EXERCISE 2.4

State with reason whether the following functions have inverse:
(i) f: {1, 2, 3, 4} → {10} with f = {(1, 10), (2, 10), (3, 10), (4, 10)}
(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} with g = {(5, 4), (6, 3), (7, 4), (8, 2)}
(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} with h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Solution:
(i) Given f: {1, 2, 3, 4} → {10} with f = {(1, 10), (2, 10), (3, 10), (4, 10)}
We have:
f (1) = f (2) = f (3) = f (4) = 10
⇒ f is not one-one.
⇒ f is not a bijection.
So, f does not have an inverse.

(ii) Given g: {5, 6, 7, 8} → {1, 2, 3, 4} with g = {(5, 4), (6, 3), (7, 4), (8, 2)}
from the question it is clear that g (5) = g (7) = 4
⇒ f is not one-one.
⇒ f is not a bijection.
So, f does not have an inverse.

(iii) Given h: {2, 3, 4, 5} → {7, 9, 11, 13} with h = {(2, 7), (3, 9), (4, 11), (5, 13)}
Here, different elements of the domain have different images in the co-domain.
⇒ h is one-one.
Also, each element in the co-domain has a pre-image in the domain.
⇒ h is onto.
⇒ h is a bijection.
Therefore h inverse exists.
⇒ h has an inverse and it is given by
h^{-1} = {(7, 2), (9, 3), (11, 4), (13, 5)}

2. Find f^{-1} if it exists: f: A → B, where
(i) A = {0, −1, −3, 2}; B = {−9, −3, 0, 6} and f(x) = 3 x.
(ii) A = {1, 3, 5, 7, 9}; B = {0, 1, 9, 25, 49, 81} and f(x) = x^2

Solution:
(i) Given A = {0, −1, −3, 2}; B = {−9, −3, 0, 6} and f(x) = 3 x.
So, f = {(0, 0), (-1, -3), (-3, -9), (2, 6)}

Here, different elements of the domain have different images in the co-domain.
Clearly, this is one-one.
Range of f = Range of f = B
so, f is a bijection and,
Thus, f^{-1} exists.
Hence, f^{-1}= {(0, 0), (-3, -1), (-9, -3), (6, 2)}

(ii) Given A = {1, 3, 5, 7, 9}; B = {0, 1, 9, 25, 49, 81} and f(x) = x^2
So, f = {(1, 1), (3, 9), (5, 25), (7, 49), (9, 81)}
Here, different elements of the domain have different images in the co-domain.
Clearly, f is one-one.
But this is not onto because the element 0 in the co-domain (B) has no pre-image in the domain (A)
⇒ f is not a bijection.
So, f^{-1}does not exist.

3. Consider f: {1, 2, 3} → {a, b, c} and g: {a, b, c} → {apple, ball, cat} defined as f (1) = a, f (2) = b, f (3) = c, g (a) = apple, g (b) = ball and g (c) = cat. Show that f, g and gof are invertible. Find f^{-1}, g^{-1} and $gof^{-1}$and show that $(gof)^{-1}$ = f^{-1}o g^{-1}

Solution:
Given f = {(1, a), (2, b), (c , 3)} and g = {(a , apple) , (b , ball) , (c , cat)} Clearly , f and g are bijections.
So, f and g are invertible.
Now,
f^{-1} = {(a ,1) , (b , 2) , (3,c)} and g^{-1} = {(apple, a), (ball , b), (cat , c)}
So, f^{-1} o g^{-1}= {apple, 1), (ball, 2), (cat, 3)}..........(1)
f: {1,2,3,} → {a, b, c} and g: {a, b, c} → {apple, ball, cat}
So, gof: {1, 2, 3} → {apple, ball, cat}
⇒ (gof) (1) = g (f (1)) = g (a) = apple
(gof) (2) = g (f (2))
= g (b)
= ball,
And (gof) (3) = g (f (3))
= g (c)
= cat
∴ gof = {(1, apple), (2, ball), (3, cat)}

Clearly, gof is a bijection.
So, gof is invertible.
$(gof)^{-1}$ = {(apple, 1), (ball, 2), (cat, 3)}....... (2)
Form (1) and (2), we get
$(gof)^{-1} = f^{-1} \text{ o } g^{-1}$

4. Let A = {1, 2, 3, 4}; B = {3, 5, 7, 9}; C = {7, 23, 47, 79} and f: A → B, g: B → C be defined as f(x) = 2x + 1 and g(x) = $x^2 - 2$. Express $(gof)^{-1}$ and $f^{-1} og^{-1}$ as the sets of ordered pairs and verify that $(gof)^{-1} = f^{-1} og^{-1}$.

Solution:
Given that f (x) = 2x + 1
⇒ f= {(1, 2(1) + 1), (2, 2(2) + 1), (3, 2(3) + 1), (4, 2(4) + 1)}
= {(1, 3), (2, 5), (3, 7), (4, 9)}
Also given that g(x) = x^2-2
⇒ g = {(3, 3^2-2), (5, 5^2-2), (7, 7^2-2), (9, 9^2-2)}
= {(3, 7), (5, 23), (7, 47), (9, 79)}
Clearly f and g are bijections and, hence, f^{-1}: B→ A and g^{-1}: C→ B exist.
So, f^{-1}= {(3, 1), (5, 2), (7, 3), (9, 4)}
And g^{-1}= {(7, 3), (23, 5), (47, 7), (79, 9)}
Now, (f^{-1} o g^{-1}): C→ A
f^{-1} o g^{-1} = {(7, 1), (23, 2), (47, 3), (79, 4)}...........(1)
Also, f: A→B and g: B → C,
⇒ gof: A → C, $(gof)^{-1}$: C→ A
So, f^{-1} o $g^{-1}$and $(gof)^{-1}$ have same domains.
(gof) (x) = g (f (x))
=g (2x + 1)
=$(2x +1)^2- 2$
⇒ (gof) (x) = $4x^2 + 4x + 1 - 2$
⇒ (gof) (x) = $4x^2+ 4x -1$
Then, (gof) (1) = g (f (1))
= 4 + 4 − 1
=7,
(gof) (2) = g (f (2))
= $4(2)^2 + 4(2) - 1 = 23$,
(gof) (3) = g (f (3))
= $4(3)^2 + 4(3) - 1 = 47$ and

(gof) (4) = g (f (4))
$= 4(4)^2 + 4(4) - 1 = 79$
So, gof = {(1, 7), (2, 23), (3, 47), (4, 79)}
⇒ $(gof)^{-1}$ = {(7, 1), (23, 2), (47, 3), (79, 4)}.......(2)
From (1) and (2), we get:
$(gof)^{-1} = f^{-1} \circ g^{-1}$

5. Show that the function f: Q → Q, defined by f(x) = 3x + 5, is invertible. Also, find f^{-1}

Solution:
Given function f: Q → Q, defined by f(x) = 3x + 5
Now we have to show that the given function is invertible.
Injection of f:
Let x and y be two elements of the domain (Q),
Such that f(x) = f(y)
⇒ 3x + 5 = 3y + 5
⇒ 3x = 3y
⇒ x = y
so, f is one-one.
Surjection of f:
Let y be in the co-domain (Q),
Such that f(x) = y
⇒ 3x +5 = y
⇒ 3x = y - 5
⇒ x = (y -5)/3 belongs to Q domain
⇒ f is onto.
So, f is a bijection and, hence, it is invertible.
Now we have to find f^{-1}:
Let $f^{-1}(x)$ = y… …(1)
⇒ x = f(y)
⇒ x = 3y + 5
⇒ x −5 = 3y
⇒ y = (x - 5)/3
Now substituting this value in (1) we get
So, $f^{-1}(x)$ = (x - 5)/3

6. Consider f: R → R given by f(x) = 4x + 3. Show that f is invertible. Find the inverse

of f.

Solution:

Given f: R → R given by f(x) = 4x + 3
Now we have to show that the given function is invertible.
Consider injection of f:
Let x and y be two elements of domain (R),
Such that f(x) = f(y)
⇒ 4x + 3 = 4y + 3
⇒ 4x = 4y
⇒ x = y
So, f is one-one.
Now surjection of f:
Let y be in the co-domain (R),
Such that f(x) = y.
⇒ 4x + 3 = y
⇒ 4x = y -3
⇒ x = (y-3)/4 in R (domain)
⇒ f is onto.
So, f is a bijection and, hence, it is invertible.
Now we have to find f^{-1}
Let $f^{-1}(x)$ = y....... (1)
⇒ x = f (y)
⇒ x = 4y + 3
⇒ x − 3 = 4y
⇒ y = (x -3)/4
Now substituting this value in (1) we get
So, $f^{-1}(x)$ = (x-3)/4

7. Consider f: R → R^+ → [4, ∞) given by f(x) = x^2 + 4. Show that f is invertible with inverse f^{-1} of f given by $f^{-1}(x)$ = √ (x-4) where R^+ is the set of all non-negative real numbers.

Solution:

Given f: R → R^+ → [4, ∞) given by f(x) = x^2 + 4.
Now we have to show that f is invertible,
Consider injection of f:

Let x and y be two elements of the domain (Q),
Such that f(x) = f(y)
⇒ $x^2 + 4 = y^2 + 4$
⇒ $x^2 = y^2$
⇒ x = y (as co-domain as R+)
So, f is one-one
Now surjection of f:
Let y be in the co-domain (Q),
Such that f(x) = y
⇒ $x^2 + 4 = y$
⇒ $x^2 = y - 4$
⇒ x = √ (y-4) in R
⇒ f is onto.
So, f is a bijection and, hence, it is invertible.
Now we have to find f^{-1}:
Let f^{-1} (x) = y…… (1)
⇒ x = f (y)
⇒ $x = y^2 + 4$
⇒ $x – 4 = y^2$
⇒ y = √ (x-4)
So, $f^{-1}(x)$ = √ (x-4)
Now substituting this value in (1) we get,
So, $f^{-1}(x)$ = √ (x-4)

8. If f(x) = (4x + 3)/ (6x - 4), x ≠ (2/3) show that fof(x) = x, for all x ≠ (2/3). What is the inverse of f?

Solution:
It is given that f(x) = (4x + 3)/ (6x - 4), x ≠ 2/3
Now we have to show fof(x) = x
(fof)(x) = f (f(x))
= f ((4x+ 3)/ (6x - 4))
= (4((4x + 3)/ (6x -4)) + 3)/ (6 ((4x +3)/ (6x - 4)) - 4)
= (16x + 12 + 18x - 12)/ (24x + 18 - 24x + 16)
= (34x)/ (34)
= x
Therefore, fof(x) = x for all x ≠ 2/3

=> fof = 1
Hence, the given function f is invertible and the inverse of f is f itself.

9. Consider f: $R^+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(x) = (\sqrt{x+6}-1)/3$

Solution:
Given f: $R^+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$
We have to show that f is invertible.
Injectivity of f:
Let x and y be two elements of domain (R^+),
Such that f(x) = f(y)
$\Rightarrow 9x^2 + 6x - 5 = 9y^2 + 6y - 5$
$\Rightarrow 9x^2 + 6x = 9y^2 + 6y$
$\Rightarrow x = y$ (As, $x, y \in R^+$)
So, f is one-one.
Surjectivity of f:
Let y is in the co domain (Q)
Such that f(x) = y
$\Rightarrow 9x^2 + 6x - 5 = y$
$\Rightarrow 9x^2 + 6x = y + 5$
$\Rightarrow 9x^2 + 6x + 1 = y + 6$ (By adding 1 on both sides)
$\Rightarrow (3x + 1)^2 = y + 6$
$\Rightarrow 3x + 1 = \sqrt{(y + 6)}$
$\Rightarrow 3x = \sqrt{(y + 6)} - 1$
$\Rightarrow x = (\sqrt{(y + 6)}-1)/3$ in R^+ (domain)
f is onto.
So, f is a bijection and hence, it is invertible.
Now we have to find f^{-1}
Let $f^{-1}(x) = y$. .... (1)
$\Rightarrow x = f(y)$
$\Rightarrow x = 9y^2 + 6y - 5$
$\Rightarrow x + 5 = 9y^2 + 6y$
$\Rightarrow x + 6 = 9y^2 + 6y + 1$ (adding 1 on both sides)
$\Rightarrow x + 6 = (3y + 1)^2$
$\Rightarrow 3y + 1 = \sqrt{(x + 6)}$
$\Rightarrow 3y = \sqrt{(x + 6)} - 1$

⇒ $y = (\sqrt{(x+6)}-1)/3$
Now substituting this value in (1) we get,
So, $f^{-1}(x) = (\sqrt{(x+6)}-1)/3$

10. If f: R → R be defined by $f(x) = x^3 - 3$, then prove that f^{-1} exists and find a formula for f^{-1}. Hence, find f^{-1} (24) and f^{-1} (5).

Solution:
Given f: R → R be defined by $f(x) = x^3 - 3$
Now we have to prove that f^{-1} exists
Injectivity of f:
Let x and y be two elements in domain (R),
Such that, $x^3 - 3 = y^3 - 3$
⇒ $x^3 = y^3$
⇒ $x = y$
So, f is one-one.
Surjectivity of f:
Let y be in the co-domain (R)
Such that f(x) = y
⇒ $x^3 - 3 = y$
⇒ $x^3 = y + 3$
⇒ $x = \sqrt[3]{(y+3)}$ in R
⇒ f is onto.
So, f is a bijection and, hence, it is invertible.
Finding f^{-1}:
Let $f^{-1}(x) = y$… .....(1)
⇒ $x= f(y)$
⇒ $x = y^3 - 3$
⇒ $x + 3 = y^3$
⇒ $y = \sqrt[3]{(x + 3)} = f^{-1}(x)$ [from (1)]
So, $f^{-1}(x) = \sqrt[3]{(x + 3)}$
Now, $f^{-1}(24) = \sqrt[3]{(24 + 3)}$
$= \sqrt[3]{27}$
$= \sqrt[3]{3^3}$
$= 3$
And $f^{-1}(5) = \sqrt[3]{(5 + 3)}$
$= \sqrt[3]{8}$

$= \sqrt[3]{2^3}$
$= 2$

11. A function f: R → R is defined as f(x) = x^3 + 4. Is it a bijection or not? In case it is a bijection, find f^{-1} (3).

Solution:
Given that f: R → R is defined as f(x) = $x^3 + 4$
Injectivity of f:
Let x and y be two elements of domain (R),
Such that f (x) = f (y)
$\Rightarrow x^3 + 4 = y^3 + 4$
$\Rightarrow x^3 = y^3$
$\Rightarrow x = y$
So, f is one-one.
Surjectivity of f:
Let y be in the co-domain (R),
Such that f(x) = y.
$\Rightarrow x^3 + 4 = y$
$\Rightarrow x^3 = y - 4$
$\Rightarrow x = \sqrt[3]{(y - 4)}$ in R (domain)
⇒ f is onto.
So, f is a bijection and, hence, it is invertible.
Finding f^{-1}:
Let $f^{-1}(x) = y$…… (1)
$\Rightarrow x = f(y)$
$\Rightarrow x = y^3 + 4$
$\Rightarrow x - 4 = y^3$
$\Rightarrow y = \sqrt[3]{(x-4)}$
So, $f^{-1}(x) = \sqrt[3]{(x-4)}$ [from (1)]
$f^{-1}(3) = \sqrt[3]{(3 - 4)}$
$= \sqrt[3]{-1}$
$= -1$