EXERCISE 19.19

Evaluate the following integrals:

1. $\int \frac{x}{x^2+3x+2}\,dx$

Solution:

Let

$I = \int \frac{x}{x^2+3x+2}\,dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 + 3x + 2$ and it can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2 + 3x + 2) = 2x + 3$

∴ Let, x = A (2x + 3) + B

⇒ x = 2Ax + 3A + B

On comparing both sides

We have, 2A = 1 ⇒ A = 1/2

3A + B = 0 ⇒ B = −3A = −3/2

Hence,

$I = \int \frac{\frac{1}{2}(2x+3)-\frac{3}{2}}{x^2+3x+2}\,dx$

$\therefore I = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2}\,dx - \frac{3}{2}\int \frac{1}{x^2+3x+2}\,dx$

Let, $I_1 = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2}\,dx$ and $I_2 = \frac{3}{2}\int \frac{1}{x^2+3x+2}\,dx$

Now, $I = I_1 - I_2$ ....equation 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{2}\int \frac{2x+3}{x^2+3x+2}dx$

Let $u = x^2 + 3x + 2 \Rightarrow du = (2x + 3)\,dx$

$\therefore I_1$ reduces to $\frac{1}{2}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C$

On substituting value of u, we have:

$I_1 = \frac{1}{2}\log|x^2 + 3x + 2| + C$ .... Equation 2

As, $I_2 = \frac{3}{2}\int \frac{1}{x^2+3x+2}dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will use to solve the problem.

i) $\int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \frac{3}{2}\int \frac{1}{x^2+3x+2}dx$

$\Rightarrow I_2 = \frac{3}{2}\int \frac{1}{\{x^2+2\left(\frac{3}{2}\right)x+\left(\frac{3}{2}\right)^2\}+2-\left(\frac{3}{2}\right)^2}\,dx$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \frac{3}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx$

I_2 matches with $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{3}{2}\left\{\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{\left(x+\frac{3}{2}\right)-\frac{1}{2}}{\left(x+\frac{3}{2}\right)+\frac{1}{2}}\right| + C\right\}$

$\Rightarrow I_2 = \frac{3}{2}\log\left|\frac{2x+3-1}{2x+3+1}\right| + C$

$\Rightarrow I_2 = \frac{3}{2}\log\left|\frac{2x+2}{2x+4}\right| + C = \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$ ... equation 3

From equation 1:

$I = I_1 - I_2$

Using equation 2 and equation 3:

$I = \frac{1}{2}\log|x^2+3x+2| - \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$

2. $\int \frac{x+1}{x^2+x+3}dx$

Solution:

$I = \int \frac{x+1}{x^2+x+3}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 + x + 3$ and it can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2+x+3) = 2x+1$

$\therefore$ Let, $x = A(2x+1) + B$

$\Rightarrow x = 2Ax + A + B$

On comparing both sides

We have,

$2A = 1 \Rightarrow A = 1/2$

$A + B = 0 \Rightarrow B = -A = -1/2$

Hence,

$I = \int \frac{\frac{1}{2}(2x+1)-\frac{1}{2}}{x^2+x+3} dx$

$\therefore I = \frac{1}{2}\int \frac{2x+1}{x^2+x+3} dx - \frac{1}{2}\int \frac{1}{x^2+x+3} dx$

Let, $I_1 = \frac{1}{2}\int \frac{2x+1}{x^2+x+3} dx$ and $I_2 = \frac{1}{2}\int \frac{1}{x^2+x+3} dx$

Now, $I = I_1 - I_2$ .... Equation 1

We will solve I_1 and I_2 individually.

As $I_1 = \frac{1}{2}\int \frac{2x+1}{x^2+x+3} dx$

Let $u = x^2 + x + 3 \Rightarrow du = (2x + 1)\, dx$

$\therefore I_1$ reduces to $\frac{1}{2}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C$

On substituting the value of u, we have:

$I_1 = \frac{1}{2}\log|x^2 + x + 3| + C$ ....equation 2

As, $I_2 = \frac{1}{2}\int \frac{1}{x^2+x+3} dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will help to solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \frac{1}{2}\int \frac{1}{x^2+x+3}dx$

$\Rightarrow I_2 = \frac{1}{2}\int \frac{1}{\{x^2+2(\frac{1}{2})x+(\frac{1}{2})^2\}+3-(\frac{1}{2})^2}\,dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have

$I_2 = \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}\,dx$

I_2 matches with $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{2}\{\frac{1}{\left(\frac{\sqrt{11}}{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{11}}{2}}\right) + C\}$

$\Rightarrow I_2 = \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{2x+1}{\sqrt{11}}\right) + C$ ... equation 3

From equation 1 we have

$I = I_1 - I_2$

Using equation 2 and equation 3:

$I = \frac{1}{2}\log|x^2+x+3| - \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{2x+1}{\sqrt{11}}\right) + C$

3. $\int \frac{x-3}{x^2+2x-4}\,dx$

Solution:

Let $I = \int \frac{x-3}{x^2+2x-4}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $x^2 + 2x - 4$ and it can be reduced to a fundamental integration.

As, $\frac{d}{dx}(x^2 + 2x - 4) = 2x + 2$

∴ Let, $x - 3 = A(2x + 2) + B$

$\Rightarrow x - 3 = 2Ax + 2A + B$

On comparing both sides we have, $2A = 1 \Rightarrow A = 1/2$

$2A + B = -3 \Rightarrow B = -3 - 2A = -4$

Hence, $I = \int \frac{\frac{1}{2}(2x+2)-4}{x^2+2x-4}dx$

$\therefore I = \frac{1}{2}\int \frac{2x+2}{x^2+2x-4}dx - 4\int \frac{1}{x^2+2x-4}dx$

Let, $I_1 = \frac{1}{2}\int \frac{2x+2}{x^2+2x-4}dx$ and $I_2 = \int \frac{1}{x^2+2x-4}dx$

Now, $I = I_1 - 4I_2$ ....equation 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{2}\int \frac{2x+2}{x^2+2x-4}dx$

Let $u = x^2 + 2x - 4 \Rightarrow du = (2x + 2)\,dx$

∴ I_1 reduces to $\frac{1}{2}\int \frac{du}{u}$

Hence, $I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\log|u| + C$

On substituting value of u, we have:

$I_1 = \frac{1}{2}\log|x^2 + 2x - 4| + C$ .... Equation 2

As, $I_2 = \int \frac{1}{x^2+2x-4}dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \int \frac{1}{x^2+2x-4}dx \Rightarrow I_2 = \int \frac{1}{\{x^2+2(1)x+(1)^2\}-4-(1)^2} dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$I_2 = \int \frac{1}{(x+1)^2-(\sqrt{5})^2} dx$

I_2 matches with $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

$\therefore I_2 = \frac{1}{2\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$ ... equation 3

From equation 1 we have

$I = I_1 - 4I_2$

Using equation 2 and equation 3:

$I = \frac{1}{2}\log|x^2+2x-4| - 4\left(\frac{1}{2\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right|\right) + C$

$I = \frac{1}{2}\log|x^2+2x-4| - \frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$

4. $\int \frac{2x-3}{x^2+6x+13} dx$

Solution:

Let $I = \int \frac{2x-3}{x^2+6x+13}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make a substitution for $x^2 + 6x + 13$ and It can be reduced to a fundamental integration.

As $\frac{d}{dx}(x^2+6x+13) = 2x+6$

∴ Let, $2x - 3 = A(2x + 6) + B$

⇒ $2x - 3 = 2Ax + 6A + B$

On comparing both sides

We have, $2A = 2 \Rightarrow A = 1$

$6A + B = -3 \Rightarrow B = -3-6A = -9$

Hence, $I = \int \frac{(2x+6)-9}{x^2+6x+13} dx$

∴ $I = \int \frac{2x+6}{x^2+6x+13} dx - 9\int \frac{1}{x^2+6x+13} dx$

Let, $I_1 = \int \frac{2x+6}{x^2+6x+13} dx$ and $I_2 = \int \frac{1}{x^2+6x+13} dx$

Now, $I = I_1 - 9I_2$ .... Equation 1

We will solve I_1 and I_2 individually.

As, $I_1 = \int \frac{2x+6}{x^2+6x+13} dx$

Let $u = x^2 + 6x + 13 \Rightarrow du = (2x + 6)\, dx$

∴ I_1 reduces to $\int \frac{du}{u}$

Hence, $I_1 = \int \frac{du}{u} = \log|u| + C$

On substituting value of u, we have

$I_1 = \log|x^2 + 6x + 13| + C$ ....equation 2

As, $I_2 = \int \frac{1}{x^2+6x+13} dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$\therefore I_2 = \int \frac{1}{x^2+6x+13} dx$

$\Rightarrow I_2 = \int \frac{1}{\{x^2+2(3)x+(3)^2\}+13-(3)^2} dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have $I_2 = \int \frac{1}{(x+3)^2+(2)^2} dx$

I_2 matches with $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{2} \tan^{-1}\left(\frac{x+3}{2}\right) + C$ ... equation 3

From equation 1

$I = I_1 - 9I_2$

Using equation 2 and equation 3:

$I = \log|x^2 + 6x + 13| - 9\,\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$

$I = \log|x^2 + 6x + 13| - \frac{9}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$

5. $\int \frac{x-1}{3x^2 - 4x + 3} dx$

Solution:

Let I = $\int \frac{x-1}{3x^2-4x+3}dx$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $3x^2 - 4x + 3$ and I can be reduced to a fundamental integration.

As, $\frac{d}{dx}(3x^2 - 4x + 3) = 6x - 4$

∴ Let, $x - 1 = A(6x - 4) + B$

$\Rightarrow x - 1 = 6Ax - 4A + B$

On comparing both sides

We have, $6A = 1 \Rightarrow A = 1/6$

$-4A + B = -1 \Rightarrow B = -1+4A = -2/6 = -1/3$

Hence, I = $\int \frac{\frac{1}{6}(6x-4)-\frac{1}{3}}{3x^2-4x+3}dx$

∴ I = $\frac{1}{6}\int \frac{6x-4}{3x^2-4x+3}dx - \frac{1}{3}\int \frac{1}{3x^2-4x+3}dx$

Let, $I_1 = \frac{1}{6}\int \frac{6x-4}{3x^2-4x+3}dx$ and $I_2 = \frac{1}{3}\int \frac{1}{3x^2-4x+3}dx$

Now, $I = I_1 - I_2$ ....equation 1

We will solve I_1 and I_2 individually.

As, $I_1 = \frac{1}{6}\int \frac{6x-4}{3x^2-4x+3}dx$

Let $u = 3x^2 - 4x + 3 \Rightarrow du = (6x - 4)\,dx$

∴ I_1 reduces to $\frac{1}{6}\int \frac{du}{u}$

Hence,

$I_1 = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\log|u| + C$

On substituting value of u, we have:

$I_1 = \frac{1}{6}\log|3x^2 - 4x + 3| + C$ ....equation 2

As, $I_2 = \frac{1}{3}\int \frac{1}{3x^2-4x+3}dx$ and we don't have any derivative of function present in denominator. ∴ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

i) $\int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ ii) $\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in the denominator

$\therefore I_2 = \frac{1}{9}\int \frac{1}{x^2-\frac{4}{3}x+1}dx$ {on taking 3 common from denominator}

$\Rightarrow I_2 = \frac{1}{9}\int \frac{1}{\left\{x^2-2\left(\frac{2}{3}\right)x+\left(\frac{2}{3}\right)^2\right\}+1-\left(\frac{2}{3}\right)^2}dx$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have $I_2 = \frac{1}{9}\int \frac{1}{\left(x-\frac{2}{3}\right)^2+\left(\frac{\sqrt{5}}{3}\right)^2}dx$

I_2 matches with $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\therefore I_2 = \frac{1}{9}\frac{1}{\frac{\sqrt{5}}{3}}\tan^{-1}\left(\frac{x-\frac{2}{3}}{\frac{\sqrt{5}}{3}}\right) + C$

$\therefore I_2 = \frac{3}{9\sqrt{5}}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C = \frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C$ ...equation 3

From equation 1:

$I = I_1 - I_2$

Using equation 2 and equation 3:

$$I = \frac{1}{6}\log|3x^2 - 4x + 3| - \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C$$