}}}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{8a^2}{(4a^2 - x^2)^{\frac{3}{2}}}\right] \text{ ----- (4)}$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}}$

$$\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\left[\frac{8a^2}{(4a^2 - (a\sqrt{2})^2)^{\frac{3}{2}}}\right]$$

$$\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\left[\frac{8a^2}{(4a^2 - 2a^2)^{\frac{3}{2}}}\right] = -\frac{8a^2}{(2a^2)^{\frac{3}{2}}} = -\frac{8a^2}{2\sqrt{2}\,a^3} = -\frac{2\sqrt{2}}{a}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\frac{2\sqrt{2}}{a} < 0$, so the function P is maximum at $x = a\sqrt{2}$.

Now substituting $x = a\sqrt{2}$ in equation (1):

$$y = \sqrt{4a^2 - (a\sqrt{2})^2}$$

$$y = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2}$$

$$\therefore y = a\sqrt{2}$$

As $x = y = a\sqrt{2}$ the sides of the taken rectangle are equal, we can clearly say that a rectangle with maximum perimeter which is inscribed inside a circle of radius ‘a’ is a square.

11. Question

The sum of the perimeters of a square and a circle is given. Show that the sum of their areas is least when the side of the square is equal to the diameter of the circle.

Answer

Given,

• Sum of perimeter of square and circle.

Let us consider,

• ‘x’ be the side of the square.

• ‘r’ be the radius of the circle.

• Let ‘p’ be the sum of perimeters of square and circle.

p = 4x + 2πr

Consider the sum of the perimeters of square and circle.

p = 4x + 2πr

4x = p – 2πr

$x = \frac{p-2\pi r}{4}$ ---- (1)

Sum of the area of the circle and square is

$A = x^2 + \pi r^2$

Substituting (1) in the sum of the areas,

$$A = \left(\frac{p - 2\pi r}{4}\right)^2 + \pi r^2$$

$$A = \frac{1}{16}[p^2 + 4\pi^2 r^2 - 4\pi pr] + \pi r^2 \text{ ------ (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dA}{dr} = \frac{d}{dr}\left[\frac{1}{16}[p^2 + 4\pi^2 r^2 - 4\pi pr] + \pi r^2\right]$$

$$\frac{dA}{dr} = \frac{1}{16}\frac{d}{dr}(p^2 + 4\pi^2 r^2 - 4\pi pr) + \frac{d}{dr}[\pi r^2]$$

$$\frac{dA}{dr} = \frac{1}{16}(0 + 8\pi^2 r - 4\pi p) + 2\pi r$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(\text{constant}) = 0$]

$$\frac{dA}{dr} = \frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r \text{ ------ (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dA}{dr} = \frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r = 0$$

$$\left(\frac{\pi^2}{2} + 2\pi\right) r - \frac{\pi p}{4} = 0$$

$$r = \frac{\frac{\pi p}{4}}{\frac{\pi^2}{2} + 2\pi} = \frac{2\pi p}{4(\pi^2 + 4\pi)} = \frac{\pi p}{2(\pi^2 + 4\pi)}$$

$$r = \frac{\pi p}{2\pi(\pi + 4)} = \frac{p}{2(\pi + 4)}$$

$$r = \frac{p}{2(\pi + 4)}$$

Now to check if this critical point will determine the least of the sum of the areas of square and circle, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with r:

$$\frac{d^2A}{dr^2} = \frac{d}{dx}\left[\frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r\right]$$

$$\frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{\pi^2 r}{2}\right) - \frac{d}{dr}\left(\frac{\pi p}{4}\right) + \frac{d}{dr}(2\pi r)$$

$$\frac{d^2A}{dr^2} = \frac{\pi^2}{2} - 0 + 2\pi$$

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dr^2} = \frac{\pi^2}{2} + 2\pi$ ----- (4)

Now, consider the value of $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}}$

$$\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}} = \frac{\pi^2}{2} + 2\pi$$

As $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}} = \frac{\pi^2}{2} + 2\pi > 0$, so the function A is minimum at $r = \frac{p}{2(\pi+4)}$.

Now substituting $r = \frac{p}{2(\pi+4)}$ in equation (1):

$$x = \frac{p - 2\pi\left(\frac{p}{2(\pi + 4)}\right)}{4}$$

$$x = \frac{p(\pi + 4) - \pi p}{4 \times (\pi + 4)} = \frac{\pi p + 4p - \pi p}{4\pi + 16} = \frac{4p}{4(\pi + 4)}$$

$$x = \frac{p}{\pi + 4}$$

As the side of the square,

$$x = \frac{p}{\pi + 4}$$

$$x = 2\left[\frac{p}{2(\pi + 4)}\right] = 2r$$

[as $r = \frac{p}{2(\pi+4)}$]

Therefore, side of the square, x = 2r = diameter of the circle.

12. Question

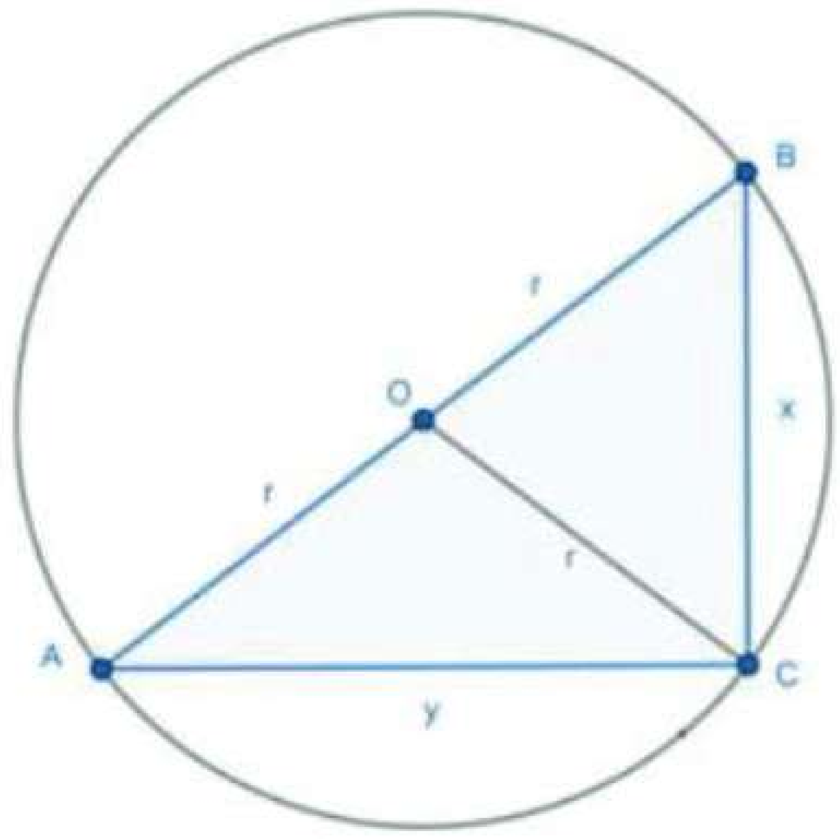
Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.

Answer

Given,

• A right angle triangle is inscribed inside the circle.

• The radius of the circle is given.



Let us consider,

• ‘r’ is the radius of the circle.

• ‘x’ and ‘y’ be the base and height of the right angle triangle.

• The hypotenuse of the ΔABC = $AB^2 = AC^2 + BC^2$

AB = 2r, AC = y and BC = x

Hence,

$4r^2 = x^2 + y^2$

$y^2 = 4r^2 - x^2$

$y = \sqrt{4r^2 - x^2}$ --- (1)

Now, Area of the ΔABC is

$A = \frac{1}{2} \times \text{base} \times \text{height}$

$A = \frac{1}{2} \times x \times y$

Now substituting (1) in the area of the triangle,

$A = \frac{1}{2} x (\sqrt{4r^2 - x^2})$

[Squaring both sides]

$Z = A^2 = \frac{1}{4} x^2 (4r^2 - x^2)$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f’(c) = 0.

Differentiating the equation (2) with respect to x:

$\frac{dZ}{dx} = \frac{d}{dx}\left[\frac{1}{4} x^2 (4r^2 - x^2)\right]$

$\frac{dZ}{dx} = \frac{1}{4}\left[(4r^2 - x^2)\frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(4r^2 - x^2)\right]$

$\frac{dZ}{dx} = \frac{1}{4}[(4r^2 - x^2) \times (2x) + x^2 (0 - 2x)]$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u.v) = v\frac{du}{dx} + u\frac{dv}{dx}$]

$$\frac{dZ}{dx} = \frac{1}{4}[\,8r^2x - \;2x^3 - 2x^3]$$

$$\frac{dZ}{dx} = \frac{1}{4}[\,8r^2x - \;4x^3] \;= \frac{4x}{4}[2r^2 - \;x^2]$$

$$\frac{dZ}{dx} = 2r^2x - \;x^3 \text{ ------ (3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = \;2r^2x - \;x^3 \;= 0$$

$$2r^2x = x^3$$

$$x^2 = \;2r^2$$

$$x = \;\pm\sqrt{2r^2}$$

$$x = r\sqrt{2}$$

[as the base of the triangle cannot be negative.]

<u>Now to check if this critical point will determine the maximum area of the triangle, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[2r^2x - \;x^3\;]$$

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}(2r^2x) - \;\frac{d}{dx}\;(x^3)$$

$$\frac{d^2Z}{dx^2} = \;2r^2 - \;3x^2 \text{ ----- (4)}$$

[Since $\frac{d}{dx}\;(x^n) = \;nx^{n-1}$]

Now, consider the value of $\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}}$

$$\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = 2r^2 - \;3(r\sqrt{2})^2 = 2\,r^2 - 6r^2 = \;-4r^2$$

As $\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} \;= \;-4r^2 \;< 0$, so the function A is maximum at $x = r\sqrt{2}$.

Now substituting $x = r\sqrt{2}$ in equation (1):

$$y = \;\sqrt{4r^2 - \;(r\sqrt{2}\,)^2}$$

$$y = \;\sqrt{4r^2 - \;2r^2} \;= \;\sqrt{\,2r^2} \;= \;r\sqrt{2}$$

As $x = \;y \;= \;r\sqrt{2}$, the base and height of the triangle are equal, which means that two sides of a right angled triangle are equal,

Hence the given triangle, which is inscribed in a circle, is an isosceles triangle with sides AC and BC equal.

13. Question

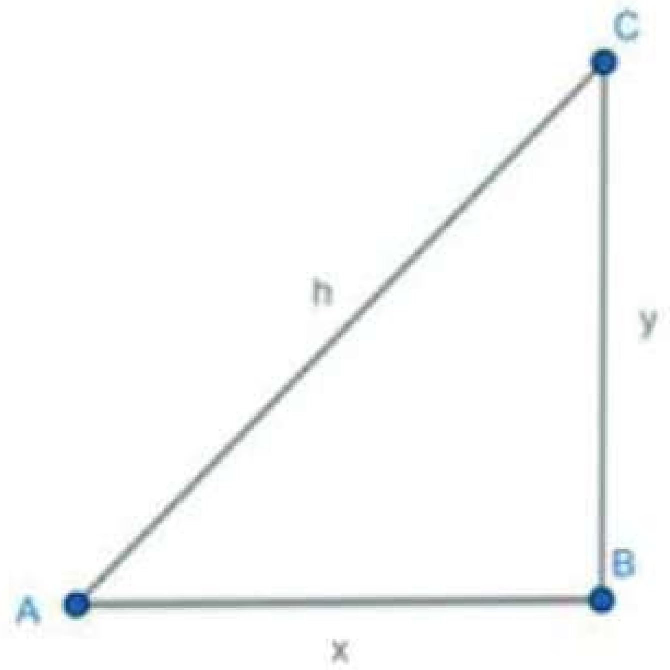
Prove that the perimeter of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.

Answer

Given,

• A right angle triangle.

• Hypotenuse of the given triangle is given.



Let us consider,

• 'h' is the hypotenuse of the given triangle.

• 'x' and 'y' be the base and height of the right angle triangle.

• The hypotenuse of the $\Delta ABC = AC^2 = AB^2 + BC^2$

AC = h, AB = x and BC = y

Hence,

$h^2 = x^2 + y^2$

$y^2 = h^2 - x^2$

$y = \sqrt{h^2 - x^2}$ --- (1)

Now, perimeter of the ΔABC is

P = h + x + y

Now substituting (1) in the area of the triangle,

$P = h + x + \sqrt{h^2 - x^2}$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[h + x + \sqrt{h^2 - x^2}\right]$$

$$\frac{dP}{dx} = \left[\frac{d}{dx}(h) + \frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{h^2 - x^2})\right]$$

$$\frac{dP}{dx} = 0 + 1 + \frac{1}{2}\left(\frac{-2x}{\sqrt{h^2 - x^2}}\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dP}{dx} = 1 - \frac{x}{\sqrt{h^2 - x^2}}$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 1 - \frac{x}{\sqrt{h^2 - x^2}} = 0$$

$$\frac{x}{\sqrt{h^2 - x^2}} = 1$$

$$x = \sqrt{h^2 - x^2}$$

[squaring on both sides]

$$x^2 = h^2 - x^2$$

$$x^2 = \frac{h^2}{2}$$

$$x = \pm\sqrt{\frac{h^2}{2}}$$

$$x = \frac{h}{\sqrt{2}}$$

[as the base of the triangle cannot be negative.]

Now to check if this critical point will determine the maximum perimeter of the triangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[1 - \frac{x}{\sqrt{h^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{x}{\sqrt{h^2 - x^2}}\right)$$

$$\frac{d^2P}{dx^2} = 0 - \left[\frac{\sqrt{h^2 - x^2}\,\frac{d}{dx}(x) - x\,\frac{d}{dx}\left(\sqrt{h^2 - x^2}\right)}{\left(\sqrt{h^2 - x^2}\right)^2}\right]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ if u and v are two functions of x, then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{h^2 - x^2}\,(1) - x\left(\frac{-2x}{2\sqrt{h^2 - x^2}}\right)}{h^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{(\sqrt{h^2 - x^2})^2 + x^2}{h^2 - x^2\sqrt{h^2 - x^2}}\right] = -\left[\frac{h^2}{(h^2 - x^2)\sqrt{h^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{h^2}{(h^2 - x^2)^{\frac{3}{2}}}\right]$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}}$

$$\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\left[\frac{h^2}{\left(h^2 - \left(\frac{h}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}}}\right] = -\left[\frac{h^2}{\left(\frac{h^2}{2}\right)^{\frac{3}{2}}}\right] = -\left[\frac{h^2}{\left(\frac{h^2}{2}\right)^{\frac{3}{2}}}\right] = -\frac{2^{\frac{3}{2}}}{h}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\frac{2^{\frac{3}{2}}}{h} < 0$, so the function A is maximum at $x = \frac{h}{\sqrt{2}}$.

Now substituting $x = \frac{h}{\sqrt{2}}$ in equation (1):

$$y = \sqrt{h^2 - \left(\frac{h}{\sqrt{2}}\right)^2}$$

$y = \sqrt{\frac{h^2}{2}} = \frac{h}{\sqrt{2}}$

As $x = y = \frac{h}{\sqrt{2}}$, the base and height of the triangle are equal, which means that two sides of a right angled triangle are equal,

Hence the given triangle is an isosceles triangle with sides AB and BC equal.

14. Question

The perimeter of a triangle is 8 cm. If one of the sides of the triangle be 3 cm, what will be the other two sides for maximum area of the triangle?

Answer

Given,

• Perimeter of a triangle is 8 cm.

• One of the sides of the triangle is 3 cm.

• The area of the triangle is maximum.

Let us consider,

• 'x' and 'y' be the other two sides of the triangle.

Now, perimeter of the ΔABC is

8 = 3 + x + y

y = 8-3-x = 5-x

y = 5-x --- (1)

Consider the Heron's area of the triangle,

$A = \sqrt{s(s-a)(s-b)(s-c)}$

Where $s = \frac{a+b+c}{2}$

As perimeter = a + b+ c = 8

$s = \frac{8}{2} = 4$

Now Area of the triangle is given by

$A = \sqrt{8\,(8-3)(8-x)(8-y)}$

Now substituting (1) in the area of the triangle,

$A = \sqrt{4(4-3)(4-x)(4-(5-x))}$

$A = \sqrt{4(4-x)(x-1)}$

$A = \sqrt{4(\,4x - 4 - \,x^2 + x)} = \sqrt{4(\,5x - \,x^2 - 4)}$

$A = \sqrt{4(\,5x - \,x^2 - 4)}$

[squaring on both sides]

$Z = A^2 = 4(5x - x^2 - 4)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}[4(5x - x^2 - 4)]$$

$$\frac{dZ}{dx} = 4\frac{d}{dx}(5x) - 4\frac{d}{dx}(x^2) - 4\frac{d}{dx}(4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dZ}{dx} = 4(5) - 4(2x) - 0$$

$\frac{dZ}{dx} = 20 - 8x$------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 20 - 8x = 0$$

20 – 8x = 0

8x = 20

$$x = \frac{5}{2}$$

Now to check if this critical point will determine the maximum area of the triangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[20 - 8x]$$

$\frac{d^2Z}{dx^2} = -8$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

As $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{5}{2}} = -8 < 0$, so the function A is maximum at $x = \frac{5}{2}$.

Now substituting $x = \frac{5}{2}$ in equation (1):

y = 5 – 2.5

y = 2.5

As x = y = 2.5, two sides of the triangle are equal,

Hence the given triangle is an isosceles triangle with two sides equal to 2.5 cm and the third side equal to 3cm.

15. Question

A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 metres. Find the dimensions of the windows to admit maximum light through it.

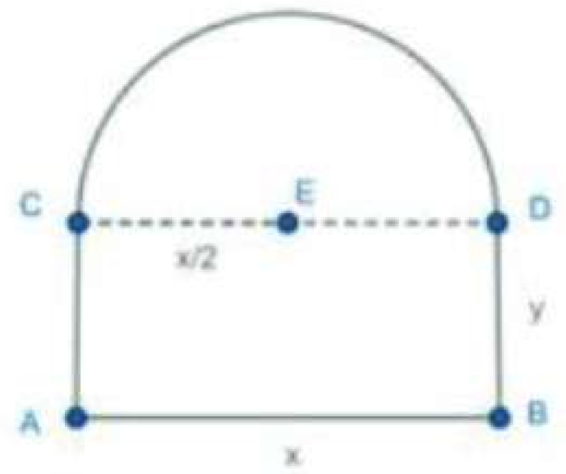
Answer

Given,

• Window is in the form of a rectangle which has a semicircle mounted on it.

• Total Perimeter of the window is 10 metres.

• The total area of the window is maximum.



Let us consider,

• The breadth and height of the rectangle be 'x' and 'y'.

• The radius of the semicircle will be half of the base of the rectangle.

Given Perimeter of the window is 10 meters:

$$10 = (x + 2y) + \frac{1}{2}\left[2\pi\left(\frac{x}{2}\right)\right]$$

[as the perimeter of the window will be equal to one side (x) less to the perimeter of rectangle and the perimeter of the semicircle.]

$$10 = (x + 2y) + \left(\frac{\pi x}{2}\right)$$

From here,

$$2y = 10 - x - \left(\frac{\pi x}{2}\right) = \frac{20 - 2x - \pi x}{2}$$

$y = \frac{20-2x-\pi x}{4}$ ----- (1)

Now consider the area of the window,

Area of the window = area of the semicircle + area of the rectangle

$$A = \frac{1}{2}\left[\pi\left(\frac{x}{2}\right)^2\right] + x\,y$$

Substituting (1) in the area equation:

$$A = \frac{1}{2}\left[\pi\left(\frac{x}{2}\right)^2\right] + x\left(\frac{20 - 2x - \pi x}{4}\right)$$

$$A = \frac{1}{8}\left[\pi x^2\right] + \left(\frac{20x - 2x^2 - \pi x^2}{4}\right)$$

$$A = \frac{\pi x^2 - 2\pi x^2 + 40x - 4x^2}{8}$$

$A = \frac{1}{8}\left[x^2(\pi - 2\pi - 4) + 40x\right]$ ------ (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{8}\left[x^2(\pi - 2\pi - 4) + 40x\right]\right]$$

$$\frac{dA}{dx} = \frac{1}{8}\frac{d}{dx}\left(x^2(\pi - 2\pi - 4)\right) + \frac{1}{8}\frac{d}{dx}(40x)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dA}{dx} = \frac{1}{8}\left[2x(-\pi - 4)\right] + \frac{1}{8}(40)$$

$\frac{dA}{dx} = \frac{1}{4}[x(-\pi - 4)] + 5$ ------ (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dA}{dx} = \frac{1}{4}[x(-\pi - 4)] + 5 = 0$$

$$\frac{1}{4}[x(-\pi - 4)] + 5 = 0$$

$$\frac{1}{4}[x(4 + \pi)] = 5$$

x (4 +π) = 20

$$x = \frac{20}{(4 + \pi)}$$

<u>Now to check if this critical point will determine the maximum area of the window, we need to check with second differential which needs to be negative.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{1}{4}[x(-\pi - 4)] + 5\right]$$

$$\frac{d^2A}{dx^2} = \frac{d}{dx}[x(-\pi - 4)] + \frac{d}{dx}(5)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dx^2} = (-\pi - 4)(1) + 0 = -(\pi + 4)$ ------ (4)

As $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{20}{(4+\pi)}} = -(\pi + 4) < 0$, so the function A is maximum at $x = \frac{20}{(4+\pi)}$.

Now substituting $x = \frac{20}{(4+\pi)}$ in equation (1):

$$y = \frac{20 - \left(\frac{20}{(4 + \pi)}\right)(\pi + 2)}{4}$$

$$y = \frac{20(4 + \pi) - (20)(\pi + 2)}{4\,(4 + \pi)} = \frac{20\,[4 + \pi - \pi - 2]}{4(4 + \pi)} = \frac{20 \times 2}{4(4 + \pi)}$$

$$y = \frac{5 \times 2}{(4 + \pi)} = \frac{10}{(4 + \pi)}$$

Hence the given window with maximum area has breadth, $x = \frac{20}{(4+\pi)}$ and height, $y = \frac{10}{(4+\pi)}$.

16. Question

A square piece of tin of side 12 cm is to be made into a box without a lid by cutting a square from each corner and folding up the flaps to form the sides. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume.

Answer

Given,

• Side of the square piece is 12 cms.

• the volume of the formed box is maximum.