

EXERCISE 3.1

Determine whether the following operation define a binary operation on the given set or not:

(i) '*' on \mathbb{N} defined by $a * b = a^b$ for all $a, b \in \mathbb{N}$.

(ii) 'O' on \mathbb{Z} defined by $a O b = a^b$ for all $a, b \in \mathbb{Z}$.

(iii) '*' on \mathbb{N} defined by $a * b = a + b - 2$ for all $a, b \in \mathbb{N}$

(iv) ' \times_6 ' on $S = \{1, 2, 3, 4, 5\}$ defined by $a \times_6 b =$ Remainder when $a b$ is divided by 6.

(v) '+ $_6$ ' on $S = \{0, 1, 2, 3, 4, 5\}$ defined by $a +_6 b$

$$= \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6, & \text{if } a + b \geq 6 \end{cases}$$

(vi) ' \odot ' on \mathbb{N} defined by $a \odot b = a^b + b^a$ for all $a, b \in \mathbb{N}$

(vii) '* on \mathbb{Q} defined by $a * b = (a - 1) / (b + 1)$ for all $a, b \in \mathbb{Q}$

Solution:

(i) Given '*' on \mathbb{N} defined by $a * b = a^b$ for all $a, b \in \mathbb{N}$.

Let $a, b \in \mathbb{N}$. Then,

$a^b \in \mathbb{N}$ [$\because a^b \neq 0$ and a, b is positive integer]

$\Rightarrow a * b \in \mathbb{N}$

Therefore,

$a * b \in \mathbb{N}, \forall a, b \in \mathbb{N}$

Thus, * is a binary operation on \mathbb{N} .

(ii) Given 'O' on \mathbb{Z} defined by $a O b = a^b$ for all $a, b \in \mathbb{Z}$.

Both $a = 3$ and $b = -1$ belong to \mathbb{Z} .

$\Rightarrow a * b = 3^{-1}$

$= 1/3 \notin \mathbb{Z}$

Thus, * is not a binary operation on \mathbb{Z} .

(iii) Given '*' on \mathbb{N} defined by $a * b = a + b - 2$ for all $a, b \in \mathbb{N}$

If $a = 1$ and $b = 1$,

$a * b = a + b - 2$

$= 1 + 1 - 2$

$= 0 \notin \mathbb{N}$

Thus, there exist $a = 1$ and $b = 1$ such that $a * b \notin \mathbb{N}$

So, * is not a binary operation on \mathbb{N} .

(iv) Given ' \times_6 ' on $S = \{1, 2, 3, 4, 5\}$ defined by $a \times_6 b =$ Remainder when $a \times b$ is divided by 6.
Consider the composition table,

\times_6	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Here all the elements of the table are not in S .

\Rightarrow For $a = 2$ and $b = 3$,

$a \times_6 b = 2 \times_6 3 =$ remainder when 6 divided by 6 = $0 \notin S$

Thus, \times_6 is not a binary operation on S .

(v) Given ' $+_6$ ' on $S = \{0, 1, 2, 3, 4, 5\}$ defined by $a +_6 b$

$$= \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6, & \text{if } a + b \geq 6 \end{cases}$$

Consider the composition table,

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Here all the elements of the table are not in S .

\Rightarrow For $a = 2$ and $b = 3$,

$$a \times_6 b = 2 \times_6 3 = \text{remainder when 6 divided by 6} = 0 \neq$$

Thus, \times_6 is not a binary operation on S.

(vi) Given ' \odot ' on N defined by $a \odot b = a^b + b^a$ for all $a, b \in \mathbb{N}$

Let $a, b \in \mathbb{N}$. Then,

$$a^b, b^a \in \mathbb{N}$$

$$\Rightarrow a^b + b^a \in \mathbb{N} \quad [\because \text{Addition is binary operation on } \mathbb{N}]$$

$$\Rightarrow a \odot b \in \mathbb{N}$$

Thus, \odot is a binary operation on N.

(vii) Given '*' on Q defined by $a * b = (a - 1) / (b + 1)$ for all $a, b \in \mathbb{Q}$

If $a = 2$ and $b = -1$ in Q,

$$a * b = (a - 1) / (b + 1)$$

$$= (2 - 1) / (-1 + 1)$$

$$= 1/0 \text{ [which is not defined]}$$

For $a = 2$ and $b = -1$

$a * b$ does not belong to Q

So, * is not a binary operation in Q.

2. Determine whether or not the definition of * given below gives a binary operation. In the event that * is not a binary operation give justification of this.

(i) On \mathbb{Z}^+ , defined * by $a * b = a - b$

(ii) On \mathbb{Z}^+ , define * by $a * b = ab$

(iii) On R, define * by $a * b = ab^2$

(iv) On \mathbb{Z}^+ define * by $a * b = |a - b|$

(v) On \mathbb{Z}^+ define * by $a * b = a$

(vi) On R, define * by $a * b = a + 4b^2$

Here, \mathbb{Z}^+ denotes the set of all non-negative integers.

Solution:

(i) Given On \mathbb{Z}^+ , defined * by $a * b = a - b$

If $a = 1$ and $b = 2$ in \mathbb{Z}^+ , then

$$a * b = a - b$$

$$= 1 - 2$$

$$= -1 \notin \mathbb{Z}^+ \text{ [because } \mathbb{Z}^+ \text{ is the set of non-negative integers]}$$

For $a = 1$ and $b = 2$,

$$a * b \notin \mathbb{Z}^+$$

Thus, $*$ is not a binary operation on Z^+ .

(ii) Given Z^+ , define $*$ by $a*b = a b$

Let $a, b \in Z^+$

$\Rightarrow a, b \in Z^+$

$\Rightarrow a * b \in Z^+$

Thus, $*$ is a binary operation on R .

(iii) Given on R , define by $a*b = ab^2$

Let $a, b \in R$

$\Rightarrow a, b^2 \in R$

$\Rightarrow ab^2 \in R$

$\Rightarrow a * b \in R$

Thus, $*$ is a binary operation on R .

(iv) Given on Z^+ define $*$ by $a * b = |a - b|$

Let $a, b \in Z^+$

$\Rightarrow |a - b| \in Z^+$

$\Rightarrow a * b \in Z^+$

Therefore,

$a * b \in Z^+, \forall a, b \in Z^+$

Thus, $*$ is a binary operation on Z^+ .

(v) Given on Z^+ define $*$ by $a * b = a$

Let $a, b \in Z^+$

$\Rightarrow a \in Z^+$

$\Rightarrow a * b \in Z^+$

Therefore, $a * b \in Z^+ \forall a, b \in Z^+$

Thus, $*$ is a binary operation on Z^+ .

(vi) Given On R , define $*$ by $a * b = a + 4b^2$

Let $a, b \in R$

$\Rightarrow a, 4b^2 \in R$

$\Rightarrow a + 4b^2 \in R$

$\Rightarrow a * b \in R$

Therefore, $a * b \in R, \forall a, b \in R$

Thus, $*$ is a binary operation on R .

3. Let $*$ be a binary operation on the set I of integers, defined by $a * b = 2a + b - 3$. Find the value of $3 * 4$.

Solution:

$$\text{Given } a * b = 2a + b - 3$$

$$3 * 4 = 2(3) + 4 - 3$$

$$= 6 + 4 - 3$$

$$= 7$$

4. Is $*$ defined on the set $\{1, 2, 3, 4, 5\}$ by $a * b = \text{LCM of } a \text{ and } b$ a binary operation? Justify your answer.

Solution:

LCM	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	5	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

In the given composition table, all the elements are not in the set $\{1, 2, 3, 4, 5\}$.

If we consider $a = 2$ and $b = 3$, $a * b = \text{LCM of } a \text{ and } b = 6 \notin \{1, 2, 3, 4, 5\}$.

Thus, $*$ is not a binary operation on $\{1, 2, 3, 4, 5\}$.

5. Let $S = \{a, b, c\}$. Find the total number of binary operations on S .

Solution:

Number of binary operations on a set with n elements is n^{n^2}

Here, $S = \{a, b, c\}$

Number of elements in $S = 3$

Number of binary operations on a set with 3 elements is 3^{3^2}