EXERCISE 4.11

1. Prove the following results:
(i) $\tan^{-1}(1/7) + \tan^{-1}(1/13) = \tan^{-1}(2/9)$
(ii) $\sin^{-1}(12/13) + \cos^{-1}(4/5) + \tan^{-1}(63/16) = \pi$
(iii) $\tan^{-1}(1/4) + \tan^{-1}(2/9) = \sin^{-1}(1/\sqrt{5})$

Solution:
(i) Given $\tan^{-1}(1/7) + \tan^{-1}(1/13) = \tan^{-1}(2/9)$

Consider LHS

$\tan^{-1}(\frac{1}{7}) + \tan^{-1}(\frac{1}{13})$

We know that, Formula

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

According to the formula, we can write as

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{13+7}{91}}{\frac{91-1}{91}}\right)$$

$$= \tan^{-1}\left(\frac{20}{90}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right)$$

= RHS
Hence, proved.

(ii) Given $\sin^{-1}(12/13) + \cos^{-1}(4/5) + \tan^{-1}(63/16) = \pi$
Consider LHS

$\sin^{-1}(\frac{12}{13}) + \cos^{-1}\frac{4}{5} + \tan^{-1}(\frac{63}{16})$

We know that, Formula

$\sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$\cos^{-1}x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

Now, by substituting the formula we get,

$= \tan^{-1}\left(\frac{\frac{12}{13}}{\sqrt{1-(\frac{12}{13})^2}}\right) + \tan^{-1}\left(\frac{\sqrt{1-(\frac{4}{5})^2}}{\frac{4}{5}}\right) + \tan^{-1}(\frac{63}{16})$

$= \tan^{-1}(\frac{12}{5}) + \tan^{-1}(\frac{3}{4}) + \tan^{-1}(\frac{63}{16})$

Again we know that,

$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\frac{x+y}{1-xy}$

Again by substituting, we get

$= \pi + \tan^{-1}(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}) + \tan^{-1}(\frac{63}{16})$

$= \pi + \tan^{-1}(-\frac{63}{16}) + \tan^{-1}(\frac{63}{16})$

We know that,

$\tan^{-1}(-x) = -\tan^{-1}x$

$= \pi - \tan^{-1}(\frac{63}{16}) + \tan^{-1}(\frac{63}{16})$

$= \pi$

So, $\sin^{-1}(\frac{12}{13}) + \cos^{-1}\frac{4}{5} + \tan^{-1}(\frac{63}{16}) = \pi$

Hence, proved.

(iii) Given $\tan^{-1}(1/4) + \tan^{-1}(2/9) = \sin^{-1}(1/\sqrt{5})$

$\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{2}{9})$

We know that,

$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\frac{x+y}{1-xy}$

By substituting this formula we get,

$= \tan^{-1}\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}$

$= \tan^{-1}\frac{\frac{17}{36}}{\frac{34}{36}}$

$= \tan^{-1}\frac{\frac{17}{36}}{\frac{34}{36}}$

$= \tan^{-1}\frac{1}{2}$

Now let, $\tan\theta = \frac{1}{2}$

Therefore, $\sin\theta = \frac{1}{\sqrt{5}}$

So, $\theta = \sin^{-1}\frac{1}{\sqrt{5}}$

$\Rightarrow \tan^{-1}(\frac{1}{2}) = \sin^{-1}(\frac{1}{\sqrt{5}}) = \text{RHS}$

$\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{2}{9}) = \sin^{-1}(\frac{1}{\sqrt{5}})$

Hence, Proved.

2. Find the value of $\tan^{-1}$ (x/y) – $\tan^{-1}$ {(x-y)/(x + y)}

Solution:

Given $\tan^{-1}$ (x/y) – $\tan^{-1}$ {(x-y)/(x + y)}

We know that,

$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy}$

Now by substituting the formula, we get

$= \tan^{-1} \frac{\frac{x}{y} - \left(\frac{x-y}{x+y}\right)}{1 + \frac{x}{y} \times \left(\frac{x-y}{x+y}\right)}$

$= \tan^{-1} \frac{x(x+y) - y(x-y)}{y(x+y) + x(x-y)}$

$= \tan^{-1} \frac{x^2 + y^2}{x^2 + y^2}$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$

So,

$\tan^{-1}(\frac{x}{y}) - \tan^{-1}(\frac{x - y}{x + y}) = \frac{\pi}{4}$