EXERCISE 4.14

1. Evaluate the following:

(i) tan {2 $\tan^{-1}$ (1/5) – π/4}

(ii) Tan {1/2 $\sin^{-1}$ (3/4)}

(iii) Sin {1/2 $\cos^{-1}$ (4/5)}

(iv) Sin (2 $\tan^{-1}$ 2/3) + cos ($\tan^{-1}$ √3)

Solution:

(i) Given tan {2 $\tan^{-1}$ (1/5) – π/4}

We know that,

$$2\tan^{-1}(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } |x| < 1$$

And $\frac{\pi}{4}$ can be written as $\tan^{-1}(1)$

Now substituting these values we get,

$$= \tan\left\{\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) - \tan^{-1} 1\right\}$$

$$= \tan\left\{\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1} 1\right\}$$

Again we know that,

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$$

Now substituting this formula, we get

$$= \tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right\}$$

$= \tan\left\{\tan^{-1}\left(\frac{-7}{17}\right)\right\}$

$= -\frac{7}{17}$

(ii) Given tan {1/2 $\sin^{-1}$ (3/4)}

Let $\frac{1}{2}\sin^{-1}\frac{3}{4} = t$

Therefore,

$\Rightarrow \sin^{-1}\frac{3}{4} = 2t$

$\Rightarrow \sin 2t = \frac{3}{4}$

Now, by Pythagoras theorem, we have

$\Rightarrow \sin 2t = \frac{3}{4} = \frac{\text{perpendicular}}{\text{hypotenuse}}$

$\Rightarrow \cos 2t = \frac{\sqrt{4^2 - 3^2}}{4} = \frac{\text{Base}}{\text{hypotenuse}}$

$\Rightarrow \cos 2t = \frac{\sqrt{7}}{4}$

By considering, given question

$\tan\left\{\frac{1}{2}\sin^{-1}\frac{3}{4}\right\}$

$= \tan(t)$

We know that,

$$\tan(x) = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$$

$$= \sqrt{\frac{1-\cos 2t}{1+\cos 2t}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{7}}{4}}{1+\frac{\sqrt{7}}{4}}}$$

$$= \sqrt{\frac{4-\sqrt{7}}{4+\sqrt{7}}}$$

Now by rationalizing the denominator, we get

$$= \sqrt{\frac{(4-\sqrt{7})(4-\sqrt{7})}{(4+\sqrt{7})(4-\sqrt{7})}}$$

$$= \sqrt{\frac{(4-\sqrt{7})^2}{9}}$$

$$= \frac{4-\sqrt{7}}{3}$$

Hence

$$\tan\left\{\frac{1}{2}\sin^{-1}\frac{3}{4}\right\} = \frac{4-\sqrt{7}}{3}$$

(iii) Given sin {1/2 $\cos^{-1}$ (4/5)}

We know that

$$\cos^{-1}x = 2\sin^{-1}\left(\pm\sqrt{\frac{1-x}{2}}\right)$$

Now by substituting this formula we get,

$$\sin\left(\frac{1}{2}2\sin^{-1}\left(\pm\sqrt{\frac{1-\frac{4}{5}}{2}}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\pm\sqrt{\frac{1}{2\times5}}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\pm\frac{1}{\sqrt{10}}\right)\right)$$

As we know that

$$\sin(\sin^{-1}x) = x \text{ as } n \in [-1, 1]$$

$$= \pm\frac{1}{\sqrt{10}}$$

Hence, $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) = \pm\frac{1}{\sqrt{10}}$

(iv) Given Sin (2 tan $^{-1}$ 2/3) + cos (tan^{-1} √3)

We know that

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}(x);$$

$$\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \tan^{-1}(x);$$

Now by substituting these formulae we get,

$$= \sin\left(\sin^{-1}\left(\frac{2\times\frac{2}{3}}{1+\frac{4}{9}}\right)\right) + \cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+3}}\right)\right)$$

$= \sin\left(\sin^{-1}\left(\frac{12}{13}\right)\right) + \cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$

$= \frac{12}{13} + \frac{1}{2}$

$= \frac{37}{26}$

Hence,

$\sin\left(2\tan^{-1}(\frac{2}{3})\right) + \cos(\tan^{-1}\sqrt{3}) = \frac{37}{26}$

2. Prove the following results:
(i) $2 \sin^{-1} (3/5) = \tan^{-1} (24/7)$
(ii) $\tan^{-1} ¼ + \tan^{-1} (2/9) = ½ \cos^{-1} (3/5) = ½ \sin^{-1} (4/5)$
(iii) $\tan^{-1} (2/3) = ½ \tan^{-1} (12/5)$
(iv) $\tan^{-1} (1/7) + 2 \tan^{-1} (1/3) = \pi/4$
(v) $\sin^{-1} (4/5) + 2 \tan^{-1} (1/3) = \pi/2$
(vi) $2 \sin^{-1} (3/5) – \tan^{-1} (17/31) = \pi/4$
(vii) $2 \tan^{-1} (1/5) + \tan^{-1} (1/8) = \tan^{-1} (4/7)$
(viii) $2 \tan^{-1} (3/4) - \tan^{-1} (17/31) = \pi/4$
(ix) $2 \tan^{-1} (1/2) + \tan^{-1} (1/7) = \tan^{-1} (31/17)$
(x) $4 \tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4$

Solution:
(i) Given $2 \sin^{-1} (3/5) = \tan^{-1} (24/7)$

Consider LHS

$2\sin^{-1}\frac{3}{5}$

We know that

$\sin^{-1}(x) = \tan^{-1}(\frac{x}{\sqrt{1-x^2}})$

Now by substituting the above formula we get,

$= 2 \times \tan^{-1}(\frac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}})$

$= 2 \times \tan^{-1}(\frac{\frac{3}{5}}{\frac{4}{5}})$

$= 2 \times \tan^{-1}(\frac{3}{4})$

Again we know that

$2\tan^{-1}(x) = \tan^{-1}(\frac{2x}{1-x^2}), \text{if } |x| < 1$

Therefore,

$= \tan^{-1}(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}})$

$= \tan^{-1}(\frac{\frac{3}{2}}{\frac{7}{16}})$

$= \tan^{-1}(\frac{24}{7})$

= RHS

So, $2\sin^{-1}\frac{3}{5} = \tan^{-1}(\frac{24}{7})$

Hence, proved.

(ii) Given $\tan^{-1}$ ¼ + $\tan^{-1}$ (2/9) = ½ $\cos^{-1}$ (3/5) = ½ $\sin^{-1}$ (4/5)

Consider LHS

$= \tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{2}{9})$

We know that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

Now by substituting this formula, we get

$= \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)$

$= \tan^{-1}\left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right)$

$= \tan^{-1}\left(\frac{17}{34}\right)$

$= \tan^{-1}\left(\frac{1}{2}\right)$

Multiplying and dividing by 2

$= \frac{1}{2}\left\{2\tan^{-1}\left(\frac{1}{2}\right)\right\}$

Again we know that

$$2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$= \frac{1}{2}\cos^{-1}\left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right)$

$= \frac{1}{2}\cos^{-1}\left(\frac{\frac{3}{4}}{\frac{5}{4}}\right)$

$= \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

= RHS

So, $\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{2}{9}) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

Now,

$= \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

We know that,

$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$

By substituting this, we get

$= \frac{1}{2}\sin^{-1}\sqrt{1-\frac{9}{25}}$

$= \frac{1}{2}\sin^{-1}\sqrt{\frac{16}{25}}$

$= \frac{1}{2}\sin^{-1}\frac{4}{5}$

= RHS

So, $\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{2}{9}) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right) = \frac{1}{2}\sin^{-1}\frac{4}{5}$

Hence, proved.

(iii) Given $\tan^{-1}(2/3) = ½ \tan^{-1}(12/5)$

Consider LHS

$= \tan^{-1}(\frac{2}{3})$

Now, Multiplying and dividing by 2, we get

$= \frac{1}{2}\left\{2\tan^{-1}\left(\frac{2}{3}\right)\right\}$

We know that

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

By substituting the above formula we get

$= \frac{1}{2}\tan^{-1}\left(\frac{2\times\frac{2}{3}}{1-\frac{4}{9}}\right)$

$= \frac{1}{2}\tan^{-1}\left(\frac{\frac{4}{3}}{\frac{5}{9}}\right)$

$= \frac{1}{2}tan^{-1}\left(\frac{12}{5}\right)$

= RHS

So, $tan^{-1}(\frac{2}{3}) = \frac{1}{2}tan^{-1}\left(\frac{12}{5}\right)$

Hence, proved.

(iv) Given $\tan^{-1}(1/7) + 2\tan^{-1}(1/3) = \pi/4$

Consider LHS

$= \tan^{-1}(\frac{1}{7}) + 2\tan^{-1}(\frac{1}{3})$

We know that,

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

By substituting the above formula we get,

$= \tan^{-1}(\frac{1}{7}) + \tan^{-1}(\frac{2\times\frac{1}{3}}{1-\frac{1}{9}})$

$= \tan^{-1}(\frac{1}{7}) + \tan^{-1}(\frac{\frac{2}{3}}{\frac{8}{9}})$

$= \tan^{-1}(\frac{1}{7}) + \tan^{-1}(\frac{3}{4})$

Again we know that

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$$

$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\times\frac{3}{4}}\right)$

$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

= RHS

So, $\tan^{-1}(\frac{1}{7}) + 2\tan^{-1}(\frac{1}{3}) = \frac{\pi}{4}$

Hence, proved.

(v) Given $\sin^{-1}(4/5) + 2\tan^{-1}(1/3) = \pi/2$

Consider LHS

$= \sin^{-1}(\frac{4}{5}) + 2\tan^{-1}(\frac{1}{3})$

We know that,

$\sin^{-1}(x) = \tan^{-1}(\frac{x}{\sqrt{1-x^2}})$

And, $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now by substituting the formula we get,

$= \tan^{-1}(\frac{\frac{4}{5}}{\sqrt{1-\frac{16}{25}}}) + \tan^{-1}(\frac{2\times\frac{1}{3}}{1-\frac{1}{9}})$

$= \tan^{-1}(\frac{\frac{4}{5}}{\sqrt{\frac{9}{25}}}) + \tan^{-1}(\frac{\frac{2}{3}}{\frac{8}{9}})$

$= \tan^{-1}(\frac{4}{3}) + \tan^{-1}(\frac{3}{4})$

We know that,

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{3}{4}}{1-\frac{4}{3}\times\frac{3}{4}}\right)$

$= \tan^{-1}\left(\frac{\frac{25}{12}}{0}\right)$

$= \tan^{-1}(\infty)$

$= \frac{\pi}{2}$

= RHS

So, $\sin^{-1}(\frac{4}{5}) + 2\tan^{-1}(\frac{1}{3}) = \frac{\pi}{2}$

Hence Proved

(vi) Given 2 $\sin^{-1}$ (3/5) – $\tan^{-1}$ (17/31) = π/4

Consider LHS

$= 2\sin^{-1}(\frac{3}{5}) - \tan^{-1}(\frac{17}{31})$

We know that

$$\sin^{-1}(x) = \tan^{-1}(\frac{x}{\sqrt{1-x^2}})$$

According to the formula we have,

$$= 2\tan^{-1}(\frac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}}) - \tan^{-1}(\frac{17}{31})$$

$$= 2\tan^{-1}(\frac{\frac{3}{5}}{\sqrt{\frac{16}{25}}}) - \tan^{-1}(\frac{17}{31})$$

$$= 2\tan^{-1}(\frac{3}{4}) - \tan^{-1}(\frac{17}{31})$$

Again we know that,

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

By substituting this formula, we get

$$= \tan^{-1}(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}) - \tan^{-1}(\frac{17}{31})$$

$= \tan^{-1}(\frac{\frac{3}{2}}{\frac{7}{16}}) - \tan^{-1}(\frac{17}{31})$

$= \tan^{-1}(\frac{24}{7}) - \tan^{-1}(\frac{17}{31})$

Again we have,

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)$

$= \tan^{-1}\left(\frac{\frac{744-119}{217}}{\frac{217+408}{217}}\right)$

$= \tan^{-1}\left(\frac{625}{625}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$ = RHS

So, $2\sin^{-1}(\frac{3}{5}) - \tan^{-1}(\frac{17}{31}) = \frac{\pi}{4}$

Hence the proof.

(vii) Given $2\tan^{-1}(1/5) + \tan^{-1}(1/8) = \tan^{-1}(4/7)$

Consider LHS

$= 2\tan^{-1}(\frac{1}{5}) + \tan^{-1}(\frac{1}{8})$

We know that

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now by substituting the formula we get,

$= \tan^{-1}(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}) + \tan^{-1}(\frac{1}{8})$

$= \tan^{-1}(\frac{\frac{2}{5}}{\frac{24}{25}}) + \tan^{-1}(\frac{1}{8})$

$= \tan^{-1}(\frac{5}{12}) + \tan^{-1}(\frac{1}{8})$

Again from the formula we have,

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{1}{8}}{1-\frac{5}{12}\times\frac{1}{8}}\right)$

$= \tan^{-1}\left(\frac{\frac{10+3}{24}}{\frac{96-5}{96}}\right)$

$= \tan^{-1}\left(\frac{13}{24}\times\frac{96}{91}\right)$

$= \tan^{-1}\left(\frac{4}{7}\right)$

= RHS

So, $2\tan^{-1}(\frac{1}{5}) + \tan^{-1}(\frac{1}{8}) = \tan^{-1}(\frac{4}{7})$

Hence, proved.

(viii) Given 2 $\tan^{-1}$ (3/4) - $\tan^{-1}$ (17/31) = π/4

Consider LHS

$= 2\tan^{-1}(\frac{3}{4}) - \tan^{-1}(\frac{17}{31})$

We know that,

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Now by substituting the formula we get,

$= \tan^{-1}(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}) - \tan^{-1}(\frac{17}{31})$

$= \tan^{-1}(\frac{3}{2}\times\frac{16}{7}) - \tan^{-1}(\frac{17}{31})$

$= \tan^{-1}(\frac{24}{7}) - \tan^{-1}(\frac{17}{31})$

We know that,

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$$

Again by substituting the formula we get,

$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)$

$= \tan^{-1}\left(\frac{\frac{744-119}{217}}{\frac{217+408}{217}}\right)$

$= \tan^{-1}\left(\frac{625}{625}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

= RHS

So, $2\tan^{-1}(\frac{3}{4}) - \tan^{-1}(\frac{17}{31}) = \frac{\pi}{4}$

Hence, proved.

(ix) Given 2 $\tan^{-1}$ (1/2) + $\tan^{-1}$ (1/7) = $\tan^{-1}$ (31/17)

Consider LHS

$= 2\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{7})$

We know that,

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Now by substituting the formula we get,

$= \tan^{-1}(\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}) + \tan^{-1}(\frac{1}{7})$

$= \tan^{-1}(\frac{\frac{2}{2}}{\frac{3}{4}}) + \tan^{-1}(\frac{1}{7})$

$= \tan^{-1}(\frac{4}{3}) + \tan^{-1}(\frac{1}{7})$

Again by using the formula, we can write as

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{1}{7}\times\frac{4}{3}}\right)$

$= \tan^{-1}\left(\frac{\frac{31}{21}}{\frac{17}{21}}\right)$

$= \tan^{-1}\left(\frac{31}{17}\right)$

= RHS

So, $2\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{7}) = \tan^{-1}(\frac{31}{17})$

Hence, proved.

(x) Given 4 $\tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4$

Consider LHS

$= 4\tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239})$

We know that,

$$4\tan^{-1}x = \tan^{-1}\left(\frac{4x - 4x^3}{1 - 6x^2 + x^4}\right)$$

Now by substituting the formula, we get

$= \tan^{-1}\left(\frac{4\times\frac{1}{5}-4\left(\frac{1}{5}\right)^3}{1-6\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^4}\right) - \tan^{-1}(\frac{1}{239})$

$= \tan^{-1}(\frac{120}{119}) - \tan^{-1}(\frac{1}{239})$

Again we know that,

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x - y}{1 + xy}$$

$= \tan^{-1}\left(\frac{\frac{120}{119}-\frac{1}{239}}{1-\frac{120}{119}\times\frac{1}{239}}\right)$

$= \tan^{-1}\left(\frac{120\times239-119}{119\times239+120}\right)$

$= \tan^{-1}\left(\frac{28561}{28561}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

= RHS

So,

$4\tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239}) = \frac{\pi}{4}$

Hence, proved.

3. If $\sin^{-1}(2a/1 + a^2) - \cos^{-1}(1 - b^2/1 + b^2) = \tan^{-1}(2x/1 - x^2)$, then prove that $x = (a - b)/(1 + a b)$

Solution:

Given $\sin^{-1}(2a/1 + a^2) - \cos^{-1}(1 - b^2/1 + b^2) = \tan^{-1}(2x/1 - x^2)$

Consider,

$\Rightarrow \sin^{-1}(\frac{2a}{1+a^2}) - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}(\frac{2x}{1-x^2})$

We know that,

$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now by applying these formulae in given equation we get,

$\Rightarrow 2\tan^{-1}(a) - 2\tan^{-1}(b) = 2\tan^{-1}(x)$

$\Rightarrow 2(\tan^{-1}(a) - \tan^{-1}(b)) = 2\tan^{-1}(x)$

$\Rightarrow \tan^{-1}(a) - \tan^{-1}(b) = \tan^{-1}(x)$

Again we know that,

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy}$$

Now by substituting this in above equation we get,

$\Rightarrow \tan^{-1}(\frac{a-b}{1+ab}) = \tan^{-1}(x)$

On comparing we get,

$\Rightarrow x = \frac{a-b}{1+ab}$

Hence, proved.

4. Prove that:
(i) $\tan^{-1}\{(1 - x^2)/ 2x)\} + \cot^{-1}\{(1 - x^2)/ 2x)\} = \pi/2$
(ii) $\sin \{\tan^{-1} (1 - x^2)/ 2x) + \cos^{-1} (1 - x^2)/ (1 + x^2)\} = 1$

Solution:
(i) Given $\tan^{-1}\{(1 - x^2)/ 2x)\} + \cot^{-1}\{(1 - x^2)/ 2x)\} = \pi/2$

Consider LHS

$= \tan^{-1} \frac{1-x^2}{2x} + \cot^{-1} \frac{1-x^2}{2x}$

We know that,

$$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$$

Now by applying the above formula we get,

$= \tan^{-1}\left(\frac{1-x^2}{2x}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Again we know

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

By substituting this we get,

$= \tan^{-1}\left(\frac{\left(\frac{1-x^2}{2x}\right)+\left(\frac{2x}{1-x^2}\right)}{1-\left(\frac{1-x^2}{2x}\right)\times\left(\frac{2x}{1-x^2}\right)}\right)$

$= \tan^{-1}\left(\frac{\frac{1+x^4-2x^2+4x^2}{2x(1-x^2)}}{\frac{2x(1-x^2)-2x(1-x^2)}{2x(1-x^2)}}\right)$

$= \tan^{-1}\left(\frac{1+x^4+2x^2}{0}\right)$

$= \tan^{-1}(\infty)$

$= \frac{\pi}{2}$ = RHS

$$\tan^{-1}\frac{1-x^2}{2x} + \cot^{-1}\frac{1-x^2}{2x} = \frac{\pi}{2}$$

Hence, proved.

(ii) Given sin {$\tan^{-1} (1 – x^2)/ 2x) + \cos^{-1} (1 - x^2)/ (1 + x^2)$}

Consider LHS

$= \sin\left(\tan^{-1}\frac{1-x^2}{2x} + \cos^{-1}\frac{1-x^2}{1+x^2}\right)$

We know that,

$$2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Now by applying the formula in above question we get,

$$= \sin\left(\tan^{-1}\frac{1-x^2}{2x} + 2\tan^{-1}x\right)$$

Again, we have

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Now by substituting the formula we get,

$$= \sin\left(\tan^{-1}\frac{1-x^2}{2x} + \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)$$

Again we know that,

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

Now by applying the formula,

$$= \sin\left(\tan^{-1}\left(\frac{\frac{1-x^2}{2x}+\left(\frac{2x}{1-x^2}\right)}{1-\frac{1-x^2}{2x}\times\left(\frac{2x}{1-x^2}\right)}\right)\right)$$

$$= \sin\left(\tan^{-1}\left(\frac{\frac{1+x^4-2x^2+4x^2}{2x(1-x^2)}}{\frac{2x(1-x^2)-2x(1-x^2)}{2x(1-x^2)}}\right)\right)$$

$$= \sin\left(\tan^{-1}\left(\frac{\frac{1+x^4-2x^2+4x^2}{2x(1-x^2)}}{0}\right)\right)$$

$$= \sin(\tan^{-1}(\infty))$$

$= \sin\left(\frac{\pi}{2}\right)$

= 1

= RHS

So,

$$\sin^{-1}\left(\tan^{-1}\frac{1-x^2}{2x}+\cos^{-1}\frac{1-x^2}{1+x^2}\right)=1$$

Hence, proved.

5. If $\sin^{-1}$ (2a/ 1+ a^2) + $\sin^{-1}$ (2b/ 1+ b^2) = 2 $\tan^{-1}$ x, prove that x = (a + b/ 1 – a b)

Solution:

Given $\sin^{-1}$ (2a/ 1+ a^2) + $\sin^{-1}$ (2b/ 1+ b^2) = 2 $\tan^{-1}$ x

Consider

$$\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}(x)$$

We know that,

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Now by applying the above formula we get,

$$\Rightarrow 2\tan^{-1}(a)+2\tan^{-1}(b) = 2\tan^{-1}(x)$$

$$\Rightarrow 2(\tan^{-1}(a)+\tan^{-1}(b)) = 2\tan^{-1}(x)$$

$$\Rightarrow \tan^{-1}(a)+\tan^{-1}(b) = \tan^{-1}(x)$$

Again we have,

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

Now by substituting, we get

$\Rightarrow \tan^{-1}(\frac{a+b}{1-ab}) = \tan^{-1}(x)$

On comparing we get,

$\Rightarrow x = \frac{a+b}{1-ab}$

Hence, proved.