EXERCISE 20.3

Find the sum of the following geometric progressions:
(i) 2, 6, 18, … to 7 terms
(ii) 1, 3, 9, 27, … to 8 terms
(iii) 1, -1/2, ¼, -1/8, …
(iv) $(a^2 – b^2)$, (a - b), (a-b)/(a+b), … to n terms
(v) 4, 2, 1, ½ … to 10 terms
Solution:
(i) 2, 6, 18, … to 7 terms
We know that, sum of GP for n terms = $a(r^n – 1)/(r – 1)$
Given:
$a = 2, r = t_2/t_1 = 6/2 = 3, n = 7$
Now let us substitute the values in
$a(r^n – 1)/(r – 1) = 2 (3^7 - 1)/(3-1)$
$= 2 (3^7 - 1)/2$
$= 3^7 – 1$
$= 2187 – 1$
$= 2186$

(ii) 1, 3, 9, 27, … to 8 terms
We know that, sum of GP for n terms = $a(r^n – 1)/(r – 1)$
Given:
$a = 1, r = t_2/t_1 = 3/1 = 3, n = 8$
Now let us substitute the values in
$a(r^n – 1)/(r – 1) = 1 (3^8 - 1)/(3-1)$
$= (3^8 - 1)/2$
$= (6561 – 1)/2$
$= 6560/2$
$= 3280$

(iii) 1, -1/2, ¼, -1/8, …
We know that, sum of GP for infinity = $a/(1 - r)$
Given:
$a = 1, r = t_2/t_1 = (-1/2)/1 = -1/2$
Now let us substitute the values in
$a/(1 - r) = 1/(1 – (-1/2))$
$= 1/(1 + 1/2)$
$= 1/((2+1)/2)$

= 1/(3/2)
= 2/3

(iv) $(a^2 – b^2)$, (a - b), (a-b)/(a+b), … to n terms
We know that, sum of GP for n terms = $a(r^n – 1)/(r – 1)$
Given:
$a = (a^2 – b^2)$, $r = t_2/t_1 = (a-b)/(a^2 – b^2) = (a-b)/(a-b)(a+b) = 1/(a+b)$, n = n
Now let us substitute the values in
$a(r^n – 1)/(r – 1)$ =

$$= (a^2 - b^2)\left(\frac{1-\left(\frac{1}{a+b}\right)^n}{1-\left(\frac{1}{a+b}\right)}\right)$$

$$= (a^2 - b^2)\left(\frac{\left(\frac{(a+b)^n-1}{(a+b)^n}\right)}{\frac{(a+b)-1}{a+b}}\right)$$

$$= \frac{(a+b)(a-b)}{(a+b)^{n-1}}\left(\frac{(a+b)^n-1}{(a+b)-1}\right)$$

$$= \frac{(a-b)}{(a+b)^{n-2}}\left(\frac{(a+b)^n-1}{(a+b)-1}\right)$$

(v) 4, 2, 1, ½ … to 10 terms
We know that, sum of GP for n terms = $a(r^n – 1)/(r – 1)$
Given:
$a = 4$, $r = t_2/t_1 = 2/4 = 1/2$, n = 10
Now let us substitute the values in
$a(r^n – 1)/(r – 1) = 4 ((1/2)^{10} - 1)/((1/2)-1)$
$= 4 ((1/2)^{10} - 1)/((1-2)/2)$
$= 4 ((1/2)^{10} - 1)/(-1/2)$
$= 4 ((1/2)^{10} - 1) \times -2/1$
= -8 [1/1024 -1]
= -8 [1 - 1024]/1024
= -8 [-1023]/1024
= 1023/128

2. Find the sum of the following geometric series :
(i) 0.15 + 0.015 + 0.0015 + … to 8 terms;
(ii) √2 + 1/√2 + 1/2√2 + …. to 8 terms;
(iii) 2/9 – 1/3 + ½ - ¾ + … to 5 terms;
(iv) $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2 y + xy^2 + y^3)$ + …. to n terms ;

(v) $3/5 + 4/5^2 + 3/5^3 + 4/5^4 + \ldots$ to 2n terms;
Solution:
(i) 0.15 + 0.015 + 0.0015 + … to 8 terms
Given:
a = 0.15
$r = t_2/t_1 = 0.015/0.15 = 0.1 = 1/10$
n = 8
By using the formula,
Sum of GP for n terms = $a(1 - r^n)/(1 - r)$
$a(1 - r^n)/(1 - r) = 0.15\ (1 – (1/10)^8) / (1 – (1/10))$
$= 0.15\ (1 – 1/10^8) / (1- (1/10))$
$= 1/6\ (1 – 1/10^8)$

(ii) √2 + 1/√2 + 1/2√2 + …. to 8 terms;
Given:
a = √2
$r = t_2/t_1 = (1/\sqrt{2})/\sqrt{2} = 1/2$
n = 8
By using the formula,
Sum of GP for n terms = $a(1 - r^n)/(1 - r)$
$a(1 - r^n)/(1 - r) = \sqrt{2}\ (1 – (1/2)^8) / (1 - (1/2))$
$= \sqrt{2}\ (1 – 1/256) / (1/2)$
$= \sqrt{2}\ ((256 - 1)/256) \times 2$
$= \sqrt{2}\ (255\times2)/256$
$= (255\sqrt{2})/128$

(iii) 2/9 – 1/3 + ½ - ¾ + … to 5 terms;
Given:
a = 2/9
$r = t_2/t_1 = (-1/3) / (2/9) = -3/2$
n = 5
By using the formula,
Sum of GP for n terms = $a(1 - r^n)/(1 - r)$
$a(1 - r^n)/(1 - r) = (2/9)\ (1 – (-3/2)^5) / (1 – (-3/2))$
$= (2/9)\ (1 + (3/2)^5) / (1 + 3/2)$
$= (2/9)\ (1 + (3/2)^5) / (5/2)$
$= (2/9)\ (1 + 243/32) / (5/2)$
$= (2/9)\ ((32+243)/32) / (5/2)$
$= (2/9)\ (275/32) \times 2/5$

= 55/72

(iv) $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2 y + xy^2 + y^3) + \ldots$ to n terms;

Let $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2 y + xy^2 + y^3) + \ldots$ to n terms

Let us multiply and divide by (x – y) we get,

$S_n = 1/(x - y) [(x + y) (x - y) + (x^2 + xy + y^2) (x - y) \ldots$ upto n terms]

$(x – y) S_n = (x^2 – y^2) + x^3 + x^2y + xy^2 – x^2y – xy^2 – y^3$..upto n terms

$(x – y) S_n = (x^2 + x^3 + x^4 + \ldots$n terms$) – (y^2 + y^3 + y^4 + \ldots$n terms)

By using the formula,

Sum of GP for n terms = $a(1 - r^n)/(1 - r)$

We have two G.Ps in above sum, so,

$(x - y) S_n = x^2 [(x^n - 1)/ (x - 1)] – y^2 [(y^n - 1)/ (y - 1)]$

$S_n = 1/(x-y) \{x^2 [(x^n - 1)/ (x - 1)] – y^2 [(y^n - 1)/ (y - 1)]\}$

(v) $3/5 + 4/5^2 + 3/5^3 + 4/5^4 + \ldots$ to 2n terms;

The series can be written as:

$3 (1/5 + 1/5^3 + 1/5^5 + \ldots$ to n terms$) + 4 (1/5^2 + 1/5^4 + 1/5^6 + \ldots$ to n terms)

Firstly let us consider $3 (1/5 + 1/5^3 + 1/5^5 + \ldots$ to n terms)

So, a = 1/5

$r = t_2/t_1 = 1/5^2 = 1/25$

By using the formula,

Sum of GP for n terms = $a(1 - r^n)/(1 - r)$

$$3\left(\frac{1}{5}+\frac{1}{5^3}+\frac{1}{5^5}+\cdots n \text{ terms}\right)=3\cdot\frac{\frac{1}{5}\left(1-\left(\frac{1}{25}\right)^n\right)}{1-\frac{1}{25}}$$

$$=\frac{5}{8}\left(1-\frac{1}{5^{2n}}\right)$$

Now, Let us consider $4 (1/5^2 + 1/5^4 + 1/5^6 + \ldots$ to n terms)

So, a = 1/25

$r = t_2/t_1 = 1/5^2 = 1/25$

By using the formula,

Sum of GP for n terms = $a(1 - r^n)/(1 - r)$

$$4\left(\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\cdots n \text{ terms}\right)=4\cdot\frac{\frac{1}{25}\left(1-\left(\frac{1}{25}\right)^n\right)}{1-\frac{1}{25}}$$

$$= \frac{1}{6}\left(1 - \frac{1}{5^{2n}}\right)$$

Now,

$$\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \cdots \ 2n \text{ terms} = \frac{5}{8}\left(1 - \frac{1}{5^{2n}}\right) + \frac{1}{6}\left(1 - \frac{1}{5^{2n}}\right)$$

$= 19/24\ (1 - 1/5^{2n})$

3. Evaluate the following:

(i) $\sum_{n=1}^{11} (2 + 3^n)$

(ii) $\sum_{k=1}^{n} (2^k + 3^{k-1})$

(iii) $\sum_{n=2}^{10} 4^n$

Solution:

(i) $\sum_{n=1}^{11} (2 + 3^n)$

$= (2 + 3^1) + (2 + 3^2) + (2 + 3^3) + \ldots + (2 + 3^{11})$

$= 2 \times 11 + 3^1 + 3^2 + 3^3 + \ldots + 3^{11}$

$= 22 + 3(3^{11} - 1)/(3 - 1)$ [by using the formula, $a(1 - r^n)/(1 - r)$]

$= 22 + 3(3^{11} - 1)/2$

$= [44 + 3(177147 - 1)]/2$

$= [44 + 3(177146)]/2$

$= 265741$

(ii) $\sum_{k=1}^{n} (2^k + 3^{k-1})$

$= (2 + 3^0) + (2^2 + 3) + (2^3 + 3^2) + \ldots + (2^n + 3^{n-1})$

$= (2 + 2^2 + 2^3 + \ldots + 2^n) + (3^0 + 3^1 + 3^2 + \ldots. + 3^{n-1})$

Firstly let us consider,

$(2 + 2^2 + 2^3 + \ldots + 2^n)$

Where, $a = 2$, $r = 2^2/2 = 4/2 = 2$, $n = n$

By using the formula,

Sum of GP for n terms $= a(r^n - 1)/(r - 1)$

$= 2\ (2^n - 1)/(2 - 1)$

$= 2\ (2^n - 1)$

Now, let us consider

$(3^0 + 3^1 + 3^2 + \ldots + 3^n)$

Where, $a = 3^0 = 1$, $r = 3/1 = 3$, $n = n$

By using the formula,

Sum of GP for n terms $= a(r^n - 1)/(r - 1)$

$= 1 (3^n - 1)/ (3 - 1)$

$= (3^n - 1)/2$

So,

$$\sum_{k=1}^{n} (2^k + 3^{k-1})$$

$= (2 + 2^2 + 2^3 + \ldots + 2^n) + (3^0 + 3^1 + 3^2 + \ldots + 3^n)$

$= 2 (2^n - 1) + (3^n - 1)/2$

$= ½ [2^{n+2} + 3^n – 4 - 1]$

$= ½ [2^{n+2} + 3^n - 5]$

$(iii) \sum_{n=2}^{10} 4^n$

$= 4^2 + 4^3 + 4^4 + \ldots + 4^{10}$

Where, $a = 4^2 = 16$, $r = 4^3/4^2 = 4$, $n = 9$

By using the formula,

Sum of GP for n terms $= a(r^n - 1)/(r - 1)$

$= 16 (4^9 - 1)/(4 - 1)$

$= 16 (4^9 - 1)/3$

$= 16/3 [4^9 - 1]$

4. Find the sum of the following series :

(i) 5 + 55 + 555 + … to n terms.

(ii) 7 + 77 + 777 + … to n terms.

(iii) 9 + 99 + 999 + … to n terms.

(iv) 0.5 + 0.55 + 0.555 + …. to n terms

(v) 0.6 + 0.66 + 0.666 + …. to n terms.

Solution:

(i) 5 + 55 + 555 + … to n terms.

Let us take 5 as a common term so we get,

5 [1 + 11 + 111 + … n terms]

Now multiply and divide by 9 we get,

5/9 [9 + 99 + 999 + … n terms]

$5/9 [(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots$ n terms]

$5/9 [(10 + 10^2 + 10^3 + \ldots$ n terms) - n]

So the G.P is
5/9 [$(10 + 10^2 + 10^3 + \dots$ n terms) - n]
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
Where, a = 10, r = $10^2/10$ = 10, n = n
$a(r^n - 1)/(r - 1)$ =

$$= \frac{5}{9}\left\{10 \times \frac{(10^n-1)}{10-1} - n\right\}$$
$$= \frac{5}{9}\left\{\frac{10}{9}(10^n - 1) - n\right\}$$
$$= \frac{5}{81}\left\{10^{n+1} - 9n - 10\right\}$$

(ii) 7 + 77 + 777 + … to n terms.
Let us take 7 as a common term so we get,
7 [1 + 11 + 111 + … to n terms]
Now multiply and divide by 9 we get,
7/9 [9 + 99 + 999 + … n terms]
7/9 [$(10 - 1) + (10^2 - 1) + (10^3 - 1) + \dots + (10^n - 1)$]
7/9 [$(10 + 10^2 + 10^3 + \dots + 10^n)$] – 7/9 [(1 + 1 + 1 + … to n terms)]
So the terms are in G.P
Where, a = 10, r = $10^2/10$ = 10, n = n
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
7/9 [10 $(10^n - 1)/(10-1)$] – n
7/9 [10/9 $(10^n - 1)$ - n]
7/81 [10 $(10^n - 1)$ - n]
7/81 $(10^{n+1} - 9n - 10)$

(iii) 9 + 99 + 999 + … to n terms.
The given terms can be written as
(10 - 1) + (100 - 1) + (1000 - 1) + … + n terms
$(10 + 10^2 + 10^3 + \dots$ n terms) – n
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
Where, a = 10, r = 10, n = n
$a(r^n - 1)/(r - 1)$ = [10 $(10^n - 1)/(10-1)$] – n
= 10/9 $(10^n - 1)$ – n
= 1/9 [$10^{n+1} - 10 - 9n$]
= 1/9 [$10^{n+1} - 9n - 10$]

(iv) 0.5 + 0.55 + 0.555 + …. to n terms
Let us take 5 as a common term so we get,
5(0.1 + 0.11 + 0.111 + ...n terms)
Now multiply and divide by 9 we get,
5/9 [0.9 + 0.99 + 0.999 + …+ to n terms]
5/9 [9/10 + 9/100 + 9/1000 + … + n terms]
This can be written as
5/9 [(1 – 1/10) + (1 – 1/100) + (1 – 1/1000) + … + n terms]
5/9 [n – {1/10 + $1/10^2$ + $1/10^3$ + … + n terms}]
5/9 [n – 1/10 {1-$(1/10)^n$}/{1 – 1/10}]
5/9 [n – 1/9 (1 – $1/10^n$)]

(v) 0.6 + 0.66 + 0.666 + …. to n terms.
Let us take 6 as a common term so we get,
6(0.1 + 0.11 + 0.111 + ...n terms)
Now multiply and divide by 9 we get,
6/9 [0.9 + 0.99 + 0.999 + …+ n terms]
6/9 [9/10 + 9/100 + 9/1000 + …+ n terms]
This can be written as
6/9 [(1 – 1/10) + (1 – 1/100) + (1 – 1/1000) + … + n terms]
6/9 [n – {1/10 + $1/10^2$ + $1/10^3$ + … + n terms}]
6/9 [n – 1/10 {1-$(1/10)^n$}/{1 – 1/10}]
6/9 [n – 1/9 (1 – $1/10^n$)]

5. How many terms of the G.P. 3, 3/2, ¾, ... Be taken together to make 3069/512 ?
Solution:
Given:
Sum of G.P = 3069/512
Where, a = 3, r = (3/2)/3 = 1/2, n = ?
By using the formula,
Sum of GP for n terms = a(r^n - 1)/(r - 1)
3069/512 = 3 ($(1/2)^n$ - 1)/ (1/2 - 1)
3069/512×3×2 = 1 – $(1/2)^n$
3069/3072 – 1 = - $(1/2)^n$
(3069 – 3072)/3072 = - $(1/2)^n$
-3/3072 = - $(1/2)^n$
1/1024 = $(1/2)^n$
$(1/2)^{10}$ = $(1/2)^n$

$10 = n$
∴ 10 terms are required to make 3069/512

6. How many terms of the series 2 + 6 + 18 + …. Must be taken to make the sum equal to 728?
Solution:
Given:
Sum of GP = 728
Where, $a = 2$, $r = 6/2 = 3$, $n = ?$
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
$728 = 2(3^n - 1)/(3-1)$
$728 = 2(3^n - 1)/2$
$728 = 3^n - 1$
$729 = 3^n$
$3^6 = 3^n$
$6 = n$
∴ 6 terms are required to make a sum equal to 728

7. How many terms of the sequence √3, 3, 3√3,… must be taken to make the sum 39+ 13√3 ?
Solution:
Given:
Sum of GP = $39 + 13\sqrt{3}$
Where, $a = \sqrt{3}$, $r = 3/\sqrt{3} = \sqrt{3}$, $n = ?$
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
$39 + 13\sqrt{3} = \sqrt{3}(\sqrt{3}^n - 1)/(\sqrt{3} - 1)$
$(39 + 13\sqrt{3})(\sqrt{3} - 1) = \sqrt{3}(\sqrt{3}^n - 1)$
Let us simplify we get,
$39\sqrt{3} - 39 + 13(3) - 13\sqrt{3} = \sqrt{3}(\sqrt{3}^n - 1)$
$39\sqrt{3} - 39 + 39 - 13\sqrt{3} = \sqrt{3}(\sqrt{3}^n - 1)$
$39\sqrt{3} - 39 + 39 - 13\sqrt{3} = \sqrt{3}^{n+1} - \sqrt{3}$
$26\sqrt{3} + \sqrt{3} = \sqrt{3}^{n+1}$
$27\sqrt{3} = \sqrt{3}^{n+1}$
$\sqrt{3}^6\sqrt{3} = \sqrt{3}^{n+1}$
$6+1 = n + 1$
$7 = n + 1$
$7 - 1 = n$

6 = n
∴ 6 terms are required to make a sum of 39 + 13√3

8. The sum of n terms of the G.P. 3, 6, 12, … is 381. Find the value of n.
Solution:
Given:
Sum of GP = 381
Where, a = 3, r = 6/3 = 2, n = ?
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
$381 = 3 (2^n - 1)/ (2-1)$
$381 = 3 (2^n - 1)$
$381/3 = 2^n - 1$
$127 = 2^n - 1$
$127 + 1 = 2^n$
$128 = 2^n$
$2^7 = 2^n$
n = 7
∴ value of n is 7

9. The common ratio of a G.P. is 3, and the last term is 486. If the sum of these terms be 728, find the first term.
Solution:
Given:
Sum of GP = 728
Where, r = 3, a = ?
Firstly,
$T_n = ar^{n-1}$
$486 = a3^{n-1}$
$486 = a3^n/3$
$486 (3) = a3^n$
$1458 = a3^n$ …. Equation (i)
By using the formula,
Sum of GP for n terms = $a(r^n - 1)/(r - 1)$
$728 = a (3^n - 1)/2$
$1456 = a3^n - a$ … equation (2)
Subtracting equation (1) from (2) we get
$1458 - 1456 = a.3^n - a.3^n + a$
a = 2.

∴ The first term is 2

10. The ratio of the sum of the first three terms is to that of the first 6 terms of a G.P. is 125 : 152. Find the common ratio.

Solution:

Given:

Sum of G.P of 3 terms is 125

By using the formula,

Sum of GP for n terms = $a(r^n - 1)/(r - 1)$

$125 = a(r^n - 1)/(r-1)$

$125 = a(r^3 - 1)/(r-1)$ … equation (1)

Now,

Sum of G.P of 6 terms is 152

By using the formula,

Sum of GP for n terms = $a(r^n - 1)/(r - 1)$

$152 = a(r^n - 1)/(r-1)$

$152 = a(r^6 - 1)/(r-1)$ … equation (2)

Let us divide equation (i) by (ii) we get,

$125/152 = [a(r^3 - 1)/(r-1)] / [a(r^6 - 1)/(r-1)]$

$125/152 = (r^3 – 1)/(r^6 - 1)$

$125/152 = (r^3 – 1)/[(r^3 - 1)(r^3 + 1)]$

$125/152 = 1/(r^3 + 1)$

$125(r^3 + 1) = 152$

$125r^3 + 125 = 152$

$125r^3 = 152 – 125$

$125r^3 = 27$

$r^3 = 27/125$

$r^3 = 3^3/5^3$

$r = 3/5$

∴ The common ratio is 3/5