15. Integration Using Partial Fractions

Exercise 15A

1. Question

Evaluate:

$\int \frac{dx}{x(x+2)}$

Answer

Let $I = \int \frac{dx}{x(x+2)}$,

Putting $\frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \ldots \ldots .. (1)$

Which implies A(x+2) + Bx = 1, putting x+2=0

Therefore x=-2,

And B = -0.5

Now put x=0, A= � ,

From equation (1), we get

$$\frac{1}{x(x+2)} = \frac{1}{2} \times \frac{1}{x} - \frac{1}{2} \times \frac{1}{x+2}$$

$$\int \frac{1}{x(x+2)} dx = \frac{1}{2} \int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{x+2} dx$$

$$= \frac{1}{2} \log|x| - \frac{1}{2} \log|x+2| + c$$

$$= \frac{1}{2} [\log|x| - \log|x+2|] + c$$

$$= \frac{1}{2} \log\left|\frac{x}{x+2}\right| + c$$

2. Question

Evaluate:

$\int \frac{(2x+1)}{(x+2)(x+3)} dx$

Answer

Let $I = \int \frac{(2x+1)}{(x+2)(x+3)} dx$,

Putting $\frac{2x+1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} \ldots \ldots \ldots \ldots (1)$

Which implies 2x=1 = A(x-3) + B(x+2)

Now put x-3=0, x=3

2×3+1=A(0)+B 3+2)

So $B = \frac{7}{5}$

Now put x+2=0 ,x=-2

-4+1=A(-2-3) + B(0)

So $A = \frac{3}{5}$

From equation (1), we get ,

$$\frac{2x+1}{(x+2)(x-3)} = \frac{3}{5} \times \frac{1}{x+2} + \frac{7}{5} \times \frac{1}{x-3}$$

$$\int \frac{2x+1}{(x+2)(x-3)} dx = \frac{3}{5} \int \frac{1}{x+2} dx + \frac{7}{5} \int \frac{1}{x-3} dx$$

$$= \frac{3}{5} \log|x+2| + \frac{7}{5} \log|x-3| + c$$

3. Question

Evaluate:

$$\int \frac{x}{(x+2)(3-2x)} dx$$

Answer

Let $I = \int \frac{x}{(x+2)(3-2x)} dx,$

Putting $\frac{x}{(x+2)(3-2x)} = \frac{A}{x+2} + \frac{B}{3-2x} \dots \dots \dots (1)$

Which implies A(3-2x)+B(x+2)=x

Now put 3-2x=0

Therefore, $x = \frac{3}{2}$

$$A(0) + B\left(\frac{3}{2} + 2\right) = \frac{3}{2}$$

$$B\left(\frac{7}{2}\right) = \frac{3}{2}$$

$$B = \frac{3}{7}$$

Now put x+2=0

Therefore, x=-2

A(7)+B(0)=-2

$$A = \frac{-2}{7}$$

Now From equation (1) we get

$$\frac{x}{(x+2)(3-2x)} = \frac{-2}{7} \times \frac{1}{x+2} + \frac{3}{7} \times \frac{1}{3-2x}$$

$$\int \frac{x}{(x+2)(3-2x)} dx = \frac{-2}{7} \int \frac{1}{x+2} dx + \frac{3}{7} \int \frac{1}{3-2x} dx$$

$$= \frac{-2}{7} \log|x+2| + \frac{3}{7} \times \frac{1}{-2} \log|3-2x| + c$$

$$= \frac{-2}{7} \log|x+2| + \frac{3}{7} \times \frac{1}{-2} \log|3-2x| + c$$

$$= \frac{-2}{7} \log|x+2| - \frac{3}{14} \log|3-2x| + c$$

4. Question

Evaluate:

$\int \frac{dx}{x(x-2)(x-4)}$

Answer

Let $I = \int \frac{dx}{x(x-2)(x-4)}$,

Putting $\frac{1}{x(x-2)(x-4)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4} \ldots \ldots (1)$

Which implies,

A(x-2)(x-4)+Bx(x-4)+Cx(x-2)=1

Now put x-2=0

Therefore, x=2

A(0)+B×2(2-4)+C(0)=1

B×2(-2)=1

$B = -\frac{1}{4}$

Now put x-4=0

Therefore, x=4

A(0)+B×(0)+C×4(4-2)=1

C×4(2)=1

$C = \frac{1}{8}$

Now put x=0

A(0-2)(0-4)+B(0)+C(0)=1

$A = \frac{1}{8}$

Now From equation (1) we get

$$\frac{1}{x(x-2)(x-4)} = \frac{1}{8} \times \frac{1}{x} - \frac{1}{4} \times \frac{1}{x-2} + \frac{1}{8} \times \frac{1}{x-4}$$

$$\int \frac{dx}{x(x-2)(x-4)} = \frac{1}{8}\int \frac{1}{x}dx - \frac{1}{4}\int \frac{1}{x-2}dx + \frac{1}{8}\int \frac{1}{x-4}dx$$

$$= \frac{1}{8}\log|x| - \frac{1}{4}\log|x-2| + \frac{1}{8}\log|x-4| + c$$

5. Question

Evaluate:

$\int \frac{(2x-1)}{(x-1)(x+2)(x-3)}dx$

Answer

Let $I = \int \frac{(2x-1)}{(x-1)(x+2)(x-3)}dx$

Putting $\frac{(2x-1)}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3} \dots \dots (1)$

Which implies,

A(x+2)(x-2)+B(x-1)(x-3)+C(x-1)(x+2)=2x-1

Now put x+2=0

Therefore, x=-2

A(0)+B(-2-1)(-2-3)+C(0)=2x-2-1

B(-3)(-5)=-5

$B = -\frac{1}{3}$

Now put x-3=0

Therefore, x=3

A(0)+B(0)+C(2)(5)=5

$C = \frac{1}{2}$

Now put x-1=0

Therefore, x=1

A(3)(-2)+B(0)+C(0)=1

$A = -\frac{1}{6}$

Now From equation (1) we get,

$$\frac{(2x-1)}{(x-1)(x+2)(x-3)} = \frac{-1}{6} \times \frac{1}{x-1} - \frac{1}{3} \times \frac{1}{x+2} + \frac{1}{2} \times \frac{1}{x-3}$$

$$\int \frac{(2x-1)}{(x-1)(x+2)(x-3)} dx = \frac{-1}{6} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{1}{x+2} dx + \frac{1}{2} \int \frac{1}{x-3} dx$$

$$= \frac{-1}{6} \log|x-1| - \frac{1}{3} \log|x+2| + \frac{1}{2} \log|x-3| + c$$

6. Question

Evaluate:

$\int \frac{(2x-3)}{(x^2-1)(2x+3)} dx$

Answer

Let $I = \int \frac{(2x-3)}{(x^2-1)(2x+3)} dx$

Putting $\frac{(2x-3)}{(x-1)(x+1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3} \dots \dots (1)$

Which implies,

A(x+1)(2x+3)+B(x-1)(2x+3)+C(x-1)(x+1)=2x-3

Now put x+1=0

Therefore, x=-1

A(0)+B(-1-1)(-2+3)+C(0)=-2-3

$B=-\frac{5}{2}$

Now put x-1=0

Therefore, x=1

A(2)(2+3)+B(0)+C(0)=-1

$A=-\frac{1}{10}$

Now put 2x+3=0

Therefore, $x=-\frac{3}{2}$

$A(0)+B(0)+C\left(\frac{-3}{2}-1\right)\left(\frac{-3}{2}+1\right)=2\left(\frac{-3}{2}\right)-3$

$C(\frac{-5}{2})(\frac{-1}{2})=-3-3$

$C=-\frac{24}{5}$

.Now From equation (1) we get,

$$\frac{(2x-3)}{(x^2-1)(2x+3)}=\frac{-1}{10}\times\frac{1}{x-1}+\frac{5}{2}\times\frac{1}{x+1}-\frac{24}{5}\times\frac{1}{2x+3}$$

$$\int\frac{(2x-3)}{(x^2-1)(2x+3)}dx=\frac{-1}{10}\int\frac{1}{x-1}dx+\frac{5}{2}\int\frac{1}{x+1}dx-\frac{24}{5}\int\frac{1}{2x+3}dx$$

$$=\frac{-1}{10}\log|x-1|+\frac{5}{2}\log|x+1|-\frac{24}{5}\frac{\log|2x+3|}{2}+c$$

$$=\frac{-1}{10}\log|x-1|+\frac{5}{2}\log|x+1|-\frac{12}{5}\log|2x+3|+c$$

7. Question

Evaluate:

$$\int\frac{(2x+5)}{(x^2-x-2)}dx$$

Answer

Let $I=\int\frac{(2x+5)}{(x^2-x-2)}dx=\int\frac{(2x+5)}{(x-2)(x+1)}dx$

Putting $\frac{(2x+5)}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}\ldots\ldots(1)$

Which implies,

A(x+1)+B(x-2)=2x+5

Now put x+1=0

Therefore, x=-1

A(0)+B(-1-2)=3

B=-1

Now put x-2=0

Therefore, x=2

A(2+1)+B(0)=2×2+5=9

A=3

Now From equation (1) we get,

$$\frac{(2x+5)}{(x-2)(x+1)} = \frac{3}{x-2} + \frac{-1}{x+1}$$

$$\int \frac{(2x+5)}{(x-2)(x+1)} dx = \int \frac{3}{x-2} + \int \frac{-1}{x+1}$$

$$= 3\log|x-2| - \log|x+1| + c$$

8. Question

Evaluate:

$$\int \frac{(x^2+5x+3)}{(x^2+3x+2)} dx$$

Answer

Let $I = \int \frac{(x^2+5x+3)}{(x^2+3x+2)} dx = \int \frac{x^2+3x+2+2x+1}{(x^2+3x+2)} dx = \int \frac{x^2+3x+2}{(x^2+3x+2)} dx + \int \frac{2x+1}{(x^2+3x+2)} dx$

Which implies $I = \int dx + \int \frac{2x+1}{(x^2+3x+2)} dx$

Therefore, $I=x+I_1$

Where, $I_1=\int \frac{2x+1}{(x^2+3x+2)} dx$

Putting $\frac{(2x+1)}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ .... (1)

Which implies,

A(x+2)+B(x+1)=2x+1

Now put x+2=0

Therefore, x=-2

A(0)+B(-1)=-4+1

B=3

Now put x+1=0

Therefore, x=-1

A(-1+2)+B(0)=-2+1

A=-1

Now From equation (1) we get,

$$\frac{(2x+1)}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{3}{x+2}$$

$$\int \frac{(2x+1)}{(x+1)(x+2)} dx = -\int \frac{1}{x+1} dx + \int \frac{3}{x+2} dx$$

$$= -\log|x+1| + 3\log|x+2| + c$$

9. Question

Evaluate:

$\int \frac{(x^2+1)}{(x^2-1)}dx$

Answer

Let $I = \int \frac{x^2+1}{x^2-1}dx$

$I = \int (1 + \frac{2}{x^2-1})dx$

$I = \int dx + 2\int \frac{1}{x^2-1}dx$

$I = x + 2 \times \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$

$I = x + \log\left|\frac{x-1}{x+1}\right| + c$

10. Question

Evaluate:

$\int \frac{x^3}{(x^2-4)}dx$

Answer

Let $I = \int \frac{x^3}{x^2-4}dx$

$I = \int x + \frac{4x}{x^2-4}dx$

$I = \int x\,dx + \int \frac{4x}{x^2-4}dx$

$= \frac{x^2}{2} + \int \frac{4x}{(x-2)(x+2)}dx$

Let $I_1 = \int \frac{4x}{(x-2)(x+2)}dx$

So

$I = \frac{x^2}{2} + I_1$

Therefore $I_1 = \int \frac{4x}{x^2-4}dx$

Putting x^2-4=t

2xdx = dt

$I_1 = 2\int \frac{dt}{t}$

$I_1 = 2log|x^2-4| + c$

Putting the value of I_1 in I,

$I = \frac{x^2}{2} + 2log|x^2-4| + c$

11. Question

Evaluate:

$\int \frac{(3+4x-x^2)}{(x+2)(x-1)} dx$

Answer

Let $I = \int \frac{3+4x-x^2}{(x+2)(x-1)} dx$

$= \int \left(-1 + \frac{5x+1}{(x+2)(x-1)}\right) dx$

$= \int -dx + \int \frac{5x+1}{(x+2)(x-1)} dx$

$= -x + I_1$

$I_1 = \int \frac{5x+1}{(x+2)(x-1)} dx$

Put $\frac{5x+1}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$

A(x-1)+B(x+2)=5x+1

Now put x-1=0

Therefore, x=1

A(0)+B(1+2)=5+1=6

B=2

Now put x+2=0

Therefore, x=-2

A(-2-1)+B(0)=5×(-2)+1

A=3

Now From equation (1) we get,

$\frac{5x+1}{(x+2)(x-1)} = \frac{3}{(x+2)} + \frac{2}{(x-1)}$

$\int \frac{5x+1}{(x+2)(x-1)} dx = 3\int \frac{1}{(x+2)} dx + 2\int \frac{1}{(x-1)} dx$

$3\log|x+2| + 2\log|x-1| + c$

Therefore,

$I = -x + 3log|x+2| + 2log|x-1| + c$

12. Question

Evaluate:

$\int \frac{x^3}{(x-1)(x-2)} dx$

Answer

Let $I = \int \frac{x^3}{(x-1)(x-2)} dx$

$= \int \left\{(x+3) + \frac{7x-6}{(x-1)(x-2)}\right\} dx$

$$= \frac{x^2}{2} + 3x + \int \frac{7x - 6}{(x-1)(x-2)} dx$$

$$= \frac{x^2}{2} + 3x + I_1 \ldots\ldots..(1)$$

Where,

$$I_1 = \int \frac{7x - 6}{(x-1)(x-2)} dx$$

Putting $\frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$..........(2)

A(x-2)+B(x-1)=7x-6

Now put x-2=0

Therefore, x=2

A(0)+B(2-1)=7×2-6

B=8

Now put x-1=0

Therefore, x=1

A(1-2)+B(0)=7-6=1

A=-1

Now From equation (2) we get,

$$\frac{7x - 6}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{8}{x-2}$$

$$I_1 = \int \frac{7x - 6}{(x-1)(x-2)} dx = -\int \frac{1}{x-1} dx + 8\int \frac{1}{x-2} dx$$

$$= -\log|x-1| + 8log|x-2| + c$$

Now From equation (1) we get,

$$I = \frac{x^2}{2} + 3x - \log|x-1| + 8log|x-2| + c$$

13. Question

Evaluate:

$$\int \frac{(x^3 - x - 2)}{(1 - x^2)} dx$$

Answer

Let $I = \int \frac{(x^3-x-2)}{(1-x^2)} dx$

$$= \int \left(-x + \frac{-2}{1-x^2}\right) dx$$

$$= \int -x dx + (-2)\int \frac{1}{1-x^2} dx$$

$$= \frac{-x^2}{2} - log\left|\frac{1+x}{1-x}\right| + c$$

$$= \frac{-x^2}{2} + log\left|\frac{1-x}{1+x}\right| + c$$

14. Question

Evaluate:

$$\frac{(2x+1)}{(4-3x-x^2)}dx$$

Answer

Let $I = \int \frac{2x+1}{(4-3x-x^2)}dx$

$$= \int \frac{2x+1}{(1-x)(4+x)}dx$$

Putting $\frac{2x+1}{(1-x)(4+x)} = \frac{A}{1-x} + \frac{B}{4+x} \dots \dots \dots (1)$

A(4+x)+B(1-x)=2x+1

Now put 1-x=0

Therefore, x=1

A(5)+B(0)=3

$$A = \frac{3}{5}$$

Now put 4+x=0

Therefore, x=-4

A(0)+B(5)=-8+1=-7

$$B = \frac{-7}{5}$$

Now From equation (1) we get,

$$\frac{2x+1}{(1-x)(4+x)} = \frac{3}{5} \times \frac{1}{1-x} + \frac{-7}{5} \times \frac{1}{4+x}$$

$$\int \frac{2x+1}{(1-x)(4+x)}dx = \frac{3}{5}\int \frac{1}{1-x}dx + \frac{-7}{5}\int \frac{1}{4+x}dx$$

$$= \frac{-3}{5}log|1-x| - \frac{7}{5}log|4+x| + c$$

$$= -\frac{1}{5}[3log|1-x| + 7log|4+x|] + c$$

15. Question

Evaluate:

$$\int \frac{2x}{(x^2+1)(x^2+3)}dx$$

Answer

Put x^2=t

2xdx=dt

$$\int \frac{dt}{(1+t)(3+t)} = \frac{1}{2}\int \left(\frac{1}{1+t} - \frac{1}{3+t}\right)dt$$

$\frac{1}{2}[log|1+t| - log|3+t|] + c = \frac{1}{2}log\left|\frac{1+t}{3+t}\right| + c$

$= \frac{1}{2}log\left|\frac{1+x^2}{3+x^2}\right| + c$

16. Question

Evaluate:

$\int \frac{\cos x}{(\cos^2 x - \cos x - 2)} dx$

Answer

Let $I = \int \frac{cosx}{(1+sinx)(2+sinx)} dx$

Putting t=sin x

dt=cos x dx

$I = \int \frac{dt}{(1+t)(2+t)},$

Now putting, $\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t} \ldots\ldots\ldots (1)$

A(2+t)+B(1+t)=1

Now put t+1=0

Therefore, t=-1

A(2-1)+B(0)=1

A=1

Now put t+2=0

Therefore, t=-2

A(0)+B(-2+1)=1

B=-1

Now From equation (1) we get,

$\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} + \frac{-1}{2+t}$

$\int \frac{1}{(1+t)(2+t)} dt = \int \frac{1}{1+t} dt - \int \frac{1}{2+t} dt$

$= \log|1+t| - \log|t+2| + c$

$= log\left|\frac{t+1}{t+2}\right| + c$

So,

$I = \int \frac{cosx}{(1+sinx)(2+sinx)} dx = log\left|\frac{sinx+1}{sinx+2}\right| + c$

17. Question

Evaluate:

$\int \frac{\sec^2 x}{(2+\tan x)(3+\tan x)} dx$

Answer

Let $I = \int \frac{sec^2 x}{(2+tanx)(3+tanx)} dx$

Putting t=tanx

dt=sec^{2}xdx

$$I = \int \frac{dt}{(2+t)(3+t)},$$

Now putting, $\frac{1}{(3+t)(2+t)} = \frac{A}{2+t} + \frac{B}{3+t} \dots\dots\dots.(1)$

A(3+t)+B(2+t)=1

Now put t+2=0

Therefore, t=-2

A(3-2)+B(0)=1

A=1

Now put t+3=0

Therefore, t=-3

A(0)+B(2-3)=1

B=-1

Now From equation (1) we get,

$$\frac{1}{(2+t)(3+t)} = \frac{1}{2+t} + \frac{-1}{3+t}$$

$$\int \frac{1}{(2+t)(3+t)} dt = \int \frac{1}{2+t} dt - \int \frac{1}{3+t} dt$$

$$= \log|2+t| - \log|t+3| + c$$

$$= log\left|\frac{t+2}{t+3}\right| + c$$

So,

$$I = \int \frac{sec^2 x}{(2+tanx)(3+tanx)} dx = log\left|\frac{tanx+2}{tanx+3}\right| + c$$

18. Question

Evaluate:

$$\int \frac{\sin x \cos x}{(\cos^2 x - \cos x - 2)} dx$$

Answer

Let $I = \int \frac{sinx\ cosx}{cos^2x - cosx - 2} dx$

Putting t=cos x

dt=-sin x dx

$$I = \int \frac{(-dt)t}{t^2 - t - 2} =, - \int \frac{tdt}{(t+1)(t-2)},$$

Now putting, $\frac{-t}{(t+1)(t-2)} = \frac{A}{t+1} + \frac{B}{t-2} \ldots\ldots\ldots (1)$

A(t-2)+B(t+1)=-t

Now put t-2=0

Therefore, t=2

A(0)+B(2+1)=-2

$B = \frac{-2}{3}$

Now put t+1=0

Therefore, t=-1

A(-1-2)+B(0)=1

$A = \frac{-1}{3}$

Now From equation (1) we get,

$$\frac{-t}{(t+1)(t-2)} = \frac{-1}{3} \times \frac{1}{t+1} - \frac{2}{3} \times \frac{1}{t-2}$$

$$\int \frac{-t}{(t+1)(t-2)} dt = \frac{-1}{3} \int \frac{1}{t+1} - \frac{2}{3} \int \frac{1}{t-2}$$

$$= \frac{-1}{3} \log|t+1| - \frac{2}{3} \log|t-2| + c$$

So,

$$I = \int \frac{sinx\ cosx}{cos^2 x - cosx - 2} dx = \frac{-1}{3} \log|cosx + 1| - \frac{2}{3} \log|cosx - 2| + c$$

19. Question

Evaluate:

$\int \frac{e^x}{(e^{2x} + 5e^x + 6)} dx$

Answer

Let $I = \int \frac{e^x}{e^{2x} + 5e^x + 6} dx$

Putting $t = e^x$

$dt = e^x\, dx$

$$I = \int \frac{dt}{(t^2 + 5t + 6)},$$

Now putting, $\frac{1}{(t^2 + 5t + 6)} = \frac{A}{2+t} + \frac{B}{3+t} \ldots\ldots\ldots (1)$

A(3+t)+B(2+t)=1

Now put t+2=0

Therefore, t=-2

A(3-2)+B(0)=1

A=1

Now put t+3=0

Therefore, t=-3

A(0)+B(2-3)=1

B=-1

Now From equation (1) we get,

$$\frac{1}{(2+t)(3+t)} = \frac{1}{2+t} + \frac{-1}{3+t}$$

$$\int \frac{1}{(2+t)(3+t)} dt = \int \frac{1}{2+t} dt - \int \frac{1}{3+t} dt$$

$$= \log|2+t| - \log|t+3| + c$$

$$= log\left|\frac{t+2}{t+3}\right| + c$$

$$= log\left|\frac{e^x+2}{e^x+3}\right| + c$$

20. Question

Evaluate:

$$\int \frac{e^x}{(e^{3x} - 3e^{2x} - e^x + 3)} dx$$

Answer

Let $I = \int \frac{e^x}{e^{3x} - 3e^{2x} - e^x + 3} dx$

Putting $t=e^x$

$dt=e^x\,dx$

$$I = \int \frac{dt}{(t^3 - 3t^2 - t + 3)} =, \int \frac{dt}{(t^2)(t-3) - (t-3)} = \int \frac{dt}{(t^2-1)(t-3)}$$

Now putting, $\frac{1}{(t-1)(t+1)(t-3)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{t-3} \dots\dots\dots(1)$

A(t+1)(t-3)+B(t-1)(t-3)+C(t-1)(t+1)=1

Now put t+1=0

Therefore, t=-1

A(0)+B(-1-1)(-1-3)+C(0)=1

B(-2)(-4)=1

$$B = \frac{1}{8}$$

Now put t-1=0

Therefore, t=1

A(1+1)(1-3)+B(0)+C(0)=1

$$A = \frac{-1}{4}$$

Now put t-3=0

Therefore, t=3

A(0)+B(0)+C(3-1)(3+1)=1

$C = \frac{1}{8}$

Now From equation (1) we get,

$$\frac{1}{(t-1)(t+1)(t-3)} = \frac{-1}{4} \times \frac{1}{t-1} + \frac{1}{8} \times \frac{1}{t+1} + \frac{1}{8} \times \frac{1}{t-3}$$

$$\int \frac{1}{(t-1)(t+1)(t-3)} = \frac{-1}{4} \int \frac{1}{t-1} + \frac{1}{8} \int \frac{1}{t+1} + \frac{1}{8} \int \frac{1}{t-3}$$

$$= \frac{-1}{4} \log|t-1| + \frac{1}{8} \log|t+1| + \frac{1}{8} \log|t-3| + c$$

$$\int \frac{e^x}{e^{3x} - 3e^{2x} - e^x + 3} dx = \frac{-1}{4} \log|e^x - 1| + \frac{1}{8} \log|e^x + 1| + \frac{1}{8} \log|e^x - 3| + c$$

21. Question

Evaluate:

$$\int \frac{2\log x}{x\left[2(\log x)^2 - \log x - 3\right]} dx$$

Answer

Let $I = \int \frac{2logx}{x[2(logx)^2 - logx - 3]} dx$

Putting t=log x

dt=dx/x

$$I = \int \frac{2tdt}{(2t^2 - t - 3)},$$

Now putting, $\frac{2t}{(2t^2-t-3)} = \frac{A}{2t-3} + \frac{B}{t+1} \ldots\ldots\ldots(1)$

A(t+1)+B(2t-3)=2t

Now put 2t-3=0

Therefore, $t = \frac{3}{2}$

$$A\left(\frac{3}{2} + 1\right) + B(0) = 2 \times \frac{3}{2} = 3$$

$A = \frac{6}{5}$

Now put t+1=0

Therefore, t=-1

A(0)+B(-2-3)=-2

$B = \frac{2}{5}$

Now From equation (1) we get,

$$\frac{2t}{(2t^2 - t - 3)} = \frac{6}{5} \times \frac{1}{2t-3} + \frac{2}{5} \times \frac{1}{t+1}$$

$$\int \frac{2t}{(2t^2-t-3)}dt = \frac{6}{5}\int \frac{1}{2t-3}dt + \frac{2}{5}\int \frac{1}{t+1}dt$$

$$= \frac{6}{5}\log\left|\frac{6}{5} \times \frac{log\ (2t-3)}{2}\right| + \frac{2}{5}\log|\log x + 1| + c$$

$$\int \frac{2logx}{x[2(logx)^2 - logx - 3]}dx = \frac{3}{5}\log|2logx - 3| + \frac{2}{5}\log|logx + 1| + c$$

22. Question

Evaluate:

$\int \frac{\operatorname{cosec}^2 x}{(1-\cot^2 x)}dx$

Answer

Let $I = \int \frac{cosec^2 x}{(1-cot^2 x)}dx$

Putting t=cot x

dt=-cosec^{2}xdx

$$I = \int \frac{-dt}{(1-t^2)} = -\int \frac{1}{(1-t^2)}dt$$

$$= \frac{-1}{2}\,log\left|\frac{1+cotx}{1-cotx}\right| + c$$

23. Question

Evaluate:

$\int \frac{\sec^2 x}{(\tan^3 x + 4\tan x)}dx$

Answer

Let $I = \int \frac{sec^2 x}{(tan^3 x + 4tanx)}dx$

Putting t=tan x

dt=sec^{2}xdx

$$I = \int \frac{dt}{(t^3+4t)} = \int \frac{dt}{t(t^2+4)}$$

Now putting, $\frac{1}{t(t^2+4)} = \frac{A}{t} + \frac{Bt+C}{t^2+4} \ldots\ldots\ldots\ldots(1)$

A(t^2+4)+(Bt + C)t=1

Putting t=0,

A(0+4)× B(0)=1

$$A = \frac{1}{4}$$

By equating the coefficients of t^2 and constant here,

A+B=0

$$\frac{1}{4} + B = 0$$

$B = -\frac{1}{4}, C = 0$

Now From equation (1) we get,

$$\int \frac{1}{t(t^2+4)} dt = \frac{1}{4}\int \frac{dt}{t} - \frac{1}{4}\int \frac{t}{t^2+4} dt$$

$$= \frac{1}{4} logt - \frac{1}{4} \times \frac{1}{2}\log(t^2+4) + c$$

$$= \frac{1}{4}\log tanx - \frac{1}{8}\log(tan^2 x + 4) + c$$

41. Question

$\int \frac{dx}{(x^3-1)}$

Answer

Let $I = \int \frac{dx}{x^3-1}$

Put $\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \dots\dots\dots(1)$

$A(x^2+x+1)+(Bx+C)(x-1)=1$

Now putting x-1=0

X=1

A(1+1+1)+0=1

$A = \frac{1}{3}$

By equating the coefficient of x^2 and constant term, A+B=0

$\frac{1}{3} + B = 0$

$B = -\frac{1}{3}$

A-C=1

$\frac{1}{3} - C = 1$

$C = \frac{1}{3} - 1$

$C = \frac{-2}{3}$

From the equation(1), we get,

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{1}{3} \times \frac{1}{x-1} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}$$

$$I = \int \frac{1}{(x-1)(x^2+x+1)} dx$$
$$= \frac{1}{3}\int \frac{1}{x-1} dx - \frac{1}{3}\int \frac{x}{x^2+x+1} dx - \frac{2}{3}\int \frac{1}{x^2+x+1} dx$$

$$= \frac{1}{3} log|x-1| - \frac{1}{6}\int \frac{2x+1-1}{x^2+x+1} dx - \frac{2}{3}\int \frac{1}{x^2+x+1} dx$$

$$= \frac{1}{3} log|x-1| - \frac{1}{6}\int \frac{2x+1}{x^2+x+1} dx + \frac{1}{6}\int \frac{1}{x^2+x+1} dx - \frac{2}{3}\int \frac{1}{x^2+x+1} dx$$

Put $t=x^2+x+1$

$dt=(2x+1)dx$

$$I = \frac{1}{3} log|x-1| - \frac{1}{6}\int \frac{dt}{t} + \left(\frac{1}{6} - \frac{2}{3}\right)\int \frac{dx}{x^2+x+1}$$

$$= \frac{1}{3} log|x-1| - \frac{1}{6}\log t + \left(\frac{1-4}{6}\right)\int \frac{dx}{x^2+2\times\frac{1}{2}+\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}$$

$$= \frac{1}{3} log|x-1| - \frac{1}{6} log|x^2+x+1| - \frac{1}{2}\times\frac{1}{\sqrt{3}/2} tan^{-1}\frac{x+1/2}{\sqrt{3}/2} + c$$

$$= \frac{1}{3} log|x-1| - \frac{1}{6} log|x^2+x+1| - \frac{1}{\sqrt{3}} tan^{-1}\frac{2x+1}{\sqrt{3}} + c$$

42. Question

$\int \frac{dx}{(x^3+1)}$

Answer

Let $I = \int \frac{dx}{x^3+1}$

Put $\frac{1}{x^3-1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ ……….(1)

$A(x^2-x+1)+(Bx+C)(x+1)=1$

Now putting x+1=0

X=-1

A(1+1+1)+C(0)=1

$$A = \frac{1}{3}$$

By equating the coefficient of x^2 and constant term, A+B=0

$$\frac{1}{3} + B = 0$$

$$B = -\frac{1}{3}$$

A+C+=1

$$\frac{1}{3} + C = 1$$

$$C = 1 - \frac{1}{3}$$

$$C = \frac{2}{3}$$

From the equation(1), we get,

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{1}{3} \times \frac{1}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}$$

$$I = \int \frac{1}{(x+1)(x^2-x+1)} dx$$
$$= \frac{1}{3}\int \frac{1}{x+1} dx - \frac{1}{3}\int \frac{x}{x^2-x+1} dx + \frac{2}{3}\int \frac{1}{x^2-x+1} dx$$

$$= \frac{1}{3} log|x+1| - \frac{1}{6}\int \frac{2x-1+1}{x^2-x+1} dx + \frac{2}{3}\int \frac{1}{x^2-x+1} dx$$

$$= \frac{1}{3} log|x+1| - \frac{1}{6}\int \frac{2x-1}{x^2-x+1} dx - \frac{1}{6}\int \frac{1}{x^2-x+1} dx + \frac{2}{3}\int \frac{1}{x^2-x+1} dx$$

$$= \frac{1}{3} log|x+1| - \frac{1}{6} log|x^2-x+1| - \frac{1}{2} \times \frac{1}{\sqrt{3}/2} tan^{-1} \frac{x-1/2}{\sqrt{3}/2} + c$$

$$= \frac{1}{3} log|x+1| - \frac{1}{6} log|x^2-x+1| + \frac{1}{\sqrt{3}} tan^{-1} \frac{2x-1}{\sqrt{3}} + c$$

24. Question

Evaluate:

$$\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)} dx$$

Answer

Let $I = \int \frac{\sin 2x}{(1+\sin x)(2+\sin x)} dx$

Putting t=sin x

dt=cos x dx

$$I = \int \frac{2t}{(1+t)(2+t)} dt$$

Now putting, $\frac{2t}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$ ……….. (1)

A(2+t)+B(1+t)=2t

Now put t+2=0

Therefore, t=-2

A(0)+B(1-2)=-4

B=4

Now put t+1=0

Therefore, t=-1

A(2-)+B(0)=-2

A=-2

Now from equation (1), we get,

$$\frac{2t}{(1+t)(2+t)} = \frac{-2}{1+t} + \frac{4}{2+t}$$

$$\int \frac{2t}{(1+t)(2+t)} dt = -2\int \frac{1}{1+t} dt + 4\int \frac{1}{2+t} dt$$

$= 4\log|2+t| - 2\log|1+t| + c$

So,

$$\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}dx = 4\log|2+t| - 2\log|1+t| + c$$

25. Question

Evaluate:

$$\frac{e^x}{e^x(e^x-1)}dx$$

Answer

Let $I = \int \frac{e^x}{e^x(e^x-1)}dx$

Putting $t=e^x$

$dt=e^x dx$

$$I = \int \frac{dt}{t(t-1)}$$

Now putting, $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$ ……..(1)

A(t-1)+Bt=1

Now put t-1=0

Therefore, t=1

A(0)+B(1) =1

B=1

Now put t=0

A(0-1)+B(0)=1

A=-1

Now From equation (1) we get,

$$\frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\int \frac{1}{t(t-1)}dt = -\int \frac{1}{t}dt + \int \frac{1}{t-1}dt$$

$$= -\log t + \log|t-1| + c$$

$$= log\left|\frac{t-1}{t}\right| + c$$

$$= log\left|\frac{e^x-1}{e^x}\right| + c$$

43. Question

$$\int \frac{dx}{(x+1)^2(x^2+1)}$$

Answer

Let $I = \int \frac{dx}{(x^2+1)(x+1)^2}$

Put $\frac{1}{(x^2+1)(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} \dots\dots\dots..(1)$

$A(x+1)(x^2+1)+B(x^2+1)+(Cx+D)(x+1)^2=1$

Put x+1=0

X=-1

A(0)+B(1+1)+0=1

$B = \frac{1}{2}$

By equating the coefficient of x^2 and constant term, A+C=0

A+B+2C=0......(2)

$A + 2C = \frac{-1}{2} \dots\dots\dots.(3)$

A+B+D=1

Solving (2) and (3),we get,

$$\frac{1}{(x^2+1)(x+1)^2} = \frac{1}{2} \times \frac{1}{x+1} + \frac{1}{2} \times \frac{1}{(x+1)^2} + \frac{-\frac{1}{2}x + 0}{x^2+1}$$

$$\int \frac{1}{(x^2+1)(x+1)^2} dx = \frac{1}{2}\int \frac{1}{x+1} dx + \frac{1}{2}\int \frac{1}{(x+1)^2} dx - \frac{1}{2}\int \frac{x}{x^2+1} dx$$

$$= \frac{1}{2} log|x+1| - \frac{1}{2} \times \frac{1}{x+1} - \frac{1}{4} \log|x^2+1| + c$$

26. Question

Evaluate:

$\int \frac{dx}{x(x^4-1)}$

Answer

Let $I = \int \frac{dx}{x(x^4-1)} dx$

Putting $t=x^4$

$dt=4x^3dx$

$$I = \int \frac{x^3 dx}{x^4(x^4-1)} = \frac{1}{4} \times \int \frac{dt}{t(t-1)}$$

Now putting, $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1} \dots\dots..(1)$

A(t-1)+Bt=1

Now put t-1=0

Therefore, t=1

A(0)+B(1) =1

B=1

Now put t=0

A(0-1)+B(0)=1

A=-1

Now From equation (1) we get,

$$\frac{1}{t(t-1)}=\frac{-1}{t}+\frac{1}{t-1}$$

$$\frac{1}{4}\int\frac{1}{t(t-1)}dt=-\frac{1}{4}\int\frac{1}{t}dt+\frac{1}{4}\int\frac{1}{t-1}dt$$

$$=-\frac{1}{4}\log t+\frac{1}{4}\log|t-1|+c$$

$$=-\frac{1}{4}\log x^4+\frac{1}{4}\log|x^4-1|+c$$

$$=-\log|x|+\frac{1}{4}\log|x^4-1|+c$$

44. Question

$\int\frac{17}{(2x+1)(x^2+4)}dx$

Answer

Let $I=\int\frac{17}{(2x+1)(x^2+4)}dx$

Put $\frac{17}{(2x+1)(x^2+4)}=\frac{A}{2x+1}+\frac{Bx+C}{x^2+4}\ldots\ldots\ldots(1)$

$A(x^2+4)+(Bx+C)(2x+1)=17$

Put 2x+1=0

$$x=-\frac{1}{2}$$

$$A\left(\frac{1}{4}+4\right)+0=17$$

$$A\left(\frac{17}{4}\right)=17$$

A=4

By equating the coefficient of x^2 and constant term,

A+2B=0

4+2B=0

B=-2

4A+C=17

4×4+C=17

C=1

From the equation(1), we get,

$$\frac{17}{(2x+1)(x^2+4)}=\frac{4}{2x+1}+\frac{-2x+1}{x^2+4}$$

$$\int\frac{17}{(2x+1)(x^2+4)}dx=4\int\frac{1}{2x+1}dx-2\int\frac{2x}{x^2+4}dx+\int\frac{1}{x^2+2^2}dx$$

$$= \frac{4\log|2x+1|}{2} - \log|x^2+4| + \frac{1}{2}tan^{-1}\frac{x}{2} + c$$

$$= 2\log|2x+1| - \log|x^2+4| + \frac{1}{2}tan^{-1}\frac{x}{2} + c$$

27. Question

Evaluate:

$$\int \frac{(1-x^2)}{x(1-2x)}dx$$

Answer

Let $I = \int \frac{(x^2-1)}{x(2x-1)}dx = \int \left(\frac{1}{2} + \frac{\left(\frac{1}{2}x-1\right)}{x(2x-1)}\right)dx = \int \frac{1}{2}dx + \int \frac{x}{x(2x-1)}dx - \int \frac{1}{x(2x-1)}dx$

$$I = \frac{1}{2}x + \frac{1}{2} \times \frac{\log|2x-1|}{2} - I_1 \dots \dots (1)$$

Where $I_1 = \int \frac{1}{x(2x-1)}dx$........(2)

Now putting, $\frac{1}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$

A(2x-1)+Bx=1

Putting 2x-1=0

$$x = \frac{1}{2}$$

$$A(0) + B\left(\frac{1}{2}\right) = 1$$

B=2

Putting x=0,

A(0-1)+B(0)=1

A=-1

From equation (2), we get,

$$\frac{1}{x(2x-1)} = -\frac{1}{x} + \frac{2}{2x-1}$$

$$\int \frac{1}{x(2x-1)}dx = -\int \frac{1}{x}dx + 2\int \frac{1}{2x-1}dx$$

$$= -\log|x| + \frac{2\log|2x-1|}{2} + c$$

$$= \log|2x-1| - \log x + c$$

From equation (1),

$$I = \frac{1}{2}x + \frac{1}{4}\log|2x-1| - \log|2x-1| + logx + c$$

$$= \frac{1}{2}x - \frac{3}{4}\log|1-2x| + log|x| + c$$

45. Question

$\int \frac{dx}{(x^2+2)(x^2+4)}$

Answer

Let $I = \int \frac{dx}{(x^2+2)(x^2+4)} dx$

Put $\frac{1}{(x^2+2)(x^2+4)} = \frac{1}{(t+2)(t+4)} = \frac{A}{t+2} + \frac{B}{t+4} \dots \dots \dots .. (1)$

A(t+4)+B(t+2) = 1

Put t+4=0

t=-4

A(0)+B(-4+2)=1

$B = -\frac{1}{2}$

Put t+2=0

t=-2

A(-2+4)+B(0)=1

$A = \frac{1}{2}$

From equation(1),we get,

$$\frac{1}{(t+2)(t+4)} = \frac{1}{2} \times \frac{1}{t+2} - \frac{1}{2} \times \frac{1}{t+4}$$

$$\int \frac{1}{(x^2+2)(x^2+4)} dx = \frac{1}{2}\int \frac{1}{x^2+2} dx - \frac{1}{2}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{2} tan^{-1}\frac{x}{2} + c$$

$$= \frac{1}{4} tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{4} tan^{-1}\frac{x}{2} + c$$

28. Question

Evaluate:

$\int \frac{(x^2+x+1)}{(x+2)(x+1)^2} dx$

Answer

Let $I = \int \frac{x^2+x+1}{(x+2)(x+1)^2} dx$

Now putting, $\frac{x^2+x+1}{(x+2)(x+1)^2} = \frac{A}{(x+2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} \dots \dots .. (1)$

$A(x+1)^2+B(x+2)(x+1)+C(x+2)=x^2+x+1$

Now put x+1=0

Therefore, x=-1

A(0)+B(0)+C(-1+2) =1-1+1=1

C=1

Now put x+2=0

Therefore, x=-2

$A(-2+1)^2+B(0)+C(0) = 4-2+1=3$

A=3

Equating the coefficient of x^2,A+B=1

3+B=1

B=-2

Form equation (1),we get,

$$\frac{x^2+x+1}{(x+2)(x+1)^2} = \frac{3}{(x+2)} - \frac{2}{(x+1)} + \frac{1}{(x+1)^2}$$

So,

$$\int \frac{x^2+x+1}{(x+2)(x+1)^2} dx = \int \frac{3}{(x+2)} dx - \int \frac{2}{(x+1)} dx + \int \frac{1}{(x+1)^2} dx$$

$$= 3\log|x+2| - 2\log|x+1| - \frac{1}{1+x} + c$$

46. Question

$$\frac{x^2+1}{(x^2+4)(x^2+25)} dx$$

Answer

Let $I = \int \frac{x^2+1}{(x^2+4)(x^2+25)} dx$

Putting $\frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{t+1}{(t+4)(t+25)} = \frac{A}{t+4} + \frac{B}{t+25} \ldots\ldots\ldots (1)$

Where $t=x^2$

(A+B)t+(25A+4B)=t+1

A+B=1............(1)

25A+4B=1..........(2)

Solving equation (1)and(2), we get,

$$A = \frac{-1}{7} \; and \; B = \frac{8}{7}$$

Now,

$$\frac{t+1}{(t+4)(t+25)} = \frac{-1}{7} \times \frac{1}{t+4} + \frac{8}{7} \times \frac{1}{t+25}$$

$$\frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{-1}{7} \times \frac{1}{x^2+4} + \frac{8}{7} \times \frac{1}{x^2+25}$$

$$\int \frac{x^2+1}{(x^2+4)(x^2+25)} dx = \frac{-1}{7} \int \frac{1}{x^2+2^2} dx + \frac{8}{7} \int \frac{1}{x^2+5^2} dx$$

$$= -\frac{1}{7} \times \frac{1}{2} tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{7} \times \frac{1}{5} tan^{-1}\left(\frac{x}{5}\right) + c$$

$= -\frac{1}{14}tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35}tan^{-1}\left(\frac{x}{5}\right) + c$

29. Question

Evaluate:

$\int \frac{(2x+9)}{(x+2)(x-3)^2} dx$

Answer

Let $I = \int \frac{2x+9}{(x+2)(x-3)^2} dx$

Now putting, $\frac{2x+9}{(x+2)(x-3)^2} = \frac{A}{(x+2)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2} \dots \dots (1)$

$A(x-3)^2+B(x+2)(x-3)+C(x+2)=2x+9$

Now put x-3=0

Therefore, x=3

A(0)+B(0)+C(3+2) =6+9=15

C=3

Now put x+2=0

Therefore, x=-2

$A(-2-3)^2+B(0)+C(0) = -4+9=5$

$A = \frac{1}{5}$

Equating the coefficient of x^2,we get,

A+B=0

$\frac{1}{5} + B = 0$

$B = -\frac{1}{5}$

From equation (1), we get,

$$\frac{2x+9}{(x+2)(x-3)^2} = \frac{1}{5} \times \frac{1}{(x+2)} - \frac{1}{5} \times \frac{1}{(x-3)} + \frac{3}{(x-3)^2}$$

$$\int \frac{2x+9}{(x+2)(x-3)^2} dx = \frac{1}{5}\int \frac{1}{(x+2)} dx - \frac{1}{5}\int \frac{1}{(x-3)} dx + 3\int \frac{1}{(x-3)^2} dx$$

$$= \frac{1}{5}\log|x+2| - \frac{1}{5}\log|x-3| - \frac{3}{x-3} + c$$

47. Question

$\int \frac{dx}{(e^x-1)^2}$

Answer

putting $t=e^x-1$

$e^x=t+1$

$dt= e^x\, dx$

$\frac{dt}{e^x} = dx$

$\frac{dt}{t+1} = dx$

Put $\frac{1}{(1+t)t^2} = \frac{A}{t+1} + \frac{Bt+C}{t^2} \dots \dots (1)$

A(t^2)+(Bt+C)(t+1)=1

Put t+1=0

t=-1

A=1

Equating coefficients

A+B=0

1+B=0

B=-1

C=1

From equation (1),we get,

$$\frac{1}{(1+t)t^2} = \frac{1}{t+1} + \frac{-t+1}{t^2}$$

$$\int \frac{1}{(1+t)t^2} dt = \int \frac{1}{t+1} dt - \int \frac{t}{t^2} dt + \int \frac{1}{t^2} dt$$

$$= \log|t+1| - \int \frac{1}{t} dt + \int \frac{1}{t^2} dt$$

$$= \log|t+1| - log|t| - \frac{1}{t} + c$$

$$\int \frac{1}{(e^x-1)^2} dx = log\,|e^x| - \log|e^x - 1| - \frac{1}{e^x-1} + c$$

48. Question

$\int \frac{dx}{x(x^5+1)}$

Answer

Let $I = \int \frac{dx}{x(x^5+1)}$

Put t=x^5

dt=5x^4dx

$$\int \frac{dt}{\frac{(5x^4)}{x(t+1)}} = \frac{1}{5}\int \frac{dt}{x^5(t+1)} = \frac{1}{5}\int \frac{dt}{t(t+1)}$$

Putting $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \dots \dots ..(1)$

A(t+1)+Bt=1

Now put t+1=0

t=-1

A(0)+B(-1)=1

B=-1

Now put t=0

A(0+1)+B(0)=1

A=1

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$\int \frac{1}{t(t+1)} dt = \int \frac{1}{t} dt - \int \frac{1}{t+1} dt$$

$$= log t - \log|t+1| + c$$

$$= log\left|\frac{t}{t+1}\right| + c$$

$$\int \frac{dx}{x(x^5+1)} = \frac{1}{5}\int \frac{dt}{t(t+1)} = \frac{1}{5} log\left|\frac{x^5}{x^5+1}\right| + c$$

$$= log x - \frac{1}{5} log|x^5+1| + c$$

30. Question

Evaluate:

$$\int \frac{(x^2+1)}{(x-1)^2(x+3)} dx$$

Answer

Let $I = \int \frac{x^2+1}{(x+3)(x-1)^2} dx$

Now putting, $\frac{x^2+1}{(x+3)(x-1)^2} = \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \dots\dots (1)$

$A(x-1)^2+B(x+3)(x-1)+C(x+3)=x^2+1$

Now put x-1=0

Therefore, x=1

A(0)+B(0)+C(4) =2

$$C = \frac{1}{2}$$

Now put x+3=0

Therefore, x=-3

$A(-3-1)^2+B(0)+C(0) =9+1=10$

$$A = \frac{5}{8}$$

By equating the coefficient of x^2, we get, A+B=1

$$\frac{5}{8} + B = 1$$

$$B = 1 - \frac{5}{8} = \frac{3}{8}$$

From equation (1), we get,

$$\frac{x^2+1}{(x+3)(x-2)^2}=\frac{5}{8}\times\frac{1}{(x+3)}+\frac{3}{8}\times\frac{1}{(x-2)}+\frac{1}{(x-2)^2}$$

$$\int\frac{x^2+1}{(x+3)(x-2)^2}dx=\frac{5}{8}\int\frac{1}{(x+3)}dx+\frac{3}{8}\int\frac{1}{(x-2)}dx+\int\frac{1}{(x-2)^2}dx$$

$$=\frac{5}{8}\log|x+3|+\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+c$$

31. Question

Evaluate:

$$\int\frac{(x^2+1)}{(x+3)(x-1)}dx$$

Answer

Let $I=\int\frac{x^2+1}{(x-3)(x-1)^2}dx$

Now putting, $\frac{x^2+1}{(x-3)(x-1)^2}=\frac{A}{(x-3)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}\dots\dots..(1)$

$A(x-1)^2+B(x-3)(x-1)+C(x-3)=x^2+1$

Putting x-1=0,

X=1

A(0)+B(0)+C(1-3)=1+1

C=-1

Putting x-3=0,

X=3

$A(3-1)^2+B(0)+C(0)=9+1$

A(4)=10

$$A=\frac{5}{2}$$

Equating the coefficient of x^2

A+B=1

$$\frac{5}{2}+B=1$$

$$B=1-\frac{5}{2}=\frac{-3}{2}$$

From (i) $\int\frac{x^2+1}{(x-3)(x-1)^2}dx=\frac{5}{2}\int\frac{1}{x-3}dx-\frac{3}{2}\int\frac{1}{x-1}dx-\int\frac{1}{(x-1)^2}dx$

$$=\frac{5}{2}log|x-3|-\frac{3}{2}\log|x-1|+\frac{1}{x-1}+C$$

49. Question

$$\int\frac{dx}{x(x^6+1)}$$

Answer

Let $I = \int \frac{dx}{x(x^6+1)}$

Put $t=x^6$

$dt=6x^5dx$

$$\int \frac{\frac{dt}{(6x^5)}}{x(t+1)} = \frac{1}{6}\int \frac{dt}{x^6(t+1)} = \frac{1}{6}\int \frac{dt}{t(t+1)}$$

Putting $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \ldots\ldots(1)$

A(t+1)+Bt=1

Now put t+1=0

t=-1

A(0)+B(-1)=1

B=-1

Now put t=0

A(0+1)+B(0)=1

A=1

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$\int \frac{1}{t(t+1)}dt = \int \frac{1}{t}dt - \int \frac{1}{t+1}dt$$

$$= logt - \log|t+1| + c$$

$$= log\left|\frac{t}{t+1}\right| + c$$

$$\int \frac{dx}{x(x^6+1)} = \frac{1}{6}\int \frac{dt}{t(t+1)} = \frac{1}{6}log\left|\frac{x^6}{x^6+1}\right| + c$$

$$= logx - \frac{1}{6}log|x^6+1| + c$$

32. Question

Evaluate:

$$\int \frac{(x^2+x+1)}{(x+2)(x^2+1)}dx$$

Answer

Let $I = \int \frac{x^2+x+1}{(x+2)(x^2+1)}dx$

Now putting, $\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{(x+2)} + \frac{Bx+C}{(x^2+1)}$

$A(x^2+1)+(Bx+C)(x+2) = x^2+x+1$

$Ax^2+A+Bx^2+Cx+2Bx+2C = x^2+x+1$

$(A+B)x^2+(C+2B)x+(A+2C) = x^2+x+1$

Equating coefficients A+B=1.......(i)

A+2C=1

A=1-2C......(ii)

2B+C=1

2B=1-C

$B=\frac{1-C}{2}\dots\dots(iii)$

$(1-2C)+\frac{1-C}{2}=1$

2-4C+1-C=2

3-5C=2

-5C=-1

$C=\frac{1}{5}$

And $2B=1-\frac{1}{5}=\frac{4}{5}$

$B=\frac{2}{5}$

$A=1-2\times\frac{1}{5}$

$=1-\frac{2}{5}$

$=\frac{3}{5}$

$I=\int\frac{x^2+x+1}{(x+2)(x^2+1)}dx=\int\frac{A}{(x+2)}dx+\int\frac{Bx+C}{(x^2+1)}dx$

$=\frac{3}{5}\times\int\frac{1}{(x+2)}dx+\frac{1}{5}\times\int\frac{2x+1}{(x^2+1)}dx$

$=\frac{3}{5}\log|x+2|+\frac{1}{5}I_1+C_1$

$I_1=\int\frac{2x+1}{(x^2+1)}dx=\int\frac{2x}{(x^2+1)}dx+\int\frac{1}{(x^2+1)}dx$

$=log|x^2+1|+tan^{-1}x+C_2$

$I=\int\frac{x^2+x+1}{(x+2)(x^2+1)}dx=\frac{3}{5}\log|x+2|+\frac{1}{5}log|x^2+1|+\frac{1}{5}tan^{-1}x+C$

50. Question

$\int\frac{dx}{\sin x(3+2\cos x)}$

Answer

let $I=\int\frac{dx}{sinx\ (3+2cosx)}$

Put t=cosx

dt=-sinxdx

$$\frac{dt}{-sinx} = dx$$

$$I = \int \frac{\frac{dt}{-sinx}}{sinx(3+2t)}$$

$$= -\int \frac{dt}{sin^2x(3+2t)} = -\int \frac{dt}{(1-cos^2x)(3+2t)}$$

$$= -\int \frac{dt}{(1-t^2)(3+2t)}$$

$$\frac{1}{(1-t^2)(3+2t)} = \frac{1}{(1-t)(1+t)(3+2t)}$$

Putting $\frac{1}{(1-t)(1+t)(3+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{3+2t} \ldots\ldots(1)$

A(1+t)(3+2t)+B(1-t)(3+2t)+C(1+t)(1-t)=1

Now Putting 1+t=0

t=-1

A(0)+B(2)(3-2)+C(0)=1

$$B = \frac{1}{2}$$

Now Putting 1-t=0

t=1

A(2)(5)+B(0)+C(0)=1

$$A = \frac{1}{10}$$

Now Putting 3+2t=0

$$t = -\frac{3}{2}$$

$$A(0) + B(0) + C\left(1 - \frac{9}{4}\right) = 1$$

$$C = \frac{-4}{5}$$

$$\frac{1}{(1-t)(1+t)(3+2t)} = \frac{1}{10} \times \frac{1}{1-t} + \frac{1}{2} \times \frac{1}{1+t} - \frac{4}{5} \times \frac{1}{3+2t}$$

$$\int \frac{1}{(1-t)(1+t)(3+2t)} dt = \frac{1}{10}\int \frac{1}{1-t} dt + \frac{1}{2}\int \frac{1}{1+t} dt - \frac{4}{5}\int \frac{1}{3+2t} dt$$

$$= -\frac{1}{10} log|1-t| + \frac{1}{2} log|1+t| - \frac{4}{5} \times \frac{log|3+2t|}{2} + c$$

$$= -\frac{1}{10} log|1-cosx| + \frac{1}{2} log|1+cosx| - \frac{2}{5} log|3+2cosx| + c$$

33. Question

Evaluate:

$$\int \frac{2x}{(2x+1)^2} dx$$

Answer

Let $I = \int \frac{2x}{(2x+1)^2} dx$

Now putting, $\frac{2x}{(2x+1)^2} = \frac{A}{(2x+1)} + \frac{B}{(2x+1)^2} \ldots \ldots \ldots \ldots (1)$

A(2x+1)+B = 2x

Putting 2x+1=0,

$$x = \frac{-1}{2}$$

A(0)+B=-1

B=-1

By equating the coefficient of x,

2A=2

A=1

From equation (1),we get,

$$\frac{2x}{(2x+1)^2} = \frac{1}{(2x+1)} - \frac{1}{(2x+1)^2}$$

$$\int \frac{2x}{(2x+1)^2} dx = \int \frac{1}{(2x+1)} dx - \int \frac{1}{(2x+1)^2} dx$$

$$= \frac{\log|2x+1|}{2} + \frac{1}{2(2x+1)} + c$$

$$= \frac{1}{2}\left[\log|2x+1| + \frac{1}{2x+1}\right] + c$$

51. Question

$\int \frac{dx}{\cos x(5-4\sin x)}$

Answer

let $I = \int \frac{dx}{cosx\,(5-4sinx)}$

Put t=sinx

dt=cosxdx

$$I = \int \frac{dt}{(1-sin^2x)(5-4t)} = \int \frac{dt}{(1-t^2)(5-4t)}$$

$$\frac{1}{(1-t^2)(5-4t)} = \frac{1}{(1-t)(1+t)(5-4t)}$$

Putting $\frac{1}{(1-t)(1+t)(5-4t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{5-4t} \ldots \ldots . (1)$

A(1+t)(5-4t)+B(1-t)(5-4t)+C(1+t)(1-t)=1

Now Putting 1+t=0

t=-1

A(0)+B(2)(9)+C(0)=1

$$B = \frac{1}{18}$$

Now Putting 1-t=0

t=1

A(2) +B(0)+C(0)=1

$A = \frac{1}{2}$

Now Putting 5-4t=0

$t = \frac{5}{4}$

$A(0) + B(0) + C\left(1 - \frac{25}{16}\right) = 1$

$C = \frac{-16}{9}$

From equation(1),we get,

$$\frac{1}{(1-t)(1+t)(5-4t)} = \frac{1}{2} \times \frac{1}{1-t} + \frac{1}{18} \times \frac{1}{1+t} - \frac{16}{9} \times \frac{1}{5-4t}$$

$$\int \frac{1}{(1-t)(1+t)(5-4t)} dt = \frac{1}{2}\int \frac{1}{1-t} dt + \frac{1}{18}\int \frac{1}{1+t} dt - \frac{16}{9}\int \frac{1}{5-4t} dt$$

$$= -\frac{1}{2} log|1-t| + \frac{1}{18} log|1+t| - \frac{16}{9} \times \frac{log|5-4t|}{-4} + c$$

$$= -\frac{1}{2} log|1-sinx| + \frac{1}{18} log|1+sinx| + \frac{4}{9} log|5-4sinx| + c$$

34. Question

Evaluate:

$\int \frac{3x+1}{(x+2)(x-2)^2} dx$

Answer

Let $I = \int \frac{3x+1}{(x+2)(x-2)^2} dx$

Now putting, $\frac{3x+1}{(x+2)(x-2)^2} = \frac{A}{(x+2)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} \ldots\ldots (1)$

$A(x-2)^2+B(x+2)(x-2)+C(x+2)=3x+1$

Putting x-2=0,

X=2

A(0)+B(0)+C(2+1)=3×2+1

$C = \frac{7}{4}$

Putting x+2=0,

X=-2

$A(-4)^2+B(0)+C(0)=-6+1=-5$

$A = \frac{-5}{16}$

By equation the coefficient of x^2, we get, A+B=0

$$\frac{-5}{16}+B=0$$

$$B=\frac{5}{16}$$

$$I=-\frac{5}{16}log|x+2|+\frac{5}{16}log|x-2|-\frac{7}{4(x-2)}+c$$

52. Question

$\int\frac{dx}{\sin x\cos^2 x}$

Answer

Let $I=\int\frac{1}{sinx\times cos^2x}dx=\int\frac{sin^2x+cos^2x}{sinx\times cos^2x}dx=\int\frac{sin^2x}{sinx\times cos^2x}dx+\int\frac{cos^2x}{sinx\times cos^2x}dx$

$$=\int\frac{sinx}{cos^2x}dx+\int\frac{1}{sinx}dx$$

$$=\int(\tan x\sec x+cosec\,x)dx$$

$$=\sec x-\frac{1}{2}logcot^2\frac{x}{2}=\sec x-\frac{1}{2}\log\left(\frac{1+cosx}{1-cosx}\right)+c$$

53. Question

$\int\frac{\tan x}{(1-\sin x)}dx$

Answer

let $I=\int\frac{\tan x}{(1-sinx)}$dx $=\int\frac{\sin x}{\cos x(1-sinx)}$dx

Put t=sinx

dt=cosxdx

$$I=\int\frac{\sin x\times cosx}{cos^2x\,(1-sinx)}dx=\int\frac{tdt}{(1-sin^2x)(1-t)}=\int\frac{tdt}{(1-t^2)(1-t)}$$

Putting $\frac{t}{(1-t)(1+t)(1-t)}=\frac{A}{1+t}+\frac{B}{1-t}+\frac{C}{(1-t)^2}\dots\dots(1)$

$A(1+t)^2+B(1-t)(1+t)+C(1+t)=t$

Now Putting 1-t=0

t=1

A(0)+B(0)+C(1+1)=1

$$C=\frac{1}{2}$$

Now Putting 1+t=0

t=-1

$A(2)^2+B(0)+C(0)=-1$

$$A=-\frac{1}{4}$$

By equating the coefficient of t^2,we get,A-B=0